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# ELEMENTE DER MATHEMATIK

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## Note on Packing of 19 Equal Circles on a Sphere

Consider the problem of determining the largest angular diameter  $a_n$  of  $n$  equal circles (or spherical caps) which can be packed on the surface of a sphere without overlapping. For  $n = 20$  van der Waerden [4] has published an arrangement as conjectured solution with circle diameter  $a_{20} = 47^\circ 26'$ . The graph of this packing may be seen in a simplified stereographic projection in Fig. 1. The vertices of the graph are the centres of the spherical circles and the edges of the graph are great-circle arcs joining the centres of the touching spherical circles. Goldberg [1] has thought that after removing an appropriate circle from van der Waerden's arrangement the packing of the remaining 19 circles can be improved. Goldberg has come very near to finding a correct improved packing, however, he has probably overlooked something and obtained faulty results. In [2], this mistake has been mentioned but not corrected. Therefore, at present, no packing is known for 19 circles better than that obtained by removing a circle from van der Waerden's packing of 20 circles.

The aim of this paper is to correct Goldberg's results and to present a packing of 19 equal circles on a sphere, in which  $a_{19}$  is greater than  $a_{20}$  due to van der Waerden.

In fact, two errors are made in [1]: first, to sharpen van der Waerden's result  $a_{20}$  to  $47^\circ 24' 51''$ , though its correct value is  $a_{20} = 47^\circ 25' 51.7''$ ; second, to compute the circle diameter  $a_{19}$  from such an arrangement which contains some overlapping circles. This second can be shown, e.g., in the following way.

Goldberg has discovered that by removing the isolated points  $M$  and  $N$  from the graph in Fig. 1 and reflecting the vertices  $A, C, E$  and  $G, I, K$  in the great-circle arcs  $FB, BD, DF$  and  $LH, HJ, JL$ , respectively, a packing with the same circle diameter can be obtained for  $n = 18$ . It is obvious that a packing for  $n = 19$  is obtained with the same circle diameter if this process is only executed on one of the two hemispheres (Fig. 2). Goldberg has considered the graph in Fig. 2 with additional edges  $GN, IN, KN$  and obtained that the circle diameter in this arrangement is  $a_{19} = 47^\circ 25' 22''$ . Since the graph in Fig. 2 is rigid in Danzerian sense [3] the edges  $GN, IN, KN$  cannot be added to the graph by moving the graph and preserving all of its edges. The removal of the isolation of the isolated point  $N$  with preservation of the

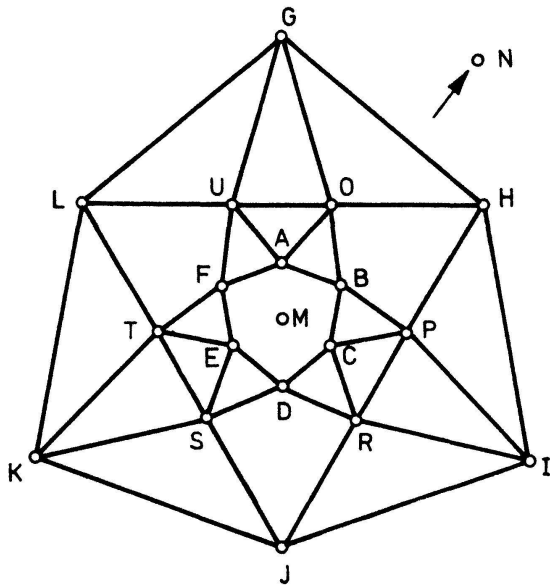


Figure 1

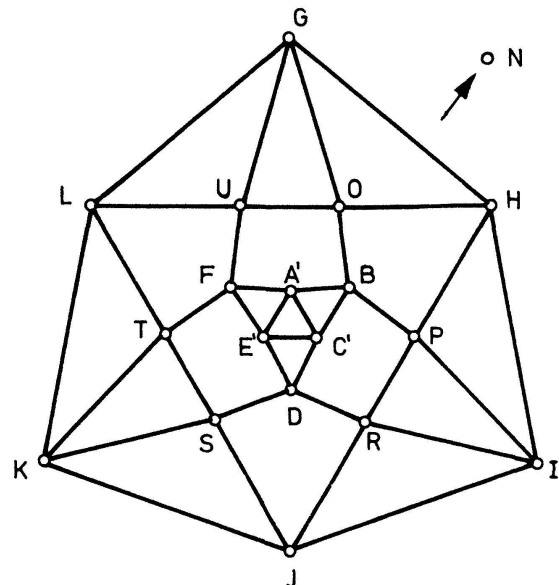


Figure 2

three-fold rotational symmetry of the graph can be done only by removing some of the edges of the graph in Fig. 2 and with decreased edge-lengths. It can be done, e.g., as shown in Fig. 3 where  $a_{19} = 47^\circ 25' 20.2''$ , or in Fig. 4 where  $a_{19} = 47^\circ 25' 21.4''$  and the graph is obtained by moving the graph of Fig. 3. Thus, this kind of process for improvement is unsuccessful.

Goldberg ([1], Fig. 4) has considered in bypassing also a further, less symmetrical arrangement of 19 circles. Now a slight modification of the graph of that arrangement indeed leads to a good packing. The improving process in this case can be the following. Let the isolated point  $M$  and the edges  $AU$ ,  $AO$ ,  $BP$ ,  $CR$ ,  $ES$ ,  $FT$ ,  $IP$ ,  $KT$  be removed from the graph of our Fig. 1. Reflect the vertex  $A$  with the edges  $AB$ ,  $AF$  in the great-circle arc  $BF$  (snap

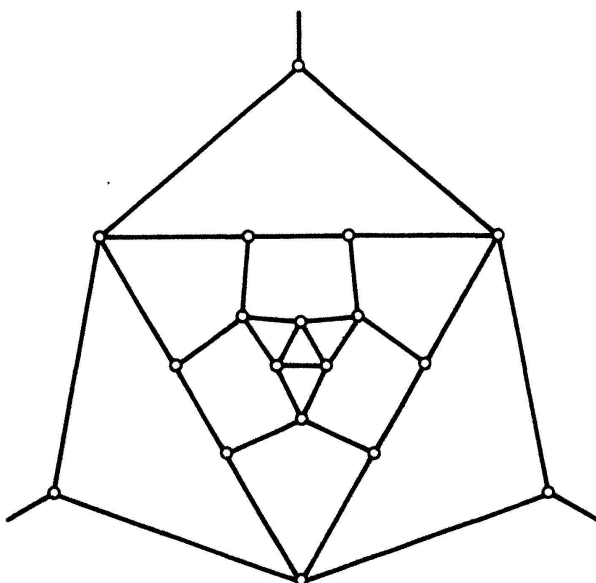


Figure 3

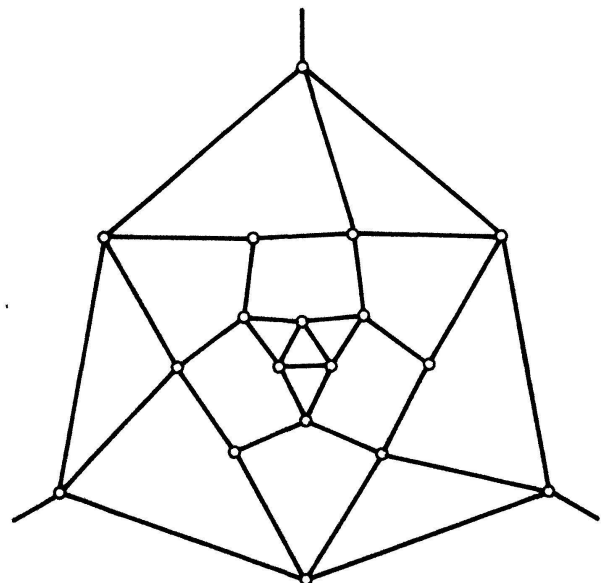


Figure 4

the pair of edges  $AB, AF$  through between the vertices  $B, F$ ). In this way the graph in Fig. 5 is obtained, which is not rigid in Danzerian sense and so can be moved with a simultaneous increase in the length of the edges. The edge-length, that is, the diameter of the circles can be increased until the distances between points  $A'$  and  $C, A'$  and  $E, G$  and  $N, I$  and  $N, K$  and  $N$  are equal to the increased edge-length in the graph. So, five additional edges appear in the graph, which make the graph rigid in Danzerian sense and prevent any further increase in the edge-length. In this way we obtained a new packing of 19 equal circles on the sphere, at which the angular diameter of the circles is

$$a_{19} = 47^\circ 40' 33.3''.$$

The graph of the new arrangement may be seen in a simplified stereographic projection in Fig. 6. The edge-length of the graph in Fig. 6 was computed by spherical trigonometry and iteration using the fact that the arrangement has a plane of symmetry, but the details are omitted here.

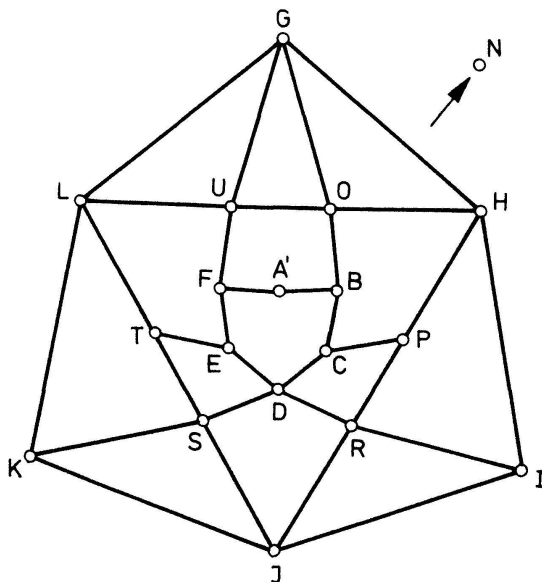


Figure 5

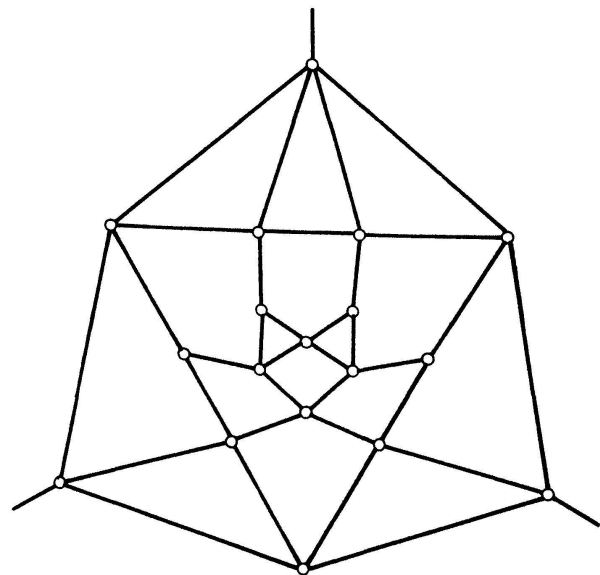


Figure 6

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