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Note on Packing of 19 Equal Circles on a Sphere

Consider the problem of determining the largest angular diameter a_n of n equal circles (or spherical caps) which can be packed on the surface of a sphere without overlapping. For $n = 20$ van der Waerden [4] has published an arrangement as conjectured solution with circle diameter $a_{20} = 47° 26'$. The graph of this packing may be seen in a simplified stereographic projection in Fig. 1. The vertices of the graph are the centres of the spherical circles and the edges of the graph are great-circle arcs joining the centres of the touching spherical circles. Goldberg [1] has thought that after removing an appropriate circle from van der Waerden's arrangement the packing of the remaining 19 circles can be improved. Goldberg has come very near to finding a correct improved packing, however, he has probably overlooked something and obtained faulty results. In [2], this mistake has been mentioned but not corrected. Therefore, at present, no packing is known for 19 circles better than that obtained by removing a circle from van der Waerden's packing of 20 circles.

The aim of this paper is to correct Goldberg's results and to present a packing of 19 equal circles on a sphere, in which a_{19} is greater than a_{20} due to van der Waerden.

In fact, two errors are made in [1]: first, to sharpen van der Waerden's result a_{20} to $47° 24' 51''$, though its correct value is $a_{20} = 47° 25' 51.7''$; second, to compute the circle diameter a_{19} from such an arrangement which contains some overlapping circles. This second can be shown, e.g., in the following way.

Goldberg has discovered that by removing the isolated points M and N from the graph in Fig. 1 and reflecting the vertices A, C, E and G, I, K in the great-circle arcs FB, BD, DF and LH, HJ, JL, respectively, a packing with the same circle diameter can be obtained for $n = 18$. It is obvious that a packing for $n = 19$ is obtained with the same circle diameter if this process is only executed on one of the two hemispheres (Fig. 2). Goldberg has considered the graph in Fig. 2 with additional edges GN, IN, KN and obtained that the circle diameter in this arrangement is $a_{19} = 47° 25' 22''$. Since the graph in Fig. 2 is rigid in Danzerian sense [3] the edges GN, IN, KN cannot be added to the graph by moving the graph and preserving all of its edges. The removal of the isolation of the isolated point N with preservation of the

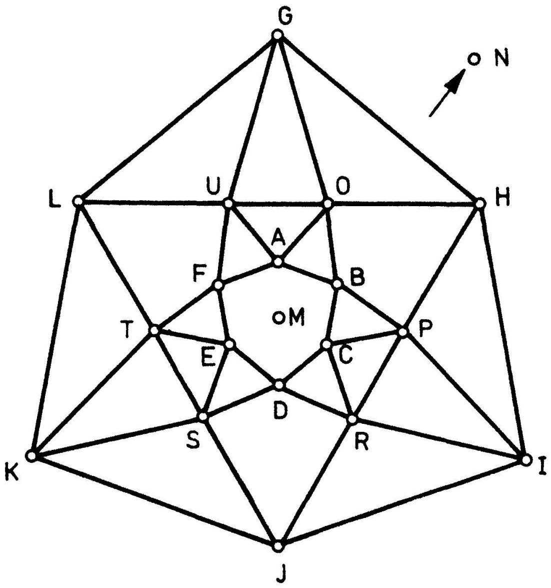

Figure 1

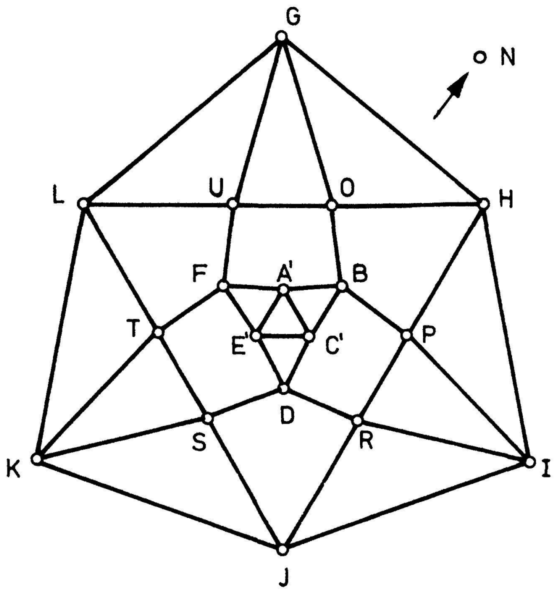

Figure 2

three-fold rotational symmetry of the graph can be done only by removing some of the edges of the graph in Fig. 2 and with decreased edge-lengths. It can be done, e.g., as shown in Fig. 3 where $a_{19} = 47°\,25'20.2''$, or in Fig. 4 where $a_{19} = 47°\,25'21.4''$ and the graph is obtained by moving the graph of Fig. 3. Thus, this kind of process for improvement is unsuccessful.

Goldberg ([1], Fig. 4) has considered in bypassing also a further, less symmetrical arrangement of 19 circles. Now a slight modification of the graph of that arrangement indeed leads to a good packing. The improving process in this case can be the following. Let the isolated point M and the edges AU, AO, BP, CR, ES, FT, IP, KT be removed from the graph of our Fig. 1. Reflect the vertex A with the edges AB, AF in the great-circle arc BF (snap

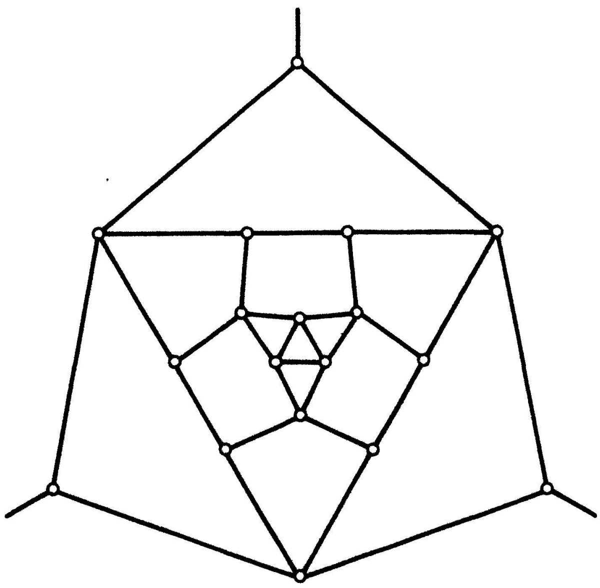

Figure 3

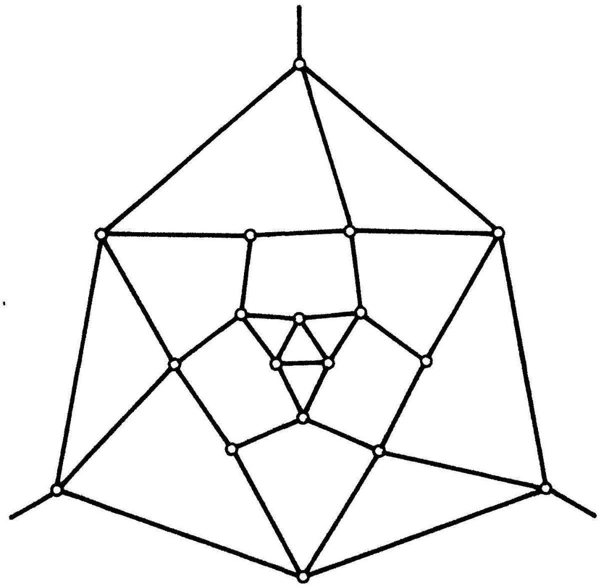

Figure 4

the pair of edges AB, AF through between the vertices B, F). In this way the graph in Fig. 5 is obtained, which is not rigid in Danzerian sense and so can be moved with a simultaneous increase in the length of the edges. The edge-length, that is, the diameter of the circles can be increased until the distances between points A' and C, A' and E, G and N, I and N, K and N are equal to the increased edge-length in the graph. So, five additional edges appear in the graph, which make the graph rigid in Danzerian sense and prevent any further increase in the edge-length. In this way we obtained a new packing of 19 equal circles on the sphere, at which the angular diameter of the circles is

$$a_{19} = 47° 40' 33.3''.$$

The graph of the new arrangement may be seen in a simplified stereographic projection in Fig. 6. The edge-length of the graph in Fig. 6 was computed by spherical trigonometry and iteration using the fact that the arrangement has a plane of symmetry, but the details are omitted here.

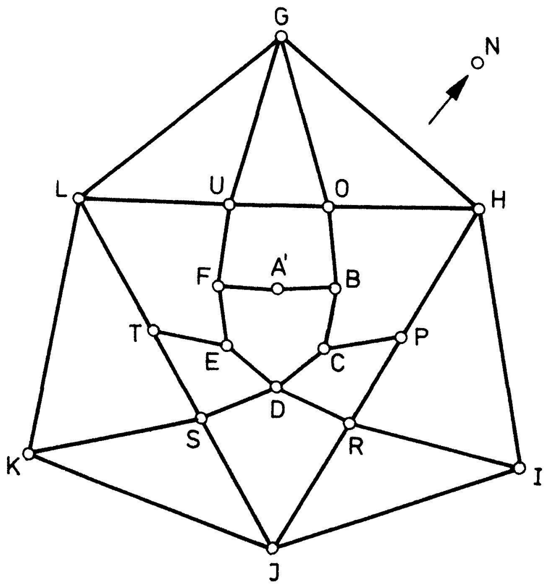

Figure 5

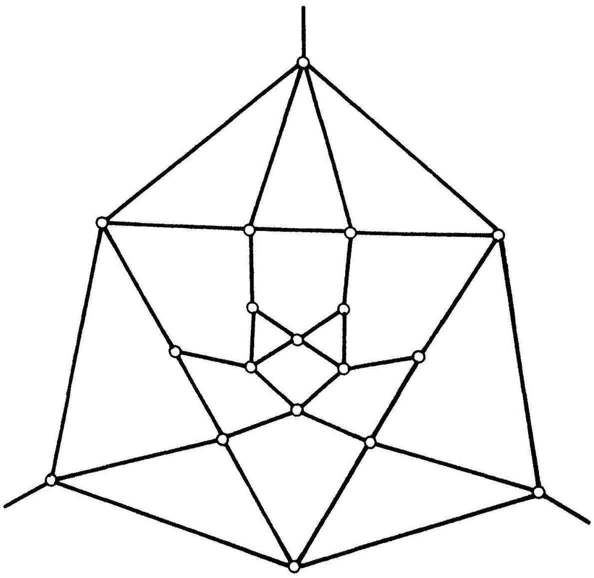

Figure 6

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REFERENCES

1 M. Goldberg: Packing of 19 equal circles on a sphere. Elemente der Mathematik *22,* 108–110 (1967).
2 E. Székely: Sur le problème de Tammes. Annales Univ. Sci. Budapest, Sect. Math. *17,* 157–175 (1974).
3 T. Tarnai and Zs. Gáspár: Improved packing of equal circles on a sphere and rigidity of its graph. Math. Proc. of the Cambridge Phil. Soc. *93,* 191–218 (1983).
4 B. L. van der Waerden: Punkte auf der Kugel. Drei Zusätze. Math. Annalen *125,* 213–222 (1952).

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