Packing of 180 equal circles on a sphere

Autor(en): **Tarnai, Tibor**

Objekttyp: **Article**

Zeitschrift: **Elemente der Mathematik**

Band (Jahr): **38 (1983)**

Heft 5

PDF erstellt am: **29.04.2024**

Persistenter Link: https://doi.org/10.5169/seals-37194

Satz. *Ist die 2π-periodische Funktion f lokal von beschränkter Schwankung und ist ihr «erster» Fourierkoeffizient normiert, d.h.*

$$c_1 = \frac{1}{2\pi} \int_0^{2\pi} f(\delta) e^{-i\delta} d\delta = 1,$$

so ist die Länge ihres geschlossenen Graphen $\bar{\Gamma}_f$ (d.i. die totale Variation von f auf $[0, 2\pi]$) dann und nur dann am kleinsten, wenn $\bar{\Gamma}_f$ eine geschlossene konvexe Kurve und der Parameter δ die Normalenrichtung ihrer Stützgeraden ist.

Albert Pfluger, Zürich

LITERATURVERZEICHNIS

1 G. Polya: Untersuchungen über Lücken und Singularitäten von Potenzreihen. Math. Z. *29,* 549–640 (1929).

2 R. Heine: Zur Theorie der konvexen Körper. Die Monotonie von $\int^{\varphi} h(x)dx + h'(\varphi)$. Math. Ann. *115* 130–131 (1938).

 0013-6018/83/030113-07$1.50+0.20/0

Packing of 180 equal circles on a sphere

The Dutch botanist Tammes raised the following problem: To determine the largest angular diameter of n equal circles (spherical caps) which can be packed on the surface of a sphere without overlapping; or what is the same, to arrange n points on the unit sphere so as to maximize the minimum distance between any two of the points. The Tammes problem has a vast literature which is surveyed, e.g., in Fejes Tóth's book [4], and more recent results are incorporated in a review paper by Melnyk, Knop and Smith [6] where the solutions and conjectured solutions of this problem are summarized for $n=2$ to 60 and 80, 110, 119, 120, 122. Some of the spherical circle-packings listed in [6] have been improved by Danzer [2] ($n=17$, 32) and, more recently, by Tarnai and Gáspár [10, 11] ($n=18$, 27, 34, 35, 40, 80, 122); and new packings for $n=54$, 72, 132 have also been constructed [9].

Studying these packings we have found that among them, in general, those packings have great density which have octahedral or icosahedral symmetry in rotation. (The density of packing is defined as the ratio of the total area of the surface of the spherical caps and the surface area of the sphere.) This fact has suggested us to search for packings, for certain values of n, in octahedral or icosahedral arrangement having no planes of symmetry. As a result of the research a packing of 180 equal circles on a sphere has been constructed that we present in this note.

According to Coxeter's paper [1], let us consider the regular tessellation of symbol $\{3, 5+\}_{3,2}$, which consists of equilateral triangles, five or six at each vertex, some slightly folded, such that they cover and fill the polyhedral surface of the regular icosahedron {3,5}; the suffixes 3,2 indicate that a vertex of the icosahedron can be arrived at from an adjacent one along the edges of the tessellation by three steps on the vertices in one direction then two steps after a change in direction by 60°. A part of this tessellation is shown by the continuous lines in figure 1 where the large equilateral triangle composed of dashed lines is a face of the icosahedron. Let us remove the vertices of the icosahedron and the edges drawn by thin lines in figure 1. The remaining edges display the rotational symmetries of the icosahedron and can have the same length even if the icosahedron is 'blown up' onto a sphere. This edge-system (heavy lines in figure 1) can be considered as the graph of packing of 180 equal circles on the sphere. The vertices of the graph are the centres of the spherical circles and the edges of the graph are great-circle arcs joining the centres of the touching spherical circles.

The edge-length of the graph, that is, the diameter of the circles can be computed by spherical trigonometry and iteration using only a part of the graph (figure 2) due to the icosahedral symmetry of the graph and the fact that all the rhombi are equal. In figure 2, a is the edge-length of the graph and A is a vertex of the spherical icosahedron. We started the actual computation with an approximate value of a, determined α and β from a, γ from a and β, δ from γ, ε from a and δ, then computed a new a from ε and δ. The process was repeated, starting from a weighted average of the old and new values of a. At the end we obtained that $\varepsilon=\gamma$ and the diameter of the circles in the investigated arrangement is

$$a=15°49'7.5''.$$

This circle diameter results in density 0.85617 which does not seem to be a bad value since densities greater than this one are only known in the cases of packings of 6 and 12 circles (regular octahedron and icosahedron) and Robinson's packings of 24 and 48 circles [8].

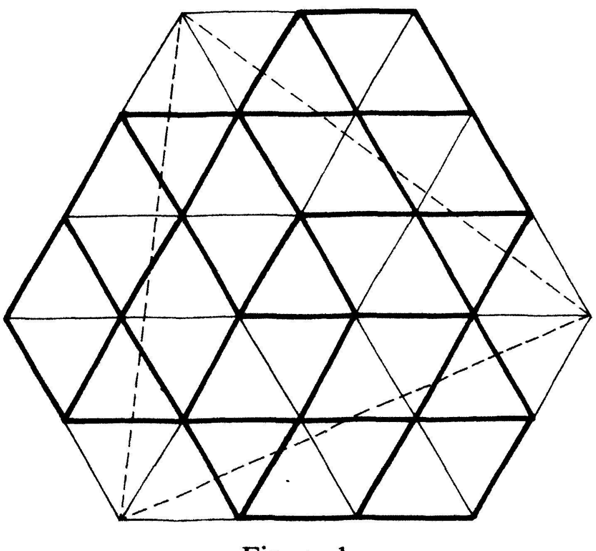

Figure 1.

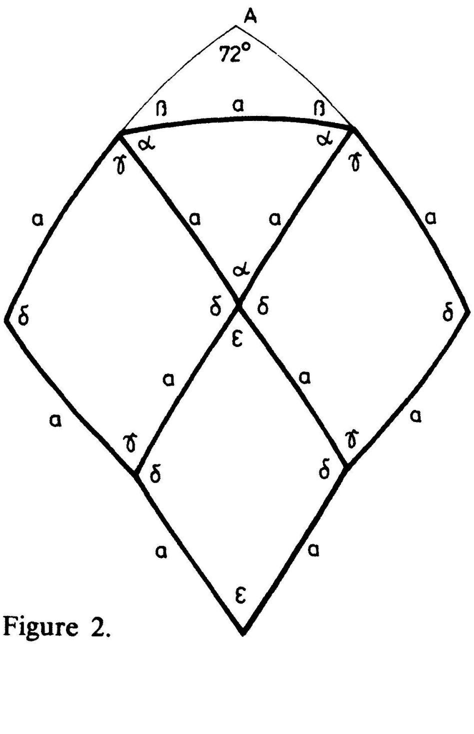

Figure 2.

The edge-system of heavy lines in figure 1 can also be used on the tetrahedron and the octahedron as graphs of packing of 36 and 72 equal circles, respectively. In this case the obtained circle diameters $34°42'12.3''$ for $n=36$ and $24°51'13.5''$ for $n=72$ are greater than those in the best packings until now constructed by Karabinta [5] and the author [9], respectively; and the densities are 0.81914 for $n=36$ and 0.84343 for $n=72$.
By these constructions we have given lower bounds for the extremal density of packing of 36, 72, 180 nonoverlapping equal circles on the sphere. We have also computed upper bounds for the extremal density by the formula (§ 9.5) of Robinson [7]. These upper bounds are as follows: 0.86559 for $n=36$; 0.87942 for $n=72$; 0.89399 for $n=180$. It may be seen that the difference between the upper and lower bounds for 72 circles is less than that for 180 circles.
Finally, we mention that Fejes Tóth [3] raised the following problem: To distribute k points in the elliptic plane so that the least distance between any two of them should be as great as possible. Since the elliptic geometry is derived from the spherical geometry by abstract identification of antipodal points, Fejes Tóth's elliptic problem for k points is the same as the Tammes problem for $2k$ points with the restriction that the set of points should be symmetric with respect to the centre of the sphere. The packings of 72 and 180 circles described in the present note are centro-symmetric sets of circles, so we also obtained approximations for the solution of Fejes Tóth's elliptic problem for 36 and 90 points, respectively.

ACKNOWLEDGMENT

I thank Professor L. Fejes Tóth for helpful discussions and valuable comments.

Tibor Tarnai, Hungarian Institute for Building Science, Budapest

REFERENCES

1 H.S.M. Coxeter: Virus Macromolecules and Geodesic Domes. A Spectrum of Mathematics (ed. J.C. Butcher). Auckland University Press and Oxford University Press 1972, p.98-107. M.R. *56*, 6547.
2 L. Danzer: Endliche Punktmengen auf der 2-Sphäre mit möglichst grossem Minimalabstand. Habilitationsschrift. Göttingen 1963.
3 L. Fejes Tóth: Distribution of Points in the Elliptic Plane. Acta Math. Acad. Sci. Hung. *16*, 437-440 (1965). M.R. *32*, 1612.
4 L. Fejes Tóth: Lagerungen in der Ebene auf der Kugel und im Raum, 2nd ed. Springer-Verlag, Berlin, Heidelberg, New York 1972. M.R. *50*, 5603.
5 A. Karabinta and E. Székely: Sur les empilements optimaux de cercles congruents sur la sphère. Ann. Univ. Sci. Budapest., Sect. Math. *16*, 143-154 (1973). M.R. *51*, 13874.
6 T.W. Melnyk, O. Knop and W.R. Smith: Extremal Arrangements of Points and Unit Charges on a Sphere: Equilibrium Configurations Revisited. Can. J. Chem. *55*, 1745-1761 (1977). M.R. *56*, 2848.
7 R.M. Robinson: Arrangement of 24 Points on a Sphere. Math. Ann. *144*, 17-48 (1961). M.R. *24*, A3565.
8 R.M. Robinson: Finite Sets of Points on a Sphere with Each Nearest to Five Others. Math. Ann. *179*, 296-318 (1969). M.R. *39*, 2068.
9 T. Tarnai: Multi-Symmetric Packing of Equal Circles on a Sphere. Ann. Univ. Sci. Budapest., Sect. Math., to appear.

10 T. Tarnai and Zs. Gáspár: Improved Packing of Equal Circles on a Sphere and Rigidity of Its Graph. Math. Proc. Camb. Phil. Soc., to appear.
11 T. Tarnai and Zs. Gáspár: Rigidity of the Graphs of the Spherical Circle Packings. Z. angew. Math. Mech., *63*, T333–T334 (1983).

 0013-6018/83/030119-04$1.50+0.20/0

A Result for the 'other' variable of Ramanujan's sum

In [1], Grytczuk shows that

$$\sum_{d \mid k} |C_d(n)| = (k,n)\, 2^{\omega(k/(k,n))}, \tag{1}$$

where $C_d(n)$ is Ramanujan's sum and $\omega(n)$ is the number of distinct prime divisors of n. In this note we derive an analogous result for the sum

$$\sum_{d \mid n} |C_k(d)|. \tag{2}$$

It is well-known (e.g. [3], p. 56) that $C_k(n)$ is multiplicative in the variable k for each fixed n. That is, if $(k,j)=1$, then $C_k(n)\,C_j(n)=C_{kj}(n)$. Grytczuk uses this multiplicative property of Ramanujan's sum to derive the identity (1). This is a standard technique for proving such identities. Ramanujan's sum is not multiplicative in its other variable and thus the same technique cannot be applied directly to evaluating the analogous sum (2). However, in [4] the following reciprocity law for Ramanujan's sum is proved. Let $\bar{k}$ be the core of k (the largest square-free divisor of k) and $k^* = k/\bar{k}$, then for all n and k,

$$\frac{\mu(\bar{k})}{k^*}\, C_k(n k^*) = \frac{\mu(\bar{n})}{n^*}\, C_n(k n^*). \tag{3}$$

It is also shown in [4] that

$$C_k(n k^*) = k^*\, C_{\bar{k}}(n). \tag{4}$$

These two results make the sum (2) tractable.
It is routine to prove the following Lemma.

Lemma 1. *Let* $F_k(n) = \sum_{d \mid n} |C_{\bar{d}}(k)|$, *then* $F_k(n)$ *is multiplicative in the variable* n *for each fixed* k.

Lemma 2. *Fix* k. *Then*

$$F_k(n) = \prod_{\substack{p^a \parallel n \\ p \nmid k}} (a+1) \prod_{\substack{p^a \parallel n \\ p \mid k}} [a(p-1)+1],$$