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Autor: Kuperberg, W.

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On minimum area quadrilaterals and triangles circumscribed about convex plane regions

1. Introduction

Throughout this note, by a *convex plane region* we will mean a closed bounded convex plane set with an interior point. For every such set K and for every integer $n \geq 3$, let $p_n(K)$ denote the maximum area n-gon contained in K and let $P_n(K)$ denote the minimum area n-gon containing K, and let

$$\phi_n(K) = \frac{|p_n(K)|}{|K|}$$

and

$$\psi_n(K) = \frac{|P_n(K)|}{|K|},$$

where $|A|$ denotes the area of the region A. Since $\phi_n(K_1) = \phi_n(K_2)$ and $\psi_n(K_1) = \psi_n(K_2)$ whenever K_1 and K_2 are affinely equivalent, and since every K is affinely equivalent to some K_1 contained in the rectangle $[0,2] \times [0,1]$ and with $|K_1| = 1$, we will restrict the domain of the functions ϕ_n and ψ_n to the set $\mathcal{K}$ whose elements are convex plane regions of area 1 contained in the rectangle. The set $\mathcal{K}$ furnished with the Hausdorff distance formula becomes a compact metric space, while ϕ_n and ψ_n turn out to be continuous functions from $\mathcal{K}$ into the real line. Therefore, for every $n \geq 3$, there exist a $K^n_{\min}$ and a $K^n_{\max}$ such that $\phi_n(K) \geq \phi_n(K^n_{\min})$ for every K and $\psi_n(K) \leq \psi_n(K^n_{\max})$ for every K. Let $a_n = \phi_n(K^n_{\min})$ and $b_n = \psi_n(K^n_{\max})$.
The values a_n are known for every $n \geq 3$, namely

$$a_n = \frac{n}{2\pi} \sin \frac{2\pi}{n} = \phi_n(S^1),$$

where S^1 is the unit circle; moreover, if $\phi_n(K) = a_n$, then K is affinely equivalent to S^1 (see [5], or [6], p. 36). It is also known that $b_3 = 2$, and if $\psi_3(K) = 2$ then K is a parallelogram (see [3]). However, the values of b_n for $n \geq 4$ are apparently unknown. Chakerian and Lange [2] (see also Chakerian [1]) proved that $b_4 \leq \sqrt{2}$ and they posed the problem of whether $b_4 < \sqrt{2}$. As we will prove in section 2 of this note, $b_4 < \sqrt{2}$ indeed. But the exact value of b_4 (and all b_n's for $n > 4$) still remains unknown.

In section 3, we are concerned with triangles whose one side is parallel to a prescribed direction. Hodges [4] proved that if K is a convex plane region of area 1 and if θ is a direction in the plane, then K contains a triangle of area $\geq 3/8$ with one side parallel to θ. Moreover, the number $3/8$ in this statement cannot be replaced by a greater one, as the case of the regular hexagon shows, θ being parallel to one of its sides. Note that

$$a_3 = \frac{3\sqrt{3}}{4\pi} > \frac{3}{8},$$

so, with the constraint that one of the sides of the triangle in K be parallel to θ, the area of the triangle is not guaranteed to be as large as that without that constraint. However, when the same constraint is imposed on triangles containing K, the conclusion is the same as without that constraint. In fact, section 3 contains the proof of the following theorem: If K is a convex plane region of area 1 and if θ is a direction in the plane, then K is contained in a triangle of area ≤ 2 with one side parallel to θ. Obviously, the number 2 cannot be replaced by a smaller one, since $b_3 = 2$.

2. Circumscribed quadrilaterals

Lemma (see [2], theorem 1, p. 58, or [6], p. 6). *Let K be a convex plane region and $n \geq 3$ a given integer. Let P be a convex n-gon of minimum area containing K. Then the midpoints of the sides of P belong to K.*

Proof: Suppose that $\overline{AB}$ is a side of a minimum area n-gon containing K and that the midpoint M of AB does not belong to K. Then some sub-segment of $\overline{AB}$ centered at M, name it U, misses K completely and one of the segments $\overline{AM}$ or $\overline{BM}$, say $\overline{AM}$, misses K completely. Choose a point C in U so that M lies between A and C. Now, a rotation of the side $\overline{AB}$ about C by a sufficiently small angle of the appropriate orientation will result in diminishing of the area of the n-gon containing K.

Theorem 1. *Every convex plane region of area 1 is contained in a quadrilateral of area smaller than $\sqrt{2}$.*

Proof: As mentioned in the Introduction, it is known that $b_4 \leq \sqrt{2}$, i.e. every convex plane region of area 1 is contained in a quadrilateral of area $\leq \sqrt{2}$. To prove the theorem, assume to the contrary that there exists a convex plane region K of area 1 such that the smallest area quadrilateral Q_0 circumscribed about K has area of $\sqrt{2}$. By the lemma, the midpoints A, B, C, D of the sides of Q_0 belong to K. Ob-

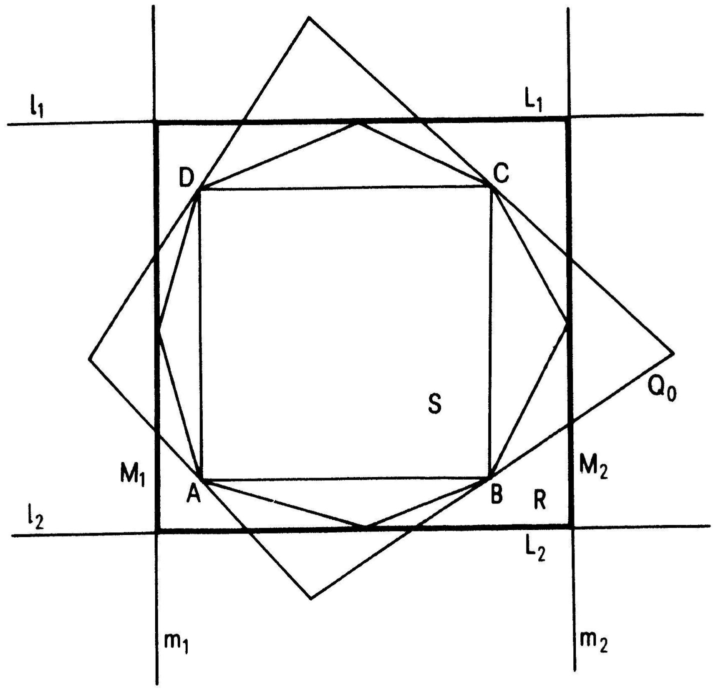

Figure 1

viously, they form a parallelogram S of area $\sqrt{2}/2$. Assume for simplicity that S is a square (S can be turned into a square by an area-preserving affine transformation). Let l_1 and l_2 be two lines tangent to K and parallel to $\overline{AB}$, and let m_1 and m_2 be two lines tangent to K and parallel to $\overline{BC}$. These four lines bound a rectangle R which contains K; let L_1, L_2, M_1, M_2 denote the sides of R with $L_i \subset l_i$ and $M_i \subset m_i$ for $i = 1, 2$ (see fig. 1). Since, on one hand, the area of R is at least $\sqrt{2}$, but, on the other hand, every side of R contains a point from K, a simple calculation shows that the area of R is exactly $\sqrt{2}$ and that R is a square (compare the proof of theorem 6 in [2]). Now, let K_1 be the convex hull of the union of the square S and the intersection of the boundary of R with K, $K_1 = \mathrm{Conv}[S \cup (\mathrm{BdR} \cap K)]$. The area of K_1 is at least 1 and K_1 is a subset of K, therefore $K_1 = K$, and thus K is a polygon (with at most 8 sides, but that is irrelevant). Notice that every side of R either contains a whole side of S or it touches K at one point only. Therefore one of the sides L_1 or L_2, say L_i, touches K at one point only, and one of the sides M_1 or M_2, say M_j, touches K at one point only. Those two points of tangency are vertices of K and midpoints of (adjacent) sides of R (the lemma is applied here again, since the area of R is $\sqrt{2}$, the minimum). Now, a rotation of the side L_i about its midpoint by a sufficiently small positive angle will turn the square R into a trapezoid T which still contains K and whose area is still $\sqrt{2}$ (the minimum). But this contradicts the lemma: one of the sides of T touches K at one point only which is not the midpoint of that side.

Corollary. *There exists a positive number $a < \sqrt{2}$ such that every convex plane region K is contained in a quadrilateral of area at most $a \cdot |K|$. In other words, $b_4 < \sqrt{2}$.*

Problem: Is $b_4 = \psi_4(P)$, where P is the regular pentagon? In other words, is it true that every convex region K is contained in a quadrilateral of area at most

$$\left(1+\frac{4}{5}\tan\frac{\pi}{5}\sin\frac{\pi}{5}\right)|K|?$$

3. Circumscribed triangles with a constraint

Theorem 2. *If K is a convex plane region of area* 1 *and if θ is a direction in the plane, then K is contained in a triangle of area at most* 2 *with one side parallel to θ.*

Proof: Let l_1 and l_2 be the two lines parallel to θ and tangent to K. Let T_i (for $i=1,2$) be a triangle containing K, with one side on l_i and of minimum area. Let b_i be the length of the side of T_i which lies on l_i and let h_i be the altitude of T_i, perpendicular

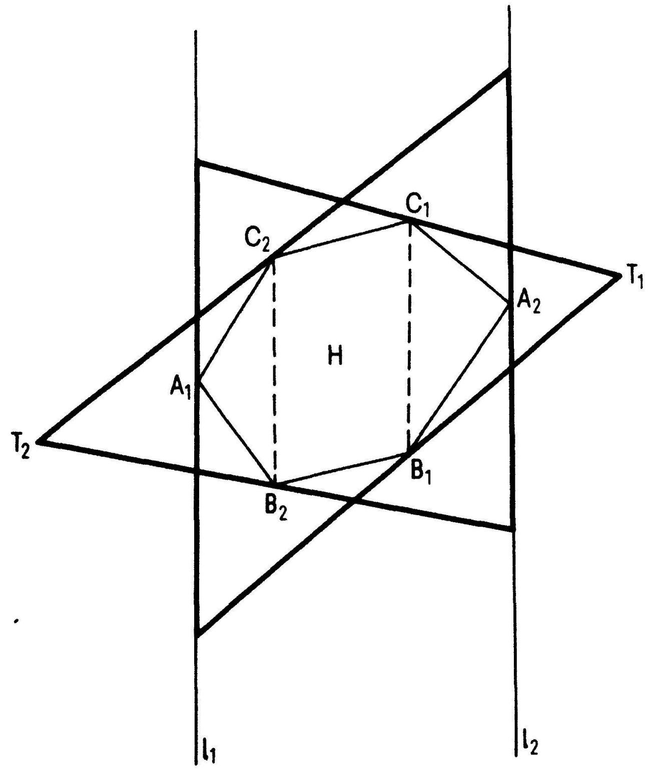

Figure 2

to θ. Let A_i be a point at which l_i touches K[1]) and let B_i and C_i be the midpoints of the sides of T_i which are not parallel to θ (see fig. 2). A similar argument to that in the proof of the lemma from section 2 can be used here to show that B_i and C_i belong to K. Thus the hexagon $H = A_1B_2B_1A_2C_1C_2$ is contained in K. Now, the area of H equals to the sum of the areas of the quadrilaterals $A_1B_2B_1C_2$ and $A_2C_1B_2B_1$, that is, $|H| = (h_1b_2 + h_2b_1)/8$. Since $H \subset K$, we get $(h_1b_2 + h_2b_1)/8 \leq 1$.

1) $l_i \cap K$ consists of a single point A_i, for almost all directions θ.

This inequality, as easily verified, implies $h_1 b_1 h_2 b_2 \leq 16$, that is, $|T_1| \cdot |T_2| \leq 4$. Therefore either $|T_1| \leq 2$ or $|T_2| \leq 2$, which completes the proof.

Remark: If for some θ, $|T_1| \cdot |T_2| = 4$, then K is a polygon with at most 6 sides.

W. Kuperberg, Auburn University, Auburn, Alabama, USA

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Das Gegenstück zur logarithmischen Spirale in der ebenen isotropen Geometrie

Die logarithmische Spirale der euklidischen Ebene wurde von Descartes und Toricelli um 1638 entdeckt und erhielt ihren Namen 1704 von Varignon. Wegen vieler markanter Eigenschaften [1–4, 9–11] gehört sie mit zu den interessantesten ebenen Kurven, und es erweist sich als lohnend, ihr Gegenstück in der isotropen Ebene zu studieren. Die isotrope Geometrie wurde wesentlich von K. Strubecker gefördert und durch Beiträge in neuerer Zeit weiterentwickelt. Sie besitzt eine von der euklidischen Geometrie abweichende Metrik. Wir verzichten auf die Darstellung dieser Grundlagen und verweisen statt dessen auf die Literatur [7]. In der Arbeit über die äquiforme Geometrie der isotropen Ebene [5] hat K. Strubecker auf die isotropen logarithmischen Spiralen hingewiesen. Wir behandeln diese Kurvenklasse im Rahmen der Bewegungsgeometrie, wobei viele klassische differentialgeometrische Ergebnisse ihre isotrope Entsprechung finden.

1. Definition und einfache Eigenschaften

Jede Isogonaltrajektorie eines Geradenbüschels mit eigentlichem Trägerpunkt Z heisst *logarithmische Spirale*. Besitzt Z die Koordinaten (b,d), dann wird eine logarithmische Spirale beschrieben durch die Differentialgleichung

$$y' = a + \frac{y-d}{x-b}, \qquad a,b,d \in \mathbf{R}, \quad a \neq 0, \quad x \neq b; \tag{1}$$

dabei ist a der *Schnittwinkel* der Isogonaltrajektorie mit den Büschelgeraden. Die Differentialgleichung (1) besitzt die allgemeine Lösung

$$y = a(x-b)\ln|x-b| + c(x-b) + d, \qquad c \in \mathbf{R}. \tag{2}$$