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Artikel: The sum of the s-th power of the integers

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Aufgabe 891. Es sei $f: [0, 1] \to \mathbf{R}$ stetig und

$$\int_0^1 f(x)\,dx = 0.$$

Man zeige, dass

$$\left(\int_0^1 x^2 f(x)\,dx\right)^2 \leqslant \frac{1}{3}\int_0^1 x^2 (f(x))^2\,dx.$$

Wann genau gilt das Gleichheitszeichen?

H.-J. Seiffert, Berlin, BRD

Aufgabe 892. Für natürliche Zahlen n sei

$$a_n := \int_0^n \exp(t^2/n)\,dt.$$

Man ermittle $\lim_{n\to\infty} (a_{n+1}/a_n)$.

U. Abel, Giessen, BRD

Aufgabe 893. Es sei p eine ungerade Primzahl, g eine Primitivwurzel $\bmod p$ und m eine prime Restklasse $\bmod p$. Man zeige, dass für $x \not\equiv 0 \pmod p$

$$\operatorname{ind}_g x \equiv \sum_{i=1}^{p-2} (1-g^i)^{p-2} x^i \pmod p.$$

Dabei bezeichnet $\operatorname{ind}_g x$ die kleinste der natürlichen Zahlen k, für die $m \equiv g^k \pmod p$ erfüllt ist.

H. Bergmann, Hamburg, BRD

Elementarmathematik und Didaktik

The sum of the s-th power of the integers

The purpose of this brief note is to present a derivation of the familiar formula for the sum of the squares of the first n integers, i.e. $\sum_{k=1}^{n} k^2$. This method is easy to remember, having an almost geometric character, and it can be iterated in a very simple manner to yield the sum of the first n integers raised to any positive integral power.

Consider the following $n \times n$ array:

$$\begin{array}{cccccc}
1 & 2 & 3 & 4 & \ldots\ldots & n \\
1 & 2 & 3 & 4 & \ldots\ldots & n \\
1 & 2 & 3 & 4 & \ldots\ldots & n \\
 & & & & & \\
1 & 2 & 3 & 4 & \ldots\; n-1 & n
\end{array}$$

The line is drawn just below the main diagonal.
Denote the sum of all the numbers in the array by S, the sum of the integers above the line by U and the sum of those below the line by B.
S and B can be evaluated by summing the rows and in both cases we have to add sums of arithmetic progressions. Indeed,

$$S = \frac{n^2(n+1)}{2} \quad \text{and} \quad B = \sum_{k=1}^{n-1} \frac{k(k+1)}{2}.$$

U is calculated by adding the columns and we find $U = \sum_{k=1}^{n} k^2$.
Since $U = S - B$ we obtain

$$\sum_{k=1}^{n} k^2 = \frac{n^2(n+1)}{2} - \sum_{k=1}^{n-1} \frac{k(k+1)}{2} = \frac{n^2(n+1)}{2} - \frac{1}{2} \sum_{k=1}^{n} k^2 + \frac{n(n+1)}{4}$$

or

$$3 \sum_{k=1}^{n} k^2 = n^2(n+1) + \frac{n(n+1)}{2} = n(n+1) \frac{(2n+1)}{2}$$

or

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

To obtain the sum of the cubes of the first n integers, replace every entry in the array by its *square* and proceed as before. In this case

$$U = \sum_{k=1}^{n} k^3, \qquad S = n \sum_{k=1}^{n} k^2 \quad \text{and} \quad B = \sum_{m=1}^{n-1} \left(\sum_{k=1}^{m} k^2 \right).$$

In general, replace every entry in the array by its $(s-1)$-th power and obtain

$$\sum_{k=1}^{n} k^s = n \sum_{k=1}^{n} k^{s-1} - \sum_{m=1}^{n-1} \left(\sum_{k=1}^{m} k^{s-1} \right).$$

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