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# ELEMENTE DER MATHEMATIK

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## The impossibility of a tessellation of the plane into equilateral triangles whose sidelengths are mutually different, one of them being minimal

**Theorem.** *There is no tessellation of the euclidean plane  $\mathbf{R}^2$  into equilateral triangles whose sidelengths are mutually different, one of them being minimal.*

Proof: Assume that there is such a tessellation of  $\mathbf{R}^2$  into equilateral triangles  $t_i, i \in I$ , where  $I$  is an arbitrary set of indices. We shall eventually see that this assumption leads to a contradiction.

For  $i \in I$ , let  $l_i$  denote the sidelength of  $t_i$ . Let  $l$  be the minimum of the sidelengths. By scaling we can attain  $l=1$ . Then the area of each triangle is at least

$$\frac{1}{4} \cdot \sqrt{3}.$$

Therefore the triangle can be enumerated and we can assume  $I=\mathbf{N}$  and  $l_1=1$ .

For each  $i \in \mathbf{N}$  let  $d_i$  denote the boundary of  $t_i$ . Further we define  $D$  to be the ‘grid’ of the tessellation:  $D = \cup \{d_i, i \in \mathbf{N}\}$ .

If  $n$  and  $\varepsilon$  are positive real numbers, the points

$$P_1 = \left( -\frac{n}{2} - \frac{\varepsilon}{2}, \frac{\varepsilon}{2} \sqrt{3} \right), \quad P_2 = \left( -\frac{n}{2}, 0 \right), \quad P_3 = \left( \frac{n}{2} - \frac{\varepsilon}{2}, \frac{\varepsilon}{2} \sqrt{3} \right)$$

and  $P_4 = \left( \frac{n}{2}, 0 \right)$

define a parallelogram in  $\mathbf{R}^2$  with the sidelengths  $\varepsilon$  and  $n$  and with angles of 60 and 120 degrees. Let  $S(n, \varepsilon)$  denote the union of the three sides  $\overline{P_1 P_2}$ ,  $\overline{P_2 P_3}$  and  $\overline{P_3 P_4}$  (fig. 1).

More generally we denote by  $S(n, \varepsilon)$  any subset of the plane that can be obtained from this special  $S(n, \varepsilon)$  by translations, rotations and reflections. If any such  $S(n, \varepsilon)$  is given, let  $R_n$  denote the interior of the associated parallelogram.

We show that there are a strongly decreasing sequence of positive numbers  $n_j, j \in \mathbf{N}$ , with  $n_j \leq n_{j-1} - 1 (j \geq 2)$  and for each  $j$  a number  $\varepsilon_j \in (0, \infty)$  as well as a set  $S_{n_j} = S(n_j, \varepsilon_j)$ , contained in  $D$ , such that  $S_{n_j}$  suffices one of the following two properties (or both):

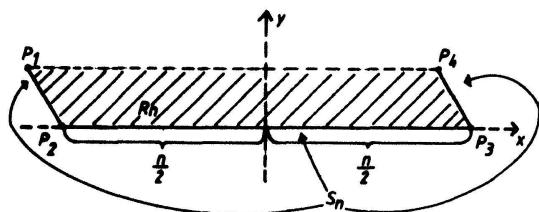


Figure 1

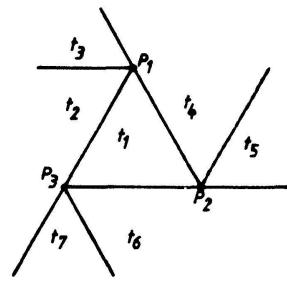


Figure 2

- a) There is a natural number  $k$  depending on  $j$  such that  $n_j = l_k$  and  $R_{l_k} \cap t_k = \emptyset$ .
- b) The 'base' of  $S_{n_j}(\overline{P_2 P_3})$  in fig. 1) is not a side of a triangle of the tessellation.

The numbers  $n_j$  and the associated sets  $S_{n_j}$  will be constructed by induction on  $j$ . Since  $n_j \leq n_{j-1} - 1$  for  $j \geq 2$ , some of the numbers  $n_j$  must be negative. But we assumed that each  $n_j$  is positive. Thus we get a contradiction and the theorem is proved.

The induction is performed in two steps.

### First step

We show that there is a subset  $S_{n_1}$  of  $D$  for some positive real number  $n = n_1$  which has property a) or property b).

Look at figure 2. Let  $P_1$ ,  $P_2$  and  $P_3$  be the vertices of  $t_1$ . Since  $t_1$  is the smallest triangle of the tessellation, each neighbor of  $t_1$  (i.e. each triangle that shares a boundary segment of positive measure with  $t_1$ ) must be larger than  $t_1$ . It is obvious that this is only possible if there are only three neighbors of  $t_1$ , say  $t_2$ ,  $t_4$  and  $t_6$ , and if (up to symmetry) they are arranged like in figure 2.

Hence there exist three triangles in the tessellation that have only a vertex in common with  $t_1$ . We can assume that these are the triangle  $t_3$ ,  $t_5$  and  $t_7$  and that the triangles  $t_2$ ,  $t_3$ ,  $t_4$ ,  $t_5$ ,  $t_6$  and  $t_7$  are arranged around  $t_1$  in a clockwise manner.

Let us call  $t_i$ ,  $i = 2, \dots, 7$ , the *surrounding triangles*. The vertices of the surrounding triangles that are not points of  $t_1$  may be called the *outer corners*. Clearly each surrounding triangle has exactly two outer corners. For each surrounding triangle  $t_k$ , let  $P_{2k}$  and  $P_{2k+1}$  be the outer corners written clockwise (see fig. 3). Since the triangles of the tessellation have pairwise different sizes, we get the inequalities  $P_5 \neq P_6$ ,

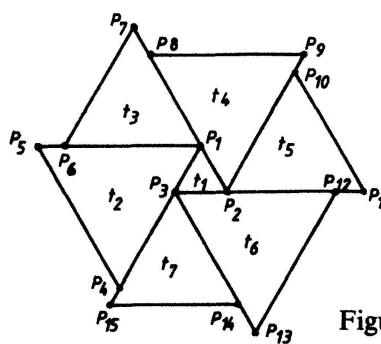


Figure 3

$P_9 \neq P_{10}$  and  $P_{13} \neq P_{14}$ .

We now have to distinguish the two cases whether  $t_2$  is larger or smaller than  $t_3$ .

**Case 1.** We assume that  $t_2 > t_3$ .

Consider figure 3. If  $D$  would contain a horizontal straight line whose right endpoint is  $P_7$ , the points  $P_6$  and  $P_7$  would define a set  $S_{l_3} \subset D$  with  $R_{l_3} \cap t_3 = \emptyset$ .

If this is not the case,  $D$  contains a straight line which elongates  $\overline{P_6P_7}$  over  $P_7$  and therefore a straight line whose left endpoint is  $P_7$ , too.

If  $P_7 \neq P_8$ , the points  $P_2$  and  $P_7$  define a set  $S_{n_1} \subset D$  that has property b). So we may assume  $P_7 = P_8$ .

Now consider all systems of straight lines that may start at  $P_9$  (fig. 4). In the cases i) and ii) of figure 4 there is a set  $S_{l_4} \subset D$  with  $R_{l_4} \cap t_4 = \emptyset$ , defined by  $P_7$  and  $P_9$ . In case iii) there is a set  $S_{l_4} \subset D$  with  $R_{l_4} \cap t_4 = \emptyset$ , defined by  $\overline{P_2P_{11}}$ ,  $\overline{P_2P_9}$  and the elongation of  $P_8P_9$  over  $P_9$ . Therefore we may assume that the lines starting from  $P_9$  look as in figure 4 iv).

This implies  $t_4 > t_5$ .

Two further quite similar argumentations – we now have  $t_4 > t_5$  like we had  $t_2 > t_3$ , and we will get  $t_6 > t_7$  – show that if there is not a set  $S_{n_1} \subset D$  with property a) or b), the conditions  $P_{11} = P_{12}$  and  $P_4 = P_{15}$  must be fulfilled. Moreover, the line systems starting at  $P_5$ ,  $P_9$  and  $P_{13}$  are of type iv) of figure 4 (see fig. 5).

Let  $d_1$ ,  $d_2$  and  $d_3$  denote the (positive) lengths of  $\overline{P_5P_6}$ ,  $\overline{P_9P_{10}}$  and  $\overline{P_{13}P_{14}}$ , respectively. Then each  $d_i$ ,  $i = 1, 2, 3$  is a sum of sidelengths of some triangles of the tessellation. Therefore  $d_i > l_1$  for  $i = 1, 2, 3$ . Hence  $d_1 + d_2 + d_3 > 3l_1$ .

Comparing the sidelength of the seven triangles  $t_1$ ,  $t_2$ ,  $t_3$ ,  $t_4$ ,  $t_5$ ,  $t_6$  and  $t_7$  yields the equations  $t_2 = l_1 + l_7$ ,  $t_2 = l_3 + d_1$ ,  $t_4 = l_1 + l_3$ ,  $t_4 = l_5 + d_2$ ,  $t_6 = l_1 + l_5$ ,  $t_6 = l_7 + d_3$ . They tell us that  $3l_1 = d_1 + d_2 + d_3$  in contradiction to the last inequality.

Therefore in case 1 there must be a set  $S_{n_1} \subset D$  with property a) or b).

**Case 2.** We assume that  $t_2 < t_3$ .

If there is not a set  $S_{l_2} \subset D$  with  $R_{l_2} \cap t_2 = \emptyset$  defined by  $P_4$  and  $P_5$ , the situation at  $P_4$  can only be as indicated in figure 6. Then the points  $P_1$  and  $P_4$  define a set  $S_{l_2} \subset D$  with  $R_{l_2} \cap t_2 = \emptyset$ .

## Second step

A set  $S_n (= S_{n_j}) \subset D$  may be given that has property a) or b). It will be shown that there is a set  $S_m \subset D$  with  $m \leq n-1$  which has property a). We then put  $n_{j+1} := m$ .

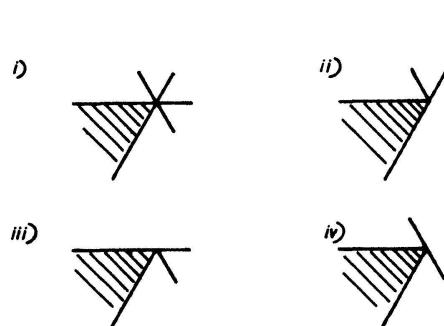


Figure 4

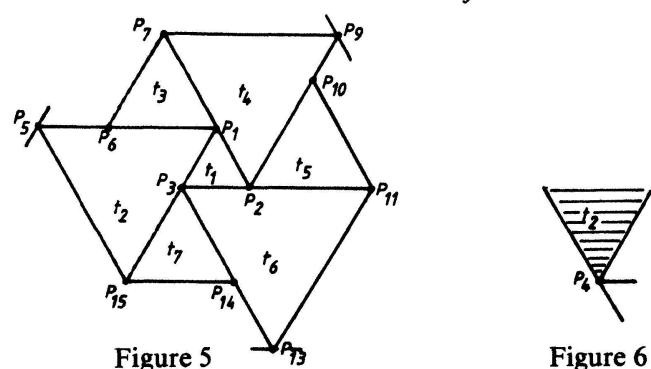


Figure 5

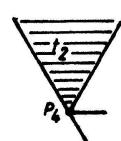


Figure 6

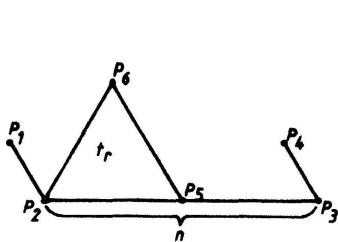


Figure 7a

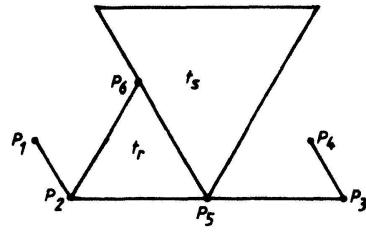


Figure 7b

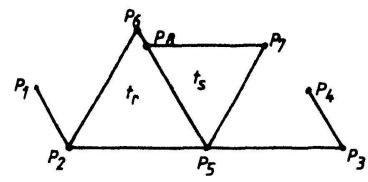


Figure 7c

After a suitable translation, rotation and reflection if necessary we can assume that the given  $S_n$  has the position shown in figure 1. With the notations of figure 1, let  $t_r$  be the triangle of the tessellation which 'stands on the base of  $S_n$  at the left corner', which means that  $t_r$  is defined by the three conditions  $P_2 \in t_r$ ,  $R_n \cap t_r \neq \emptyset$ ,  $\overline{P_2 P_3}$  contains a side of  $t_r$  (see fig. 7a). The vertices of  $t_r$  may be  $P_2$ ,  $P_5$  and  $P_6$ , where  $P_5 \in \overline{P_2 P_3}$ . Since  $S_n$  has property a) or b),  $P_3$  is different from  $P_5$ .

Let  $t_s$  denote the neighbor of  $t_r$  at the side  $\overline{P_5 P_6}$  such that  $P_5 \in t_s$ . We now have to distinguish whether  $l_s$  is greater or less than  $l_r$ .

*Case 1.* If  $l_s > l_r$ , the points  $P_2$  and  $P_6$  define a set  $S_{l_r} \subset D$  with  $R_{l_r} \cap t_r = \emptyset$  (see fig. 7 b). Moreover,  $l_s \leq n-1$  because  $S_n \subset D$  and  $l_1 = 1$ . So we can define  $m = l_r$ .

*Case 2.* If  $l_s < l_r$ , let  $P_7$  denote the vertex of  $t_s$  that is not a point of  $t_r$ , and let  $P_8$  denote its third vertex (fig. 7c). An examination of all possible line systems starting at  $P_7$  (they are listed in fig. 4) shows that in each case there is a set  $S_{l_s} \subset D$  with  $R_{l_s} \cap t_s = \emptyset$ , either defined by the points  $P_5$  and  $P_7$  or by  $P_7$  and  $P_8$ . Moreover,  $l_s < l_r \leq n-1$ . So we can define  $m = l_s$ .

Hence the induction step is completed, q.e.d.

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## Quelques considérations concernant le problème de l'aiguille de Buffon dans l'espace euclidien $E_n$

0. Soit  $E_n$  l'espace euclidien à  $n$  dimensions de coordonnées  $x_1, \dots, x_n$ .

La mesure élémentaire cinématique dans  $E_n$ , invariante par rapport au groupe de mouvements euclidiens, est [1]:

$$dK = dP \wedge dO_{n-1} \wedge \cdots \wedge dO_1, \quad (1)$$