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**Autor:** Ecker, A.  
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## On primitive roots

Baum [2] has given useful criteria for certain primitive roots. Wilansky [6] pointed out that these results can be obtained without use of quadratic reciprocity. The purpose of this note is to derive their theorems and to obtain some more results with a quite simple counting method. We shall deal with odd primes  $p$ . We assume standard results on quadratic residues and primitive roots. An integer  $a$  relatively prime to  $p$  belongs to the exponent  $k > 0$ , modulo  $p$ , if  $a^k \equiv 1 \pmod{p}$  and  $a^n \not\equiv 1 \pmod{p}$  for  $0 < n < k$ . A primitive root modulo  $p$  is a residue which belongs to the exponent  $p - 1$ . There are  $\varphi(p - 1)$  primitive roots modulo  $p$ , where  $\varphi(x)$  is the Euler phi-function or totient. Euler's totient has the following property: if  $m$  is odd then  $\varphi(2^n \cdot m) = 2^{n-1} \varphi(m)$  ( $n \geq 1$ ) and  $\varphi(2^n \cdot m) = 2^n \varphi(m)$  if  $m$  is even. A quadratic residue, modulo  $p$ , is an integer  $a \neq 0$  such that  $x^2 \equiv a \pmod{p}$  has solutions. QR (QNR) denotes the set of residues, modulo  $p$ , which are quadratic residues (non-residues). With respect to the property of being a primitive root, modulo  $p$ , these sets are denoted by PR (NPR). We note the following familiar results:  $a$  is a quadratic residue modulo  $p$  if and only if  $a^{(p-1)/2} \equiv 1 \pmod{p}$ . This result is known as Euler's criterion. From Euler's criterion it follows that  $(-1/p) = (-1)^{(p-1)/2}$ , where  $(a/p)$  is the Legendre symbol, defined by  $(a/p) = +1$  if  $a \in \text{QR}$ ,  $(a/p) = -1$  if  $a \in \text{QNR}$ . Gauss has given a theorem – known as Gauss' lemma – that puts the information contained in Euler's criterion into a slightly different form. Gauss' lemma makes it possible to evaluate  $(2/p)$ ,  $(3/p)$ ,  $(7/p)$ .

The Legendre symbol has the properties:

$$\left(\frac{a}{p}\right) \cdot \left(\frac{b}{p}\right) = \left(\frac{a \cdot b}{p}\right), \quad \left(\frac{a}{p}\right) = \left(\frac{b}{p}\right) \quad \text{if } a \equiv b \pmod{p}$$

where  $a, b$  are relatively prime to  $p$ . This makes it possible to calculate  $(-a/p)$  if  $(a/p)$  is known. We give a list of values  $(a/p)$  needed in the sequel.

$$\left(\frac{-1}{p}\right) = \begin{cases} +1 & \text{if } p \equiv 1 \pmod{4} \\ -1 & \text{if } p \equiv -1 \pmod{4} \end{cases}$$

$$\left(\frac{2}{p}\right) = \begin{cases} +1 & \text{if } p \equiv \pm 1 \pmod{8} \\ -1 & \text{if } p \equiv \pm 3 \pmod{8} \end{cases}$$

$$\left(\frac{3}{p}\right) = \begin{cases} +1 & \text{if } p \equiv \pm 1 \pmod{12} \\ -1 & \text{if } p \equiv \pm 5 \pmod{12} \end{cases}$$

$$\left(\frac{-7}{p}\right) = \begin{cases} +1 & \text{if } p \equiv 1, 2, 4 \pmod{7} \\ -1 & \text{if } p \equiv 3, 5, 6 \pmod{7} \end{cases}$$

$$\left(\frac{-2}{p}\right) = \begin{cases} +1 & \text{if } p \equiv 1, 3 \pmod{8} \\ -1 & \text{if } p \equiv -1, -3 \pmod{8} \end{cases}$$

$$\left(\frac{-3}{p}\right) = \begin{cases} +1 & \text{if } p \equiv 1 \pmod{6} \\ -1 & \text{if } p \equiv -1 \pmod{6} \end{cases}$$

Clearly no number is simultaneously a quadratic residue and a primitive root modulo  $p$ ; and there are exactly  $(p-1)/2$  quadratic residues modulo  $p$ . This means  $\text{PR} \subset \text{QNR}$  and  $|\text{QR}| = (p-1)/2$ , where  $|M|$  denotes the number of elements in a finite set  $M$ .

**Lemma 1.**  $|\text{QNR}| = |\text{PR}| + |\text{NPR} \cap \text{QNR}|$  or with  $D = |\text{NPR} \cap \text{QNR}| = |\text{QNR} \setminus \text{PR}|$

$$\frac{p-1}{2} = \varphi(p-1) + D, \quad D \geq 0. \quad (1)$$

The proof of lemma 1 is quite clear from what we said above.

**Lemma 2.** If  $p \equiv 1 \pmod{4}$ ,  $p = 4q+1$  ( $q \geq 1$ ), then  $D$  is even and  $4 \mid D$  if  $q$  is even,  $2 \parallel D$  if  $q$  is odd and  $q > 1$ .  $D$  is odd if  $p \equiv -1 \pmod{4}$ ,  $p = 2q+1$  ( $q \geq 1$ ,  $q$  odd) except for  $q=1$ ,  $p=3$  where  $D=0$ .

Proof:  $D = (p-1)/2 - \varphi(p-1)$ .

$p = 4q+1$ :  $D = 2q - \varphi(4q)$  and  $2 \mid \varphi(4q)$  proves that  $D$  is even;  $q$  even means  $\varphi(4q) = 4\varphi(q)$ ,  $4 \mid 2q$  and  $4 \mid D$  follows,  $q$  odd gives  $\varphi(4q) = 2\varphi(q)$ ,  $D = 2(q - \varphi(q))$  and  $2 \nmid (q - \varphi(q))$  except for  $q=1$ , but then  $p=5$  and  $D=0$ .  $p = 2q+1$ ,  $q$  odd:  $D = q - \varphi(q)$  and  $D$  is odd except for  $q=1$ .

Naturally we now ask whether there is – excepting 3 and 5 – any possibility that  $D=0$  happens, that means that all quadratic nonresidues are primitive roots. If

$-1 \in \text{QNR}$  or  $(-1/p) = -1$   $D=0$  is impossible, except for  $p=3$ , because  $(-1)^2 \equiv 1 \pmod{p}$  and the exponent of  $-1$  modulo  $p$  is 2,  $-1 \in \text{QNR} \setminus \text{PR}$  except in case  $p-1=2$ , that is  $p=3$ . If  $p \equiv -1 \pmod{4}$  then  $(-1/p) = -1$ , hence  $D \geq 1$  if  $p > 3$  and  $p \equiv -1 \pmod{4}$ .

**Theorem 1.**  $D=0$  if and only if  $p=2^{2^n}+1=F(n)$  ( $n \geq 0$ ), the  $n$ -th Fermat number.

Proof: If  $D=0$ , then  $p=3=F(0)$  or  $p=4q+1$  ( $q \geq 1$ ) and from lemma 1 one gets  $(p-1)/2=\varphi(p-1)$ . This given  $2q=\varphi(4q)$ ,  $q$  odd:  $q=\varphi(q)$  implies  $q=1$ .  $q$  even:  $q=2\varphi(q)$ ,  $q=2^m \cdot r$  ( $m \geq 1$ ),  $2|r$  this implies  $r=\varphi(r)$  or  $r=1$ . That means  $p=2^{m+2}+1$  ( $m \geq 1$ ).

Thus  $p=2^t+1$  ( $t \geq 1$ ), but it is well known that  $t=2^n$  ( $n \geq 0$ ) is necessary and these are the Fermat numbers. The converse follows by a simple computation.

**Corollary 1.1.** An odd prime  $p$  is a Fermat prime if and only if all quadratic non-residues, modulo  $p$ , are primitive roots.

**Corollary 1.2.**  $\pm 3$  is a primitive root modulo  $p=F(n)$  ( $n \geq 1$ ).

Proof:  $F(n) \equiv 5 \pmod{12}$  ( $n \geq 1$ ) as follows from  $4^m \equiv 4 \pmod{12}$ . From  $p \equiv 5 \pmod{12}$  we get  $(\pm 3/p) = -1$  or  $\pm 3 \in \text{QNR}$ . Corollary 1.1 completes the proof.

**Corollary 1.3.**  $\pm 7$  is a primitive root of  $p=F(n)$  ( $n \geq 2$ ).

Proof: Note that  $2^4 \equiv 2 \pmod{7}$  and  $F(n) \equiv 3, 5 \pmod{7}$  ( $n \geq 2$ ). Thus  $(-7/p) = -1$ ,  $-7 \in \text{QNR}$  and  $(-1/p) = +1$  gives  $(-1/p) \cdot (-7/p) = (7/p) = -1$ ,  $7 \in \text{QNR}$ . Corollary 1.1 then gives the conclusion.

**Theorem 2.**  $D=1$  if and only if  $p=2q+1$ , where  $q$  is an odd prime.

Proof: From lemma 2 we see that  $p=2q+1$ ,  $q$  odd ( $q > 1$ ). (1) gives  $q-1=\varphi(q)$  and  $q$  necessarily is an odd prime. The converse follows by computation.

**Corollary 2.1.** All quadratic nonresidues modulo  $p$  beside  $-1$  are primitive roots if and only if  $p=2q+1$ , where  $q$  is an odd prime.

**Corollary 2.2.** If  $p$  and  $q$  are odd primes,  $p=2q+1$ , then  $(-1)^{(q-1)/2} \cdot 2$  is a primitive root modulo  $p$ .

Proof: If  $q \equiv 1 \pmod{4}$ , then  $2q+1 \equiv 3 \pmod{8}$  but then  $(2/p) = -1$ ,  $2 \in \text{QNR}$  and  $2 \not\equiv -1 \pmod{p}$ . From corollary 2.1 we see that  $2 \in \text{PR}$ , modulo  $p$ . If  $q \equiv -1 \pmod{4}$  we get  $2q+1 \equiv -1 \pmod{8}$ , i.e.  $(-2/p) = -1$  or  $-2 \in \text{QNR}$ ,  $-2 \not\equiv -1 \pmod{p}$  and  $-2 \in \text{PR}$ .

**Corollary 2.3.** If  $p$  and  $q$  are odd primes,  $p=2q+1$ , then if  $q \equiv 1 \pmod{4}$ ,  $q+1$  is a primitive root modulo  $p$ , while if  $q \equiv -1 \pmod{4}$ ,  $q$  is a primitive root modulo  $p$ .

**Proof:** If  $q \equiv 1 \pmod{4}$ , then from corollary 2.2 we know  $(2/p) = -1$ . Since  $2(q+1) = 2q+2 \equiv 1 \pmod{p}$  we have

$$\left(\frac{2}{p}\right) \cdot \left(\frac{q+1}{p}\right) = \left(\frac{1}{p}\right) = +1 \quad \text{and thus} \quad \left(\frac{q+1}{p}\right) = -1.$$

Hence  $q+1 \in \text{QNR}$  and  $q+1 \not\equiv -1 \pmod{p}$  means  $q+1 \in \text{PR}$ . If  $q \equiv -1 \pmod{4}$  the proof is similar:  $2q \equiv -1 \pmod{p}$  or  $(-2) \cdot q \equiv 1 \pmod{p}$  implies  $(-2/p) \cdot (q/p) = +1$  and this again gives  $q \in \text{PR}$ .

Note that in corollary 2.3 one can write the conclusion as:  $(-1)^{(q+1)/2} \cdot q$  is a primitive root modulo  $p$ .

**Remark:** Theorem 1 is exercise 3.8, No. 11, in Agnew [1], p. 144, while corollary 1.2 is given by Trost [5], IV.25, p. 40, and corollary 1.3 is just problem 3, chap. 5.3, in Le Veque [4], p. 69. Corollaries 2.1, 2.2 and 2.3 are what Baum [2] proved but compare with theorem 5-6 (b), (c) in [4], p. 68.

**Theorem 3.**  $D=2$  if and only if  $p=4q+1$ , where  $q$  is an odd prime, and in this case  $a \in \text{QNR} \setminus \text{PR}$  iff  $a$  belongs to the exponent 4 modulo  $p$ .

**Proof:** Half of the theorem is trivial, we prove the rest. If  $D=2$ , then  $(p-1)/2 = \varphi(p-1)+2$  or  $2q = \varphi(4q)+2$ , because lemma 2 gives  $p=4q+1$  and  $q$  is odd. Therefore  $2q = 2\varphi(q)+2$  or  $q-1 = \varphi(q)$  and  $q$  is an odd prime. If  $a \in \text{QNR} \setminus \text{PR}$ , then  $a^k \equiv 1 \pmod{p}$ , where  $k | p-1$  ( $k \neq p-1$ ) is the exponent of  $a$  modulo  $p$ . Now  $p-1 = 4q$  and  $a^{2q} \equiv -1 \pmod{p}$  from Euler's criterion. Hence  $k \nmid 2q$  while  $q$  is an odd prime, thus  $k=4$ .

**Corollary 3.1.**  $\pm 2$  is a primitive root of  $p=4q+1$  if  $q$  is an odd prime.

**Proof:** From  $p=4q+1$ ,  $q$  an odd prime it follows that  $p \equiv -3 \pmod{8}$ . Hence  $(\pm 2/p) = -1$  or  $\pm 2 \in \text{QNR}$ . But  $2^4 \equiv 1 \pmod{p}$  means  $p=3$  or  $p=5$  while  $4q+1 \geq 13$ . An application of theorem 3 completes the proof.

**Corollary 3.2.**  $2q, 2q+1$  are primitive roots modulo  $p=4q+1$ , if  $q$  is an odd prime.

**Proof:** First of all note that  $2q+1 \equiv -2q \pmod{p}$ . Therefore it is enough to show that  $2q$  is a primitive root modulo  $p$ .  $2(2q) \equiv -1 \pmod{p}$  gives  $(2/p) \cdot (2q/p) = (-1/p) = +1$ . From corollary 3.1 we know  $(2/p) = -1$ , hence  $(2q/p) = -1$ . Next we have to prove  $(2q)^4 \not\equiv 1 \pmod{p}$ .  $(2q)^4 = (4q)^2 \cdot q^2$  and  $(4q)^2 \equiv 1 \pmod{p}$  gives  $(2q)^4 \equiv q^2 \pmod{p}$ . But  $q^2 - 1 = (q+1) \cdot (q-1)$  and from  $q^2 \equiv (2q)^4 \equiv 1 \pmod{p}$   $p \nmid q+1$  or  $p \nmid q-1$ . It is easy to see that this cannot happen. This completes the proof.

**Remark:** For corollary 3.1 see theorem 5-6 (a) in [4], p. 68.

**Theorem 4.**  $D=2^n$  ( $n \geq 2$ ) if and only if  $p=2^{n+1} \cdot r+1 = 2D \cdot r+1$ , where  $r$  is an odd prime, and in this case  $a \in \text{QNR} \setminus \text{PR}$  iff  $a$  belongs to the exponent  $2D$  modulo  $p$ .

**Proof:** From lemma 2 we see that  $p=4q+1$  and  $q=2^m \cdot r$  ( $m \geq 1$ ),  $r$  odd. Hence from (1)  $2q=4\varphi(q)+2^n$  or  $q-2^{n-1}=2\varphi(q)$ .  $q=2^m \cdot r$  gives  $2^m \cdot r-2^{n-1}=2^m \cdot \varphi(r)$  hence  $r \neq 1$  and  $2 \mid \varphi(r)$ .

If  $n-1 < m$  we rewrite  $2^{n-1}=2^m(r-\varphi(r))$  or  $1=2^{m-n+1}(r-\varphi(r))$ , a contradiction. If  $n-1 > m$  write  $2^m \cdot r=2^m(\varphi(r)+2^{n-m-1})$  where  $2 \mid (\varphi(r)+2^{n-m-1})$  but  $2 \nmid r$ , a contradiction. Hence  $m=n-1$ . This gives  $r-1=\varphi(r)$  and  $r$  is an odd prime. The converse is almost trivial. If  $a \in \text{QNR}$  and  $k$  is the exponent of a modulo  $p$ , then  $a^D \cdot r \equiv -1 \pmod{p}$  and  $k \mid D \cdot r$ . Thus  $k \nmid D \cdot r$  and  $r$  is prime. Therefore  $k=2D$  since  $k \neq 2D \cdot r$  and the proof is complete.

**Lemma 3.** Let  $p=2D \cdot r+1$ ,  $D=2^n$  ( $n \geq 2$ ) as in theorem 4. Then, except possibly for  $r=3$ ,  $\pm 3$  resp.  $\pm 6$  is a primitive root modulo  $p$  if and only if  $3^{2D} \pmod{p}$  resp.  $6^{2D} \not\equiv 1 \pmod{p}$ .

**Proof:** Note that  $4 \equiv 1 \pmod{3}$  and hence  $2^m \equiv 1 \pmod{3}$  if  $m$  is even,  $2^m \equiv 2 \pmod{3}$  if  $m$  is odd. If  $m=n+1$ , then  $p \equiv r+1$  or  $2r+1 \pmod{3}$  according as  $n$  is odd or even.  $r \equiv 1, 2 \pmod{3}$  always gives  $p \equiv 2 \pmod{3}$ ,  $p \equiv 0 \pmod{3}$  is not possible. Hence  $3 \mid p+1$ , trivially  $2 \mid p+1$ , therefore  $6 \mid p+1$  or  $p \equiv -1 \pmod{6}$ .

From our list of computed Legendre symbols we see  $(-3/p) = -1$ . Taking into account  $-1, 2 \in \text{QR}$  and theorem 4 the proof is complete.

**Corollary 4.1.** If  $p=8 \cdot r+1$ , where  $r$  is an odd prime, then  $\pm 6$  is a primitive root modulo  $p$ .  $\pm 3$  is a primitive root modulo  $p$  if  $r \neq 5$ .

**Proof:** From theorem 4 we see that  $D=4$  and lemma 3 shows that one has to consider  $3^8-1$ ,  $6^8-1$  modulo  $p$ . Note that  $r=3$  is not possible. The computation gives

$$\begin{aligned} 3^8-1 &= 6560 = 2^5 \cdot 5 \cdot 41, \\ 6^8-1 &= 1679615 = 5 \cdot 7 \cdot 37 \cdot 1297. \end{aligned}$$

It is now easy to check that (except for  $r=5$ )  $p=8r+1$  will never divide either  $3^8-1$  or  $6^8-1$ .

**Corollary 4.2.** If  $p=16 \cdot r+1$ , where  $r$  is an odd prime, then  $\pm 3$ ,  $\pm 6$  are primitive roots modulo  $p$ .

**Proof:**  $D=8$  and

$$\begin{aligned} 3^{16}-1 &= 43046720 = 2^6 \cdot 5 \cdot 17 \cdot 41 \cdot 193, \\ 6^{16}-1 &= 5 \cdot 7 \cdot 17 \cdot 37 \cdot 1297 \cdot 98801. \end{aligned}$$

( $r=3$  again is impossible). A simple consideration as in the foregoing corollary now completes the proof.

Note that  $p=2^{n+2} \cdot r+1$ ,  $r$  an integer, are just the prime divisors of  $F(n)$ .

Historical remarks: Most of the above results were known to nineteenth century mathematicians, although obtained by various quite different methods (see Dickson [3], chap. VII). M. A. Stern (J. Math. 6, 147–153, 1830) proved that, if  $p = 2q + 1$  and  $q$  are odd primes, 2 or  $-2$  is a primitive root of  $p$  according as  $p = 8n + 3$  or  $8n + 7$  (see corollary 2.2). If  $p = 4q + 1$  and  $q$  are primes, 2 and  $-2$  are primitive roots of  $p$  (see corollary 3.1). F. J. Richelot (J. Math. 9, 5, 1832) proved that, if  $p = 2^m + 1$  is a prime, every quadratic nonresidue (in particular, 3) is a primitive root of  $p$  (see corollaries 1.1, 1.2). Nearly the same results were given by P. L. Tchebychef ['Theory of congruences' (in Russian), 1849]. G. Wertheim (Acta Math. 17, 315–320, 1893) proved that any prime  $2^{4n} + 1$  has the primitive root 7 (see corollary 1.3). If  $p = 2^n \cdot q + 1$  is a prime and  $q$  is an odd prime, any quadratic nonresidue  $a$  of  $p$  is a primitive root of  $p$  if  $a^{2^n} - 1$  is not divisible by  $p$  (see lemma 3). These and other nice results on primitive roots can be derived from theorems 1–4 as corollaries (see for example in [3], p. 192, what V. Bouniakowsky proved or loc. cit., p. 199, the result of A. Cunningham).

A. Ecker,  
Hahn-Meitner-Institut für Kernforschung Berlin GmbH

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# Kleine Mitteilungen

## A homeomorphism of $\mathbf{Q}$ with $\mathbf{Q}$ as an orbit

The following result can be deduced from a theorem proved by Besicovitch [1] in which an autohomeomorphism  $h$  of the real plane is constructed such that for some  $x \in \mathbf{R}^2$

$$\{h^n(x) | n \in \mathbf{Z}\} \text{ is dense in } \mathbf{R}^2.$$

We give a direct proof for the consequence.

**Proposition.** *There exists an autohomeomorphism  $h$  of  $\mathbf{Q}$  with*

$$\{h^n(1) | n \in \mathbf{Z}\} = \mathbf{Q}.$$

**Proof:** Let  $x_1 \in [0, 1] \setminus \mathbf{Q}$  with  $2x_1 < 1$  and, for  $n \in \mathbf{Z}$ ,  $x_n = nx_1 - [nx_1]$ , [ ] designating