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## On lattice polytopes having interior lattice points

Let  $V(K)$  denote the volume of a convex body  $K$  in the Euclidean  $d$ -space  $E^d$ , and let  $G(K)$  and  $G^0(K)$  be defined by  $G(K) = \text{card}(K \cap Z^d)$  and  $G^0(K) = \text{card}(\text{int } K \cap Z^d)$ , where  $Z^d$  denotes the set of all the lattice points in  $E^d$ , i.e., points having only integral coordinates.

The following theorems of Minkowski are well known.

**Theorem A** (see [2], p. 76). *If  $K$  is a centrally symmetric convex body in  $E^d$  and  $G^0(K) = 1$ , then  $V(K) \leq 2^d$ .*

**Theorem B** (see [2], p. 79 and 96). *If  $K$  is a centrally symmetric convex body in  $E^d$  and  $G^0(K) = 1$ , then  $G(K) \leq 3^d$ ; if  $G^0(K) = 1$  and  $V(K) = 2^d$ , then  $K$  is a convex polytope and it has at most  $2^{d+1} - 2$  facets.*

There are no analogues of theorems A and B for convex bodies which are not centrally symmetric. The purpose of this paper is to give some information on  $V(P)$  and  $G(P)$  in the case where  $P$  is a lattice polytope (i.e.,  $P$  has all of its vertices in  $Z^d$ ) satisfying  $G^0(P) = n, n \geq 1$ . We will see that  $V$  and  $G$  behave very different from the centrally symmetric case.

Let  $g(d, n)$  and  $v(d, n)$  be defined for  $n \geq 1$  and  $d \geq 2$  as follows:  $g(d, n) = \sup \{G(P) \mid P \subset E^d, G^0(P) = n\}$  and  $v(d, n) = \sup \{V(P) \mid P \subset E^d, G^0(P) = n\}$ .

Scott, [3] proved that  $g(2, 1) = 10$  and that  $g(2, n) = 3n + 6$  for all  $n \geq 2$ . Our main result is the following

**Theorem.** *For all  $d, d \geq 4$ , and for all  $n, n \geq 1$ ,*

$$v(d, n) \geq \frac{n+1}{d!} 2^{2^{d-a}} \quad \text{and} \quad g(d, n) \geq \frac{n+1}{6(d-2)!} 2^{2^{d-a}},$$

where  $a = 0.5856 \dots$ ;  $v(3, n) \geq 6(n+1)$ ,  $g(3, n) \geq 16n + 23$ ,  $v(4, 1) \geq 147$  and  $g(4, 1) \geq 680$ .

We need the following

**Lemma.** *If  $(a_n)_{n=1}^{\infty}$  is defined by  $a_1 = 2$  and  $a_d = \prod_{i=1}^{d-1} a_i + 1$  for  $d \geq 2$ , then*

(i)  $a_1 = 2, a_2 = 3, a_3 = 7, a_4 = 43$  and  $a_5 = 1807$ ;

(ii) for all  $d, d \geq 1, 1 - \sum_{i=1}^d \frac{1}{a_i} = \left( \prod_{i=1}^d a_i \right)^{-1}$ ;

(iii) for all  $d, d \geq 4, \prod_{i=1}^d a_i \geq 2^{2^{d-a}}$  where  $a = 0.5856 \dots$

Proof of the lemma: (i) is trivial, (ii) is true for  $d=1$ , and if it is true for  $d$ , then by induction

$$1 - \sum_{i=1}^{d+1} \frac{1}{a_i} = \left( \prod_{i=1}^d a_i \right)^{-1} - \frac{1}{a_{d+1}} = \left( \prod_{i=1}^{d+1} a_i \right)^{-1}.$$

Let the real  $a$  be defined by  $2^{2^{d-a}} = a_1 a_2 a_3 a_4 = 1806$ , with  $d=4$ ; thus  $a=0.5856 \dots$ . Suppose (iii) holds for  $d$ , then

$$\prod_{i=1}^{d+1} a_i > \left( \prod_{i=1}^d a_i \right)^2 \geq 2^{2 \cdot 2^{d-a}} = 2^{2^{d+1-a}},$$

hence it holds for  $d+1$ .

Proof of the theorem: Let  $k_i = a_i$  for  $i=1, \dots, d-1, k_d = 2(a_d - 1)$ , and let  $S_1^d$  be the  $d$ -simplex defined by

$$S_1^d = \left\{ (x_1, \dots, x_d) \in E^d \mid x_i \geq 0, \sum_{i=1}^d \frac{x_i}{k_i} \leq 1 \right\}.$$

By (ii) we have

$$\sum_{i=1}^d \frac{1}{k_i} < 1 \quad \text{and} \quad \frac{1}{k_d} + \sum_{i=1}^d \frac{1}{k_i} = 1,$$

hence  $(1, \dots, 1) \in \text{int } S_1^d$  and  $(1, \dots, 1, 2) \notin \text{int } S_1^d$ ;  $k_1 < k_2 < \dots < k_d$  imply that  $G^0(S_1^d) = 1$ . For general  $n, n \geq 1$ , replace  $k_d = 2(a_d - 1)$  by  $k_d = (n+1)(a_d - 1)$ , and obtain the  $d$ -simplex  $S_n^d$ . It follows that the only lattice points of  $\text{int } S_n^d$  are  $(1, \dots, 1, j)$  for  $1 \leq j \leq n$ , hence  $G^0(S_n^d) = n$ .

As for the volume of  $S_n^d, V(S_n^d) = (1/d!) \prod_{i=1}^d k_i = ((n+1)/d!) (a_d - 1)^2$  and by (iii):

$$a_d - 1 \geq 2^{2^{d-1-a}}, \quad \text{therefore} \quad v(d, n) \geq \frac{n+1}{d!} 2^{2^{d-a}}.$$

To compute  $G(S_n^d)$  let  $d \geq 3$  and  $S = S_n^d \cap \{x_1 = 0\} \cap \{x_2 = 0\}$ .

It is easy to see that for each  $t \in \text{aff } S$

$G(S+t) \leq G(S)$  and equality iff  $x$  is a lattice point.

If  $C_{d-2}$  denotes the unit cube in  $\text{aff } S$  and  $V_{d-2}$  the volume in  $\text{aff } S$ , we have

$$V_{d-2}(S) = \int_{C_{d-2}} G(S+t) dt < G(S) < G(S_n^d).$$

With  $6 V_{d-2}(S) = d(d-1) V(S_n^d)$  we have

$$G(S_n^d) \leq \frac{d(d-1)}{6} V(S_n^d) = \frac{n+1}{6(d-2)!} 2^{2^{d-a}}.$$

The case  $d=3$ : Clearly:  $V(S_n^3) = 6(n+1)$ .

To compute  $G(S_n^3)$  we count the integers  $x_3 \geq 0$  satisfying

$$3(n+1)x_1 + 2(n+1)x_2 + x_3 \leq 6(n+1)$$

for  $(x_1, x_2) = (0, 0), (0, 1), (0, 2), (1, 0), (1, 1)$  and  $(2, 0)$ .

Easy computation gives  $G(S_n^3) = 16n + 23$ .

The case  $d=4$ :  $V(S_1^4) = 147$ .

To compute  $G(S_1^4)$  we count the pairs  $(x_3, x_4)$  of integers  $x_3 \geq 0, x_4 \geq 0$  satisfying

$$42x_1 + 28x_2 + 7x_3 + x_4 \leq 84$$

for  $(x_1, x_2) = (0, 0), (0, 1), (0, 2), (0, 3), (1, 0), (1, 1), (2, 0)$ .

Counting yields  $G(S_1^4) = 680$ .

This completes the proof of the theorem.

We raise the following *conjecture*:

$$v(d, n) < \infty \quad \text{for all } d \geq 3 \quad \text{and } n \geq 1.$$

We remark that the conjecture implies  $g(d, n) < \infty$  by Blichfeldt ([1], p.55):  $G(P) \leq d!V(P) + d$  for nondegenerate lattice polytopes (compare [4]).

Similar problems may be asked for the number of  $i$ -dimensional faces of convex lattice polytopes  $P$  satisfying  $G^0(P) = n$ , where  $0 \leq i \leq d-1$ .

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## REFERENCES

- 1 C.G. Lekkerkerker: Geometry of numbers. Wolters-Noordhoff, Groningen 1969.
- 2 H. Minkowski: Geometrie der Zahlen. Teubner, Leipzig 1910.
- 3 P.R. Scott: On convex lattice polygons. Bull. Austr. Math. Soc. 15, 395-399 (1976).
- 4 J.M. Wills: Gitterzahlen und innere Volumina. Comm. Math. Helv. 53, 508-524 (1978).

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## Kleine Mitteilungen

### A note on the successive remainders of the exponential series

1. For real  $x \neq 0$ , we have by Taylor's theorem

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!} \phi_n(x), \quad (1.1)$$

where

$$\phi_n(x) = e^{x\theta_n(x)}, \quad 0 < \theta_n(x) < 1. \quad (1.2)$$