

On lattice polytopes having interior lattice points

Autor(en): **Zaks, J. / Perles, M.A. / Wills, J.M.**

Objektyp: **Article**

Zeitschrift: **Elemente der Mathematik**

Band (Jahr): **37 (1982)**

Heft 2

PDF erstellt am: **21.09.2024**

Persistenter Link: <https://doi.org/10.5169/seals-36389>

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.

Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.

Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

On lattice polytopes having interior lattice points

Let $V(K)$ denote the volume of a convex body K in the Euclidean d -space E^d , and let $G(K)$ and $G^0(K)$ be defined by $G(K) = \text{card}(K \cap Z^d)$ and $G^0(K) = \text{card}(\text{int } K \cap Z^d)$, where Z^d denotes the set of all the lattice points in E^d , i.e., points having only integral coordinates.

The following theorems of Minkowski are well known.

Theorem A (see [2], p. 76). *If K is a centrally symmetric convex body in E^d and $G^0(K) = 1$, then $V(K) \leq 2^d$.*

Theorem B (see [2], p. 79 and 96). *If K is a centrally symmetric convex body in E^d and $G^0(K) = 1$, then $G(K) \leq 3^d$; if $G^0(K) = 1$ and $V(K) = 2^d$, then K is a convex polytope and it has at most $2^{d+1} - 2$ facets.*

There are no analogues of theorems A and B for convex bodies which are not centrally symmetric. The purpose of this paper is to give some information on $V(P)$ and $G(P)$ in the case where P is a lattice polytope (i.e., P has all of its vertices in Z^d) satisfying $G^0(P) = n, n \geq 1$. We will see that V and G behave very different from the centrally symmetric case.

Let $g(d, n)$ and $v(d, n)$ be defined for $n \geq 1$ and $d \geq 2$ as follows: $g(d, n) = \sup \{G(P) \mid P \subset E^d, G^0(P) = n\}$ and $v(d, n) = \sup \{V(P) \mid P \subset E^d, G^0(P) = n\}$.

Scott, [3] proved that $g(2, 1) = 10$ and that $g(2, n) = 3n + 6$ for all $n \geq 2$. Our main result is the following

Theorem. *For all $d, d \geq 4$, and for all $n, n \geq 1$,*

$$v(d, n) \geq \frac{n+1}{d!} 2^{2^{d-a}} \quad \text{and} \quad g(d, n) \geq \frac{n+1}{6(d-2)!} 2^{2^{d-a}},$$

where $a = 0.5856 \dots$; $v(3, n) \geq 6(n+1)$, $g(3, n) \geq 16n + 23$, $v(4, 1) \geq 147$ and $g(4, 1) \geq 680$.

We need the following

Lemma. *If $(a_n)_{n=1}^{\infty}$ is defined by $a_1 = 2$ and $a_d = \prod_{i=1}^{d-1} a_i + 1$ for $d \geq 2$, then*

(i) $a_1 = 2, a_2 = 3, a_3 = 7, a_4 = 43$ and $a_5 = 1807$;

(ii) for all $d, d \geq 1, 1 - \sum_{i=1}^d \frac{1}{a_i} = \left(\prod_{i=1}^d a_i \right)^{-1}$;

(iii) for all $d, d \geq 4, \prod_{i=1}^d a_i \geq 2^{2^{d-a}}$ where $a = 0.5856 \dots$

Proof of the lemma: (i) is trivial, (ii) is true for $d=1$, and if it is true for d , then by induction

$$1 - \sum_{i=1}^{d+1} \frac{1}{a_i} = \left(\prod_{i=1}^d a_i \right)^{-1} - \frac{1}{a_{d+1}} = \left(\prod_{i=1}^{d+1} a_i \right)^{-1}.$$

Let the real a be defined by $2^{2^{d-a}} = a_1 a_2 a_3 a_4 = 1806$, with $d=4$; thus $a=0.5856 \dots$. Suppose (iii) holds for d , then

$$\prod_{i=1}^{d+1} a_i > \left(\prod_{i=1}^d a_i \right)^2 \geq 2^{2 \cdot 2^{d-a}} = 2^{2^{d+1-a}},$$

hence it holds for $d+1$.

Proof of the theorem: Let $k_i = a_i$ for $i=1, \dots, d-1, k_d = 2(a_d - 1)$, and let S_1^d be the d -simplex defined by

$$S_1^d = \left\{ (x_1, \dots, x_d) \in E^d \mid x_i \geq 0, \sum_{i=1}^d \frac{x_i}{k_i} \leq 1 \right\}.$$

By (ii) we have

$$\sum_{i=1}^d \frac{1}{k_i} < 1 \quad \text{and} \quad \frac{1}{k_d} + \sum_{i=1}^d \frac{1}{k_i} = 1,$$

hence $(1, \dots, 1) \in \text{int } S_1^d$ and $(1, \dots, 1, 2) \notin \text{int } S_1^d$; $k_1 < k_2 < \dots < k_d$ imply that $G^0(S_1^d) = 1$. For general $n, n \geq 1$, replace $k_d = 2(a_d - 1)$ by $k_d = (n+1)(a_d - 1)$, and obtain the d -simplex S_n^d . It follows that the only lattice points of $\text{int } S_n^d$ are $(1, \dots, 1, j)$ for $1 \leq j \leq n$, hence $G^0(S_n^d) = n$.

As for the volume of $S_n^d, V(S_n^d) = (1/d!) \prod_{i=1}^d k_i = ((n+1)/d!) (a_d - 1)^2$ and by (iii):

$$a_d - 1 \geq 2^{2^{d-1-a}}, \quad \text{therefore} \quad v(d, n) \geq \frac{n+1}{d!} 2^{2^{d-a}}.$$

To compute $G(S_n^d)$ let $d \geq 3$ and $S = S_n^d \cap \{x_1 = 0\} \cap \{x_2 = 0\}$.

It is easy to see that for each $t \in \text{aff } S$

$G(S+t) \leq G(S)$ and equality iff x is a lattice point.

If C_{d-2} denotes the unit cube in $\text{aff } S$ and V_{d-2} the volume in $\text{aff } S$, we have

$$V_{d-2}(S) = \int_{C_{d-2}} G(S+t) dt < G(S) < G(S_n^d).$$

With $6 V_{d-2}(S) = d(d-1) V(S_n^d)$ we have

$$G(S_n^d) \leq \frac{d(d-1)}{6} V(S_n^d) = \frac{n+1}{6(d-2)!} 2^{2^{d-a}}.$$

The case $d=3$: Clearly: $V(S_n^3) = 6(n+1)$.

To compute $G(S_n^3)$ we count the integers $x_3 \geq 0$ satisfying

$$3(n+1)x_1 + 2(n+1)x_2 + x_3 \leq 6(n+1)$$

for $(x_1, x_2) = (0, 0), (0, 1), (0, 2), (1, 0), (1, 1)$ and $(2, 0)$.

Easy computation gives $G(S_n^3) = 16n + 23$.

The case $d=4$: $V(S_1^4) = 147$.

To compute $G(S_1^4)$ we count the pairs (x_3, x_4) of integers $x_3 \geq 0, x_4 \geq 0$ satisfying

$$42x_1 + 28x_2 + 7x_3 + x_4 \leq 84$$

for $(x_1, x_2) = (0, 0), (0, 1), (0, 2), (0, 3), (1, 0), (1, 1), (2, 0)$.

Counting yields $G(S_1^4) = 680$.

This completes the proof of the theorem.

We raise the following *conjecture*:

$$v(d, n) < \infty \quad \text{for all } d \geq 3 \quad \text{and } n \geq 1.$$

We remark that the conjecture implies $g(d, n) < \infty$ by Blichfeldt ([1], p.55): $G(P) \leq d!V(P) + d$ for nondegenerate lattice polytopes (compare [4]).

Similar problems may be asked for the number of i -dimensional faces of convex lattice polytopes P satisfying $G^0(P) = n$, where $0 \leq i \leq d-1$.

J. Zaks, University of Haifa and C.R.M.A., University of Montreal,
M. A. Perles, Hebrew University, Jerusalem,
J. M. Wills, University of Siegen, West Germany

REFERENCES

- 1 C.G. Lekkerkerker: Geometry of numbers. Wolters-Noordhoff, Groningen 1969.
- 2 H. Minkowski: Geometrie der Zahlen. Teubner, Leipzig 1910.
- 3 P.R. Scott: On convex lattice polygons. Bull. Austr. Math. Soc. 15, 395-399 (1976).
- 4 J.M. Wills: Gitterzahlen und innere Volumina. Comm. Math. Helv. 53, 508-524 (1978).

© 1982 Birkhäuser Verlag, Basel

0013-6016/82/020044-03\$1.50 + 0.20/0

Kleine Mitteilungen

A note on the successive remainders of the exponential series

1. For real $x \neq 0$, we have by Taylor's theorem

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!} \phi_n(x), \quad (1.1)$$

where

$$\phi_n(x) = e^{x\theta_n(x)}, \quad 0 < \theta_n(x) < 1. \quad (1.2)$$