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Benützung der zu \tilde{v}_1 parallelen Elationsachse durch P kann der Parabelscheitel nach Abschnitt 3 konstruiert werden.

H. P. Paukowitsch, TU Wien

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The number of triangles in a triangulation of a set of points in the plane¹⁾

1. Introduction

Our terminology and notation will be standard except as indicated. A good reference for undefined graph theoretic terms is [3].

In [1, 2] the authors discussed the question of the number of 3-cycles which could be present in a planar graph on p points. In this paper, we want to consider essentially the same question when the p points are in *fixed positions* in the plane. We will show that this restriction does not limit the possible range of the number of 3-cycles unless the p points are arranged in a unique, easily characterized configuration.

2. Statement of the problem and main results

Begin with a set P of $p \geq 5$ points in the plane, with no three of the points collinear. Suppose we draw straight line segments between pairs of points in P subject only to the restriction that these segments do not intersect except at the points of P themselves, until it is impossible to add more segments in this manner. We call this collection of line segments a *triangulation* of P (since all the finite regions into which these segments divide the plane are triangles). We will generally use T to denote a triangulation of P . Note in particular that the line segments comprising the boundary of the convex hull of P will be included in every triangulation T of P .

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We want to consider the number of triangles (or 3-cycles) in various triangulations of P . To this end, let h denote the number of extreme points of the convex hull of P . (For brevity in the sequel, we will term this collection of h points the *extreme points* of P .) In any triangulation T of P , it follows by Euler's well-known formula that the number of 3-cycles each of which bounds a region (i.e., contains points of P in either its interior or exterior, but not both) will be precisely $2p-h-2$ if $h > 3$, and $2p-4$ if $h=3$. In addition, however, a triangulation T of P may have 3-cycles containing points of P in both their interior and exterior. We will call such 3-cycles *separating*. In [1, 2], it was shown that the number of separating 3-cycles must be between 0 and either $p-h$ if $h > 3$ or $p-4$ if $h=3$. Our goal is to show that except for the two cases described in the statement of the theorem below, it is always possible to triangulate P so as to obtain any number of separating 3-cycles in the indicated range. We begin with the following result.

Lemma. *Let P be a set of $p \geq 5$ points in the plane, with no three of the points collinear. Suppose P has h extreme points. Then there is a triangulation T of P without separating triangles, unless $h=3$ and P has an extreme point x such that $P-x$ has $p-1$ extreme points (see fig. 1). In this exceptional case, any triangulation of P contains exactly $p-4$ separating triangles.*

Moreover, if $h < p$ and it is possible to triangulate P without separating triangles, then it is possible to obtain such a triangulation with no 'chords' between extreme points of P (i.e., with no line segments between extreme points of P except those comprising the boundary of the convex hull of P).

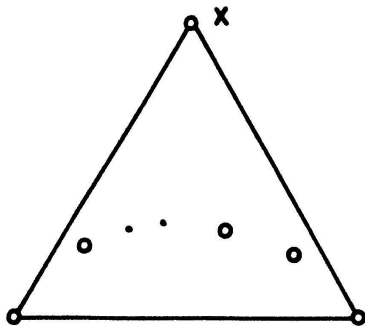


Figure 1

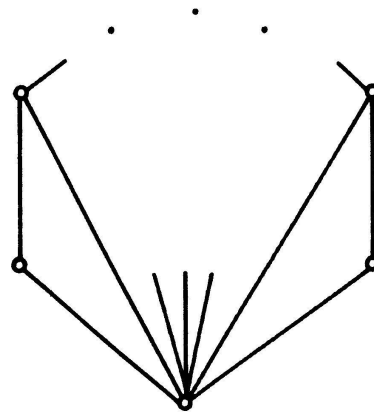


Figure 2

Proof: Observe first that if $h=p$, the desired triangulation is trivial (see fig. 2). Hence we assume $h < p$ in the rest of the proof.

Noting that the lemma is readily verified for $p=5$, we proceed by induction on p . Suppose first that $h \geq 4$. It is then easy to see that we can choose an extreme point x of P such that $P-x$ contains say $h' \geq 4$ extreme points. If $h' < p-1$, then by our induction hypothesis, we can triangulate $P-x$ without separating triangles or chords between extreme points of $P-x$. It is then a simple matter to obtain the desired triangulation of P (see fig. 3). If $h'=p-1$, then $P-x$ has extreme points y, z positioned as shown in figure 4. (If y (resp., z) did not exist, we would have $h=p$ (resp., $h=3$), contrary to what we have assumed.) It is then a simple matter to complete the desired triangulation of P (see fig. 4).

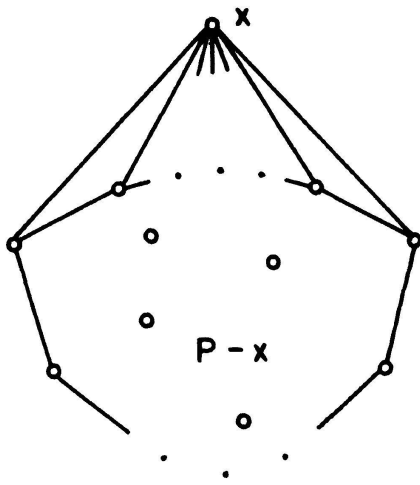


Figure 3

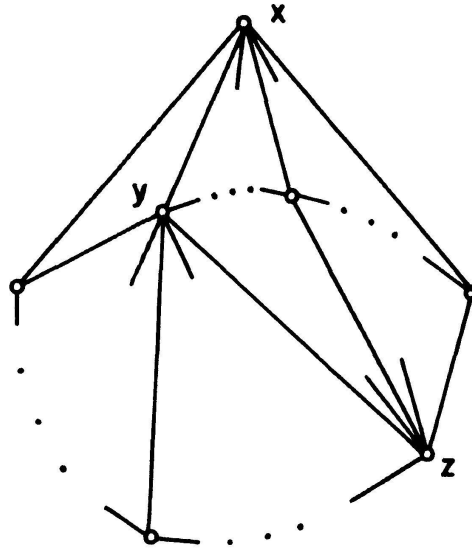


Figure 4

Suppose, therefore, that $h=3$. Let x_1, x_2, x_3 be the extreme points of P . If $P - x_i$ has h_i extreme points, where $4 \leq h_i < p-1$, for some i , then we can triangulate P without separating triangles as in the last paragraph. Otherwise, $P - x_i$ has three extreme points for each i . In particular, let x_2, x_3 and say x_4 be the extreme points of $P - x_1$. Suppose first that $P - x_1 - x_i$ does not have $p-2$ extreme points, for $i=2, 3, 4$. Then by the induction hypothesis, we can triangulate $P - x_1$ without separating 3-cycles. Call this triangulation T' . It is easy to see that there will be a point x in the interior of $x_2 x_3 x_4$ such that $x x_i x_4$ is a nonseparating 3-cycle of T' , and $x x_i x_1 x_4$ is convex, for either $i=2$ or 3 ; without loss of generality, suppose this occurs for $i=2$ (see fig. 5a). We obtain a triangulation T of P without separating 3-cycles from T' as follows: Remove the line segment $x_2 x_4$ from T' , and add the segments $x_1 x, x_1 x_2, x_1 x_3$ and $x_1 x_4$ (see fig. 5b). It is easily seen that the only way T could contain a separating 3-cycle is if the line segment $x x_3$ belonged to T . But in that case, either $x x_2 x_3$ or $x x_3 x_4$ would be a separating 3-cycle in T' , a contradiction. Suppose, therefore, that $P - x_1 - x_i$ has $p-2$ extreme points, for some $i=2, 3$ or 4 . We need to consider essentially two cases. If $P - x_1 - x_2$ has $p-2$ extreme points

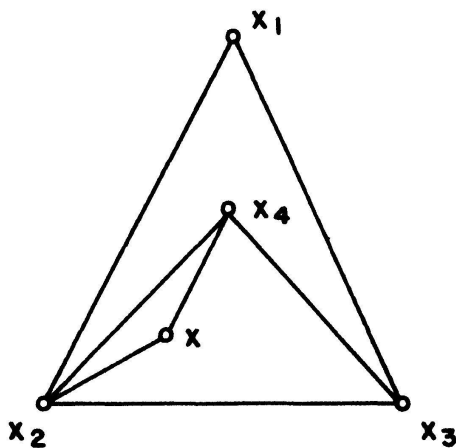


Figure 5a

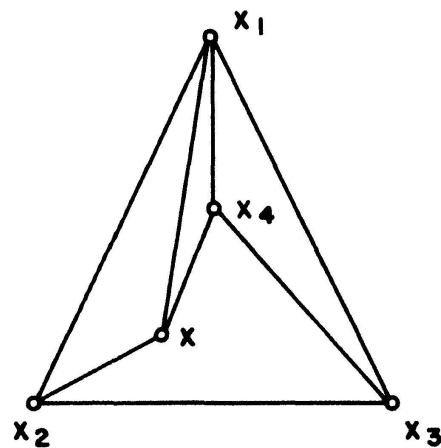


Figure 5b

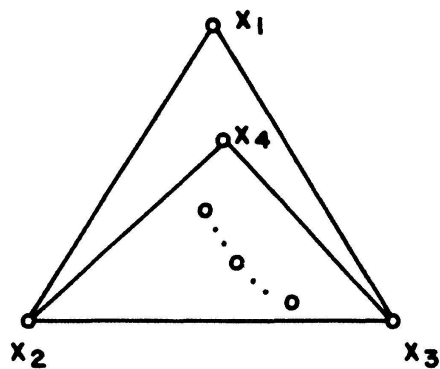


Figure 6a

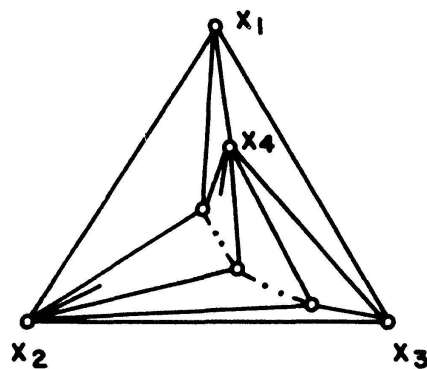


Figure 6b

(see fig. 6a), then we can triangulate P without separating 3-cycles as shown in figure 6b (assuming of course that $P - x_2$ does not have $p - 1$ extreme points).

On the other hand, if $P - x_1 - x_4$ has $p - 2$ extreme points (see fig. 7a), we can triangulate P without separating triangles as shown in figure 7b. This completes the proof for all but the exceptional case.

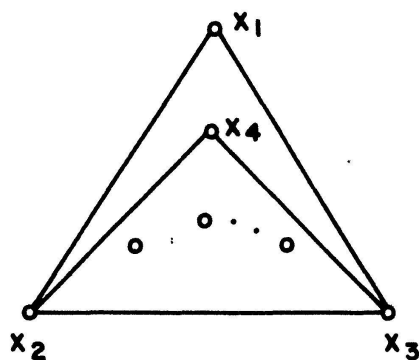


Figure 7a

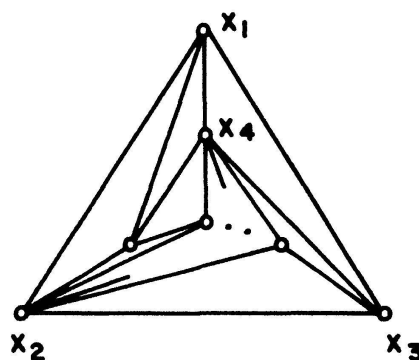


Figure 7b

Consider, therefore, the situation when $h = 3$ and P has an extreme point x such that $P - x$ contains $p - 1$ extreme points. Let y, z be the other extreme points of P . Then in any triangulation T of P , there will be a point w inside xyz such that wyz is a nonseparating 3-cycle in T (see fig. 8). It is then easy to see that the line segment wx must also belong to T .

Consider the sets of points P_1 and P_2 inside or on the 3-cycles wxy and wxz , respectively. It is easy to see that $P_i - x$ has $|P_i| - 1$ extreme points, for $i = 1, 2$. If

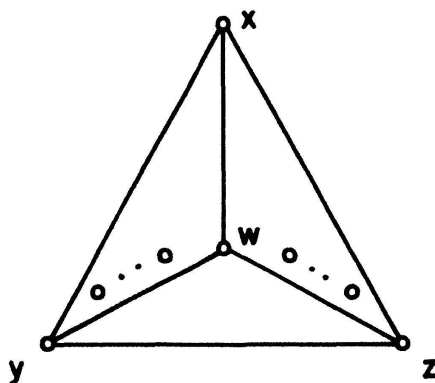


Figure 8

$|P_i| \geq 5$ for $i=1,2$, it follows by the induction hypothesis that any triangulation T_i of P_i must contain $|P_i| - 4$ separating 3-cycles. Moreover, the only other separating 3-cycles in T (besides those in T_1 and T_2) would be wxy and wxz . Hence the number of separating 3-cycles in T will be

$$(|P_1| - 4) + (|P_2| - 4) + 2 = p - 4,$$

as asserted. The cases when $|P_i| = 3$ or 4 are similar, and are, therefore, omitted. This completes the proof of the lemma.

We can now state our main result.

Theorem. *Let P be a set of $p \geq 5$ points in the plane with no three of the points collinear. Suppose P has h extreme points. Let $\Delta(T)$ denote the number of 3-cycles in a triangulation T of P . Then*

1. *If $h \geq 4$,*

$$2p - h - 2 \leq \Delta(T) \leq 3p - 2h - 2.$$

Moreover, if a is any number in the indicated range, there is a triangulation T of P with $\Delta(T) = a$.

2. *If $h = 3$,*

$$2p - 4 \leq \Delta(T) \leq 3p - 8.$$

Moreover, if a is any number in the indicated range, then there is a triangulation T of P with $\Delta(T) = a$ unless either

a) $a = 3p - 9$, or

b) P has an extreme point x such that $p - x$ has $p - 1$ extreme points. (In this case, $\Delta(T) = 3p - 8$ for every triangulation T of P .)

Proof: For case 2a, it was shown in [1] that a maximal planar graph on p points cannot contain exactly $3p - 9$ 3-cycles. Moreover, case 2b was covered in the preceding lemma. It only remains, therefore, to treat the nonexceptional cases.

Suppose first, therefore, that $h \geq 4$ as in case 1. Choose any $(3p - 2h - 2 - a)$ non-extreme points of P , and let P' denote these points together with the h extreme points of P . Triangulate P' without separating 3-cycles (this is possible by the lemma). At this stage, we have exactly $(2(3p - h - 2 - a) - h - 2)$ 3-cycles. Then recursively join each of the remaining $(a - 2p + h + 2)$ points of $P - P'$ by line segments to the three points of the triangle in which it occurs. The resulting triangulation of P contains $2(3p - h - 2 - a) - h - 2 + 3(a - 2p + h + 2) = a$ 3-cycles as desired.

Thus suppose $h = 3$ as in case 2, but $a \neq 3p - 9$. If $a = 3p - 8$, recursively draw line segments between each nonextreme point of P (taken in any order) and the three points of the triangle in which it occurs. If $a \leq 3p - 10$, let us suppose for the moment that we can choose a set S of three nonextreme points in P such that S , together

with the extreme points of P , comprise a set of six points not of the type excluded by case 2b. Then choose arbitrarily an additional $3p - 10 - a$ nonextreme points of P . These additional points, together with S and the extreme points of P , will form a set P' of $3p - 4 - a$ points which again are not of the type excluded by case 2b. We can, therefore, triangulate P' without separating 3-cycles; at this stage we have exactly $(2(3p - 4 - a) - 4)$ 3-cycles. Now recursively draw line segments between each of the remaining $(a - 2p + 4)$ points of $P - P'$ and the three points of the triangle in which it occurs to obtain a triangulation of P with exactly $2(3p - 4 - a) - 4 + 3(a - 2p + 4) = a$ 3-cycles.

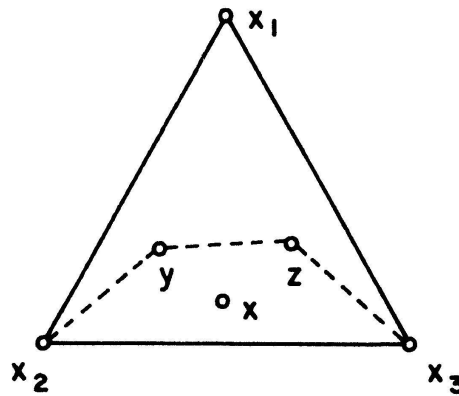


Figure 9

To complete the proof, we need to establish the existence of the set S . Let the extreme points of P be x_1, x_2, x_3 . Suppose that $P - x_i$ has three extreme points, say y_i, x_j and x_k , for each i . Then we can take $S = \{y_1, y_2, y_3\}$. Otherwise, suppose that say $P - x_1$ has at least four extreme points. Since $P - x_1$ does not have $p - 1$ extreme points, let x be a nonextreme point of $P - x_1$. Then it is easy to see there exist extreme points y, z of $P - x_1$ such that x is positioned in the interior of the convex quadrilateral $x_2 y z x_3$ (see fig. 9). We can then take $S = \{x, y, z\}$.

This completes the proof of the theorem.

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