

Zeitschrift: Elemente der Mathematik
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 34 (1979)
Heft: 6

Artikel: Diophantine representation of generalized Fibonacci numbers
Autor: Kiss, Péter
DOI: <https://doi.org/10.5169/seals-33810>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 29.04.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

ELEMENTE DER MATHEMATIK

Revue de mathématiques élémentaires – Rivista di matematica elementare

*Zeitschrift zur Pflege der Mathematik
und zur Förderung des mathematisch-physikalischen Unterrichts*

El. Math.

Band 34

Heft 6

Seiten 129–160

Basel, 10. November 1979

Diophantine representation of generalized Fibonacci numbers

Let A and B be integers such that either $A > 0$ and $B = -1$ or $A > 3$ and $B = 1$. We define a sequence $R = \{R_n\}_{n=0}^{\infty}$ by the integers $R_0 = 0$, $R_1 = 1$ and the recurrence

$$R_n = AR_{n-1} - BR_{n-2}, \quad n > 1.$$

When $A = -B = 1$ the positive terms of sequence R are the Fibonacci numbers; when $A = -B = 1$, $R_0 = 2$, $R_1 = 1$ they are the Lucas numbers.

Jones [1, 2] has proved that the set of all Fibonacci (respectively Lucas) numbers is identical with the set of positive values of the polynomial

$$y(2 - (y^2 - yx - x^2)^2),$$

respectively

$$y(1 - ((y^2 - yx - x^2)^2 - 25)^2),$$

as the variables x and y range over the positive integers.

The purpose of this paper is to extend the results of Jones on Fibonacci numbers to the generalized sequence. We prove two theorems and a lemma.

Theorem 1. For non-negative integers x, y

$$|x^2 - Axy + By^2| = 1 \tag{1}$$

if and only if x and y are consecutive terms of sequence R .

Theorem 2. The set of all terms of sequence R is identical with the set of all non-negative values of the polynomial

$$f(x, y) = y(2 - (x^2 - Axy + By^2)^2)$$

as the variables x and y range over the non-negative integers.

Lemma. For every non-negative integer n ,

$$|R_{n+1}^2 - AR_{n+1}R_n + BR_n^2| = 1.$$

Proof of the lemma: The lemma is obviously true for $n=0$. And by definition of sequence R ,

$$\begin{aligned} |R_{n+1}^2 - AR_{n+1}R_n + BR_n^2| &= |(AR_n - BR_{n-1})^2 - A(AR_n - BR_{n-1})R_n + BR_n^2| \\ &= |B(R_n^2 - AR_nR_{n-1} + BR_{n-1}^2)| = |R_n^2 - AR_nR_{n-1} + BR_{n-1}^2| \end{aligned}$$

for $n > 0$, since $|B| = 1$.

Proof of theorem 1: Equality (1) holds for $x=R_{n+1}$, $y=R_n$ by the lemma. Thus we must prove that if (1) holds for integers $x, y \geq 0$, then x and y are consecutive terms of sequence R .

Suppose that for integers $x_0, y_0 > 0$ we have

$$x_0^2 - Ax_0y_0 + By_0^2 = \varepsilon, \quad (2)$$

where $\varepsilon = 1$ or $\varepsilon = -1$. If $x_0, y_0 > 0$, we may assume $x_0 \geq y_0$. Indeed, for $B = 1$ condition (1) is symmetric in x and y ; and for $B = -1$, (2) gives

$$x_0^2 - y_0^2 \geq Ax_0y_0 - 1 \geq 0$$

(because $x_0y_0 \neq 0$).

Furthermore, if $x_0 \geq y_0 > 0$, then

$$x_0 = \frac{1}{2} (Ay_0 + \sqrt{A^2y_0^2 - 4By_0^2 + 4\varepsilon}).$$

For if

$$x_0 = \frac{1}{2} (Ay_0 - \sqrt{A^2y_0^2 - 4By_0^2 + 4\varepsilon})$$

and $y_0 > 0$, then in case $B = -1$, $A > 0$ we would get $x_0 \leq 0$; and in case $B = 1$, $A > 3$ the condition $x_0 \geq y_0$ would imply the inequality

$$Ay_0 - 2y_0 \geq \sqrt{A^2y_0^2 - 4y_0^2 + 4\varepsilon},$$

equivalent to the inequality

$$(2 - A)y_0^2 \geq \varepsilon,$$

which is impossible for $A > 3$, $y_0 > 0$ and $\varepsilon = \pm 1$.

It follows from (2) that the integers $x_1=y_0$, $y_1=AB y_0-Bx_0$ also satisfy equation (1) since $B^2=1$ gives

$$\begin{aligned} x_1^2 - Ax_1 y_1 + B y_1^2 &= y_0^2 - A y_0 (AB y_0 - B x_0) + B (AB y_0 - B x_0)^2 \\ &= B x_0^2 - AB x_0 y_0 + y_0^2 = B (x_0^2 - A x_0 y_0 + B y_0^2) = B \varepsilon. \end{aligned}$$

But if $x_0 > 0$ and $y_0 > 0$, then $x_1 > 0$ and

$$\begin{aligned} y_1 &= AB y_0 - B x_0 = AB y_0 - \frac{B}{2} (A y_0 + \sqrt{A^2 y_0^2 - 4 B y_0^2 + 4 \varepsilon}) \\ &= \frac{B}{2} (A y_0 - \sqrt{A^2 y_0^2 - 4 B y_0^2 + 4 \varepsilon}) \geq 0 \end{aligned} \tag{3}$$

($y_1=0$ only if $y_0=1$).

We show next that $y_1 < y_0$, except perhaps when $A = -B = 1$, $y_0 = 1$. In case $B = 1$, $A > 3$, using the form of y_1 given in (3), we get

$$y_1 = \frac{1}{2} (A y_0 - \sqrt{A^2 y_0^2 - 4 y_0^2 + 4 \varepsilon}) < \frac{1}{2} (A y_0 - \sqrt{A^2 y_0^2 - 4 A y_0^2 + 4 y_0^2}) = y_0$$

since $A^2 y_0^2 - 4 y_0^2 + 4 \varepsilon > A^2 y_0^2 - 4 A y_0^2 + 4 y_0^2$, i.e. $(A-2)y_0^2 > -\varepsilon$, clearly holds for $y_0 > 0$. And if $B = -1$, then

$$y_1 = \frac{1}{2} (\sqrt{A^2 y_0^2 + 4 y_0^2 + 4 \varepsilon} - A y_0) < \frac{1}{2} ((A+2)y_0 - A y_0) = y_0$$

since $4 \varepsilon < 4 A y_0^2$, except perhaps if $A = 1$, $y_0 = 1$.

Continuing this procedure we construct the strictly decreasing sequences y_0, y_1, y_2, \dots and x_0, x_1, x_2, \dots , where

$$x_i = y_{i-1} \quad \text{and} \quad y_i = AB y_{i-1} - B x_{i-1} \quad \text{for } i > 0 \tag{4}$$

and $x_i > y_i \geq 0$, if $y_{i-1} > 0$ (except perhaps if $y_{i-1} = 1$ in case $A = -B = 1$). Furthermore equality (1) holds for $x = x_i, y = y_i$.

The construction comes to an end when an index j is reached such that $y_j = 0$ (or $y_j = 1$ in case $A = -B = 1$). If $y_j = 0$, then $x_j = 1$, so that $y_j = R_0$ and $x_j = R_1$. But by (4) we can show that if $y_i = R_k$ and $x_i = R_{k+1}$ for some indices i and k , then $y_{i-1} = R_{k+1}$ and $x_{i-1} = A y_{i-1} - B y_i = A R_{k+1} - B R_k = R_{k+2}$ (since $B^2 = 1$); this shows that y_0, x_0 are also consecutive terms of sequence R . If $A = -B = 1$ and $y_j = 1$ for some index j , then $x_j = 2, 1$ or 0 . But $(y_j, x_j) = (1, 2) = (R_2, R_3)$, $(y_j, x_j) = (1, 1) = (R_1, R_2)$ and $y_j = 1, x_j = 0$ imply that $y_{j-1} = 0 = R_0, x_{j-1} = 1 = R_1$; therefore we get as above that y_0, x_0 are also consecutive terms of sequence R .

This completes the proof of theorem 1.

Proof of theorem 2: Because of the conditions imposed on A and B , we have

$$x^2 - Axy + By^2 = 0$$

for integers x and y if and only if $x = y = 0$. Therefore by theorem 1 for non-negative integers x, y , we have $f(x, y) = 0$ if and only if $y = 0$, $f(x, y) = y > 0$ if and only if x and $y > 0$ are two consecutive terms of the sequence R , and $f(x, y) < 0$ in any other cases.

Remark: One can easily see that theorem 1 is valid for cases $A = 1, B = 1$ and $A = 2, B = 1$, but sequence R is degenerate in these cases. In case $A = 3, B = 1$ theorem 1 is false since $x = 2, y = 1$ is a solution of equation (1) and 2 is not a term of sequence R .

Péter Kiss, Teachers' Training College, Eger, Hungary

REFERENCES

- 1 James P. Jones: Diophantine representation of the Fibonacci numbers. *Fibonacci Quart.* 13, 84–88 (1975).
- 2 James P. Jones: Diophantine representation of the Lucas numbers. *Fibonacci Quart.* 14, 134 (1976).

Kleine Mitteilungen

Eine merkwürdige Familie von beweglichen Stabwerken

1. Sei $ABA'B'$ ein *gelenkiges Antiparallelogramm* mit den Seitenlängen $AB = A'B' = a$ und $AB' = A'B = d > a$. Wird es in seiner Ebene so bewegt, dass der Schnittpunkt O der Langseiten und die Symmetrieachse z festbleiben (Fig. 1), dann rollt bekanntlich eine Ellipse e mit den Brennpunkten A, B und der Hauptachse d auf einer kongruenten Ellipse e' mit den Brennpunkten A', B' gleitungslos ab, wie die Betrachtung des gemeinsamen Linienelements (O, z) lehrt; diese Tatsache bildet die kinematische Grundlage für elliptische Zahnräder [2]. Alle vier Gelenke des Antiparallelogramms wandern dabei auf einer gemeinsamen, aus zwei kongruenten Ovalen bestehenden Bahnkurve 6. Ordnung, wie in [5] gezeigt wurde.

Bezeichnet $r = OA$ den Radiusvektor des Punktes A und ψ den Richtungswinkel, gemessen von der zur z -Achse normalen x -Achse aus, so hat A die kartesischen Koordinaten $x = r \cos \psi, z = r \sin \psi$ und B die Koordinaten $\bar{x} = (d - r) \cos \psi, \bar{z} = (r - d) \sin \psi$. Die auf $AB = a$ bezügliche Distanzformel liefert dann für die *Bahnsextik* k die Polargleichung

$$r(d - r) \cos^2 \psi = m^2 \quad \text{mit} \quad 4m^2 = d^2 - a^2, \quad (1)$$

welche auf die kartesische Gleichung

$$(x^2 + z^2)(x^2 + m^2)^2 = d^2 x^4 \quad (2)$$