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# ELEMENTE DER MATHEMATIK

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## Diophantine representation of generalized Fibonacci numbers

Let  $A$  and  $B$  be integers such that either  $A > 0$  and  $B = -1$  or  $A > 3$  and  $B = 1$ . We define a sequence  $R = \{R_n\}_{n=0}^{\infty}$  by the integers  $R_0 = 0$ ,  $R_1 = 1$  and the recurrence

$$R_n = AR_{n-1} - BR_{n-2}, \quad n > 1.$$

When  $A = -B = 1$  the positive terms of sequence  $R$  are the Fibonacci numbers; when  $A = -B = 1$ ,  $R_0 = 2$ ,  $R_1 = 1$  they are the Lucas numbers.

Jones [1, 2] has proved that the set of all Fibonacci (respectively Lucas) numbers is identical with the set of positive values of the polynomial

$$y(2 - (y^2 - yx - x^2)^2),$$

respectively

$$y(1 - ((y^2 - yx - x^2)^2 - 25)^2),$$

as the variables  $x$  and  $y$  range over the positive integers.

The purpose of this paper is to extend the results of Jones on Fibonacci numbers to the generalized sequence. We prove two theorems and a lemma.

**Theorem 1.** *For non-negative integers  $x, y$*

$$|x^2 - Axy + By^2| = 1 \tag{1}$$

*if and only if  $x$  and  $y$  are consecutive terms of sequence  $R$ .*

**Theorem 2.** *The set of all terms of sequence  $R$  is identical with the set of all non-negative values of the polynomial*

$$f(x, y) = y(2 - (x^2 - Axy + By^2)^2)$$

*as the variables  $x$  and  $y$  range over the non-negative integers.*

**Lemma.** *For every non-negative integer  $n$ ,*

$$|R_{n+1}^2 - AR_{n+1}R_n + BR_n^2| = 1.$$

Proof of the lemma: The lemma is obviously true for  $n=0$ . And by definition of sequence  $R$ ,

$$\begin{aligned} |R_{n+1}^2 - AR_{n+1}R_n + BR_n^2| &= |(AR_n - BR_{n-1})^2 - A(AR_n - BR_{n-1})R_n + BR_n^2| \\ &= |B(R_n^2 - AR_nR_{n-1} + BR_{n-1}^2)| = |R_n^2 - AR_nR_{n-1} + BR_{n-1}^2| \end{aligned}$$

for  $n > 0$ , since  $|B| = 1$ .

Proof of theorem 1: Equality (1) holds for  $x = R_{n+1}$ ,  $y = R_n$  by the lemma. Thus we must prove that if (1) holds for integers  $x, y \geq 0$ , then  $x$  and  $y$  are consecutive terms of sequence  $R$ .

Suppose that for integers  $x_0, y_0 > 0$  we have

$$x_0^2 - Ax_0y_0 + By_0^2 = \varepsilon, \quad (2)$$

where  $\varepsilon = 1$  or  $\varepsilon = -1$ . If  $x_0, y_0 > 0$ , we may assume  $x_0 \geq y_0$ . Indeed, for  $B = 1$  condition (1) is symmetric in  $x$  and  $y$ ; and for  $B = -1$ , (2) gives

$$x_0^2 - y_0^2 \geq Ax_0y_0 - 1 \geq 0$$

(because  $x_0y_0 \neq 0$ ).

Furthermore, if  $x_0 \geq y_0 > 0$ , then

$$x_0 = \frac{1}{2}(Ay_0 + \sqrt{A^2y_0^2 - 4By_0^2 + 4\varepsilon}).$$

For if

$$x_0 = \frac{1}{2}(Ay_0 - \sqrt{A^2y_0^2 - 4By_0^2 + 4\varepsilon})$$

and  $y_0 > 0$ , then in case  $B = -1$ ,  $A > 0$  we would get  $x_0 \leq 0$ ; and in case  $B = 1$ ,  $A > 3$  the condition  $x_0 \geq y_0$  would imply the inequality

$$Ay_0 - 2y_0 \geq \sqrt{A^2y_0^2 - 4y_0^2 + 4\varepsilon},$$

equivalent to the inequality

$$(2-A)y_0^2 \geq \varepsilon,$$

which is impossible for  $A > 3$ ,  $y_0 > 0$  and  $\varepsilon = \pm 1$ .

It follows from (2) that the integers  $x_1=y_0$ ,  $y_1=ABy_0-Bx_0$  also satisfy equation (1) since  $B^2=1$  gives

$$\begin{aligned} x_1^2 - Ax_1y_1 + By_1^2 &= y_0^2 - Ay_0(ABy_0 - Bx_0) + B(ABy_0 - Bx_0)^2 \\ &= Bx_0^2 - ABx_0y_0 + y_0^2 = B(x_0^2 - Ax_0y_0 + By_0^2) = Be. \end{aligned}$$

But if  $x_0 > 0$  and  $y_0 > 0$ , then  $x_1 > 0$  and

$$\begin{aligned} y_1 &= ABy_0 - Bx_0 = ABy_0 - \frac{B}{2}(Ay_0 + \sqrt{A^2y_0^2 - 4By_0^2 + 4\varepsilon}) \\ &= \frac{B}{2}(Ay_0 - \sqrt{A^2y_0^2 - 4By_0^2 + 4\varepsilon}) \geq 0 \end{aligned} \tag{3}$$

( $y_1 = 0$  only if  $y_0 = 1$ ).

We show next that  $y_1 < y_0$ , except perhaps when  $A = -B = 1$ ,  $y_0 = 1$ . In case  $B = 1$ ,  $A > 3$ , using the form of  $y_1$  given in (3), we get

$$\begin{aligned} y_1 &= \frac{1}{2}(Ay_0 - \sqrt{A^2y_0^2 - 4y_0^2 + 4\varepsilon}) < \frac{1}{2}(Ay_0 - \sqrt{A^2y_0^2 - 4Ay_0^2 + 4y_0^2}) = y_0 \\ \text{since } A^2y_0^2 - 4y_0^2 + 4\varepsilon &> A^2y_0^2 - 4Ay_0^2 + 4y_0^2, \text{ i.e. } (A-2)y_0^2 > -\varepsilon, \text{ clearly holds for } y_0 > 0. \text{ And if } B = -1, \text{ then} \end{aligned}$$

$$y_1 = \frac{1}{2}(\sqrt{A^2y_0^2 + 4y_0^2 + 4\varepsilon} - Ay_0) < \frac{1}{2}((A+2)y_0 - Ay_0) = y_0$$

since  $4\varepsilon < 4Ay_0^2$ , except perhaps if  $A = 1$ ,  $y_0 = 1$ .

Continuing this procedure we construct the strictly decreasing sequences  $y_0, y_1, y_2, \dots$  and  $x_0, x_1, x_2, \dots$ , where

$$x_i = y_{i-1} \quad \text{and} \quad y_i = ABy_{i-1} - Bx_{i-1} \quad \text{for } i > 0 \tag{4}$$

and  $x_i > y_i \geq 0$ , if  $y_{i-1} > 0$  (except perhaps if  $y_{i-1} = 1$  in case  $A = -B = 1$ ). Furthermore equality (1) holds for  $x = x_i$ ,  $y = y_i$ .

The construction comes to an end when an index  $j$  is reached such that  $y_j = 0$  (or  $y_j = 1$  in case  $A = -B = 1$ ). If  $y_j = 0$ , then  $x_j = 1$ , so that  $y_j = R_0$  and  $x_j = R_1$ . But by (4) we can show that if  $y_i = R_k$  and  $x_i = R_{k+1}$  for some indices  $i$  and  $k$ , then  $y_{i-1} = R_{k+1}$  and  $x_{i-1} = Ay_{i-1} - By_i = AR_{k+1} - BR_k = R_{k+2}$  (since  $B^2 = 1$ ); this shows that  $y_0, x_0$  are also consecutive terms of sequence  $R$ . If  $A = -B = 1$  and  $y_j = 1$  for some index  $j$ , then  $x_j = 2, 1$  or  $0$ . But  $(y_j, x_j) = (1, 2) = (R_2, R_3)$ ,  $(y_j, x_j) = (1, 1) = (R_1, R_2)$  and  $y_j = 1, x_j = 0$  imply that  $y_{j-1} = 0 = R_0$ ,  $x_{j-1} = 1 = R_1$ ; therefore we get as above that  $y_0, x_0$  are also consecutive terms of sequence  $R$ .

This completes the proof of theorem 1.

**Proof of theorem 2:** Because of the conditions imposed on  $A$  and  $B$ , we have

$$x^2 - Axy + By^2 = 0$$

for integers  $x$  and  $y$  if and only if  $x = y = 0$ . Therefore by theorem 1 for non-negative integers  $x, y$ , we have  $f(x, y) = 0$  if and only if  $y = 0$ ,  $f(x, y) = y > 0$  if and only if  $x$  and  $y > 0$  are two consecutive terms of the sequence  $R$ , and  $f(x, y) < 0$  in any other cases.

**Remark:** One can easily see that theorem 1 is valid for cases  $A = 1, B = 1$  and  $A = 2, B = 1$ , but sequence  $R$  is degenerate in these cases. In case  $A = 3, B = 1$  theorem 1 is false since  $x = 2, y = 1$  is a solution of equation (1) and 2 is not a term of sequence  $R$ .

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## REFERENCES

- 1 James P. Jones: Diophantine representation of the Fibonacci numbers. *Fibonacci Quart.* **13**, 84–88 (1975).
- 2 James P. Jones: Diophantine representation of the Lucas numbers. *Fibonacci Quart.* **14**, 134 (1976).

# Kleine Mitteilungen

## Eine merkwürdige Familie von beweglichen Stabwerken

1. Sei  $ABA'B'$  ein *gelenkiges Antiparallelogramm* mit den Seitenlängen  $AB = A'B' = a$  und  $AB' = A'B = d > a$ . Wird es in seiner Ebene so bewegt, dass der Schnittpunkt  $O$  der Langseiten und die Symmetriearchse  $z$  festbleiben (Fig. 1), dann rollt bekanntlich eine Ellipse  $e$  mit den Brennpunkten  $A, B$  und der Hauptachse  $d$  auf einer kongruenten Ellipse  $e'$  mit den Brennpunkten  $A', B'$  gleitungslos ab, wie die Betrachtung des gemeinsamen Linienelements  $(O, z)$  lehrt; diese Tatsache bildet die kinematische Grundlage für elliptische Zahnräder [2]. Alle vier Gelenke des Antiparallelogramms wandern dabei auf einer gemeinsamen, aus zwei kongruenten Ovalen bestehenden Bahnkurve 6. Ordnung, wie in [5] gezeigt wurde.

Bezeichnet  $r = OA$  den Radiusvektor des Punktes  $A$  und  $\psi$  den Richtungswinkel, gemessen von der zur  $z$ -Achse normalen  $x$ -Achse aus, so hat  $A$  die kartesischen Koordinaten  $x = r \cos \psi, z = r \sin \psi$  und  $B$  die Koordinaten  $\bar{x} = (d - r) \cos \psi, \bar{z} = (r - d) \sin \psi$ . Die auf  $AB = a$  bezügliche Distanzformel liefert dann für die *Bahnsextik*  $k$  die Polargleichung

$$r(d - r) \cos^2 \psi = m^2 \quad \text{mit} \quad 4m^2 = d^2 - a^2, \quad (1)$$

welche auf die kartesische Gleichung

$$(x^2 + z^2)(x^2 + m^2)^2 = d^2 x^4 \quad (2)$$