

Zeitschrift: Elemente der Mathematik
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 34 (1979)
Heft: 5

Artikel: On the diophantine equations $2y^2 = 7k + 1$ and $x^2 + 11 = 3n$
Autor: Inkeri, K.
DOI: <https://doi.org/10.5169/seals-33809>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 10.04.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

$$x = \left(\frac{r}{2} - \rho\right) \cos \omega + \frac{3r}{2} \cos \frac{\omega}{3}, \quad y = \left(\frac{r}{2} - \rho\right) \sin \omega + \frac{3r}{2} \sin \frac{\omega}{3} \quad (23)$$

beschrieben, die als Einhüllende der Spitzentangenten auftretende Astroide (vier-spitzige Hypozykloide) durch

$$x = -\frac{m}{2} \left(\cos \omega + 3 \cos \frac{\omega}{3}\right), \quad y = -\frac{m}{2} \left(\sin \omega - 3 \sin \frac{\omega}{3}\right) \quad \text{mit} \quad m = \frac{r}{2} - \rho. \quad (24)$$

Satz 3 steht überdies in gewissem Zusammenhang mit Untersuchungen von Fréchet [3], welche die Beweglichkeit einer starren Ellipse in einer sie dreifach berührenden Steiner-Zykloide erkennen lassen und die von Wunderlich [6], Meyer [5] und anderen verallgemeinert wurden. Ernst Ungethüm, Wien

LITERATURVERZEICHNIS

- 1 G. Berkhan und W.F. Meyer: Neuere Dreiecksgeometrie. Enzykl. Math. Wiss. IIIAB10.
- 2 F. Dingeldey: Kegelschnitte und Kegelschnittssysteme. Enzykl. Math. Wiss. IIIC1.
- 3 M. Fréchet: Sur quelques propriétés de l'hypocycloïde à trois rebroussements. Nouv. Ann. 61, 206-217 (1902).
- 4 R. Henke und R. Heger: Schloemilchs Handbuch der Mathematik, I., 2. Aufl. Leipzig 1904.
- 5 P. Meyer: Über Hüllkurven von Radlinien. Arch. Math. 18, 651-662 (1967).
- 6 W. Wunderlich: Über Gleitkurvenpaare aus Radlinien. Math. Nachr. 20, 373-380 (1959).
- 7 W. Wunderlich: Ebene Kinematik. Hochschultaschenbücher, Bd. 447. Mannheim, Wien, Zürich 1970.

On the diophantine equations $2y^2 = 7^k + 1$ and $x^2 + 11 = 3^n$

Certain diophantine equations have played an important role in the recent development of the mathematical theory of error-correcting codes. One of these equations is

$$2y^2 = 7^k + 1, \quad (1)$$

for which Alter [1] has proved the following result.

Theorem 1. *The only solutions in positive integers y, k of the equation (1) are (2, 1) and (5, 2).*

Alter's proof, which is based on the theory of continued fractions, is quite long and complicated. Therefore, it may be of some interest that theorem 1 can be

demonstrated very briefly by appealing to the following theorem of Ljunggren's [8].

Theorem 2. *The diophantine equation*

$$\frac{x^n - 1}{x - 1} = y^2 \quad (n > 2)$$

is impossible in integers $x, y, |x| > 1$ with the exception of the cases $n=4, x=7$ and $n=5, x=3$.

This theorem is based on a fairly comprehensive theory which is, however, completely elementary.

In order to prove theorem 1 it is enough to show that (1) is impossible for $k > 2$. Suppose that (1) holds for some integers $y > 0, k > 2$. We distinguish three cases.

If k is odd, then $2|y$, i.e. $y=2y_1$ with a positive integer y_1 . Now (1) can be written in the form

$$\frac{(-7)^k - 1}{-7 - 1} = y_1^2.$$

By theorem 2 this is impossible.

If $k=2(2h+1)$ with h an integer > 0 , then $(7^2+1)|(7^k+1)$. Hence $5|y$, i.e. $y=5y_2$, and (1) takes the form

$$\frac{(-49)^{2h+1} - 1}{-49 - 1} = y_2^2.$$

Again, by Ljunggren's theorem, this is impossible because of $h > 0$. Lastly, let $k=4h$. Now (1) becomes $(7^h)^4 + 1 = 2y^2$. This equation is impossible, since $h > 0$ and the only integer solutions of $x^4 + y^4 = 2z^2$ with $(x, y) = 1$ are given by $x^2 = y^2 = 1$ (cf. e.g. [9], p. 18). This completes the proof of theorem 1.

We shall now deal with the equation

$$x^2 + 11 = 3^n, \tag{2}$$

for which the following result holds.

Theorem 3. *The only positive integer solution of equation (2) is given by $(x, n) = (4, 3)$.*

It seems that the first proof for this theorem is given by Alter and Kubota [2]. [An error in the proof of the case $n \equiv 7 \pmod{10}$ was later corrected by the second author [5]. By the way, we may note that the correction can be demonstrated very briefly as follows: Since $3|n$, we have $n \equiv -3 \pmod{30}$ and so, by (2) and Fermat's theorem, $3^3 x^2 \equiv 3^{n+3} + 4 \cdot 11 \equiv -48 \pmod{31}$ or $(3x)^2 + 4^2 \equiv 0 \pmod{31}$, which is impossible, since $(-1/31) = -1$.]

Very recently Cohen [4] has proved theorem 3 using the interesting method developed by Hasse [6]. However, an earlier proof given by Cohen and Ljunggren [3] is much simpler. Our following proof is slightly dissimilar to the latter.

Firstly we show that $3|n$. The equation (2) can be written in the forms

$$x^2+8=3(3^{n-1}-1), \quad x^2+2=9(3^{n-2}-1).$$

The right-hand side is divisible by $3^3-1 (=2 \cdot 13)$ in the first equation for $n \equiv 1 \pmod{3}$ and in the second one for $n \equiv 2 \pmod{3}$. But $(-2/13) = -1$ and thus the cases $n \equiv 1, 2 \pmod{3}$ are excluded. Now (2) has the form $x^2+11=y^3$. We do not wish to make use of the well-known result [7] that the only solutions in positive integers of this equation are given by (4, 3) and (58, 15), but we carry through the proof completely.

In the quadratic field $\mathcal{Q}(\sqrt{-11})$ unique factorization holds, the only units are ± 1 , and $\text{g.c.d.}(x+\sqrt{-11}, x-\sqrt{-11})=1$, whence

$$x+\sqrt{-11}=\left(\frac{a+b\sqrt{-11}}{2}\right)^3, \quad 4y=a^2+11b^2, \quad (3)$$

where a, b are rational integers. From these equations it follows that

$$3a^2b-11b^3=8, \quad a^2b-by=2.$$

Hence $|b|=1$ or 2 and so $a^2=1$ or 16 , respectively. The second equation in (3) then gives $y=3, x=4$ and $y=15, x=58$ as only solutions of $x^2+11=y^3$. Since y is a power of 3 in the case we are considering, it follows that $x=4, n=3$ is the only solution of (2).

K. Inkeri, University of Turku, Finland

REFERENCES

- 1 R. Alter: On the nonexistence of close-packed double Hamming error-correcting codes on $q=7$ symbols. *J. Comput. Syst. Sci.* 2, 169-176 (1968).
- 2 R. Alter and K.K. Kubota: The diophantine equation $x^2+11=3^n$ and a related sequence. *J. Number Theory* 7, 5-10 (1975).
- 3 E.L. Cohen: Sur l'équation diophantienne $x^2+11=3^k$. *C.r. Acad. Sci. Paris (A)* 275, 5-7 (1972).
- 4 E.L. Cohen: The diophantine equation $x^2+11=3^k$ and related questions. *Math. Scand.* 38, 240-246 (1976).
- 5 E.L. Cohen: Review of [2], MR 51, No. 344, p. 48 (1976).
- 6 H. Hasse: Über eine diophantische Gleichung von Ramanujan-Nagell und ihre Verallgemeinerung. *Nagoya Math. J.* 27, 77-102 (1966).
- 7 O. Hemer: On the diophantine equation $y^2-k=x^3$. *Doct. Diss.*, Uppsala 1952.
- 8 W. Ljunggren: Noen setninger om ubestemte likninger av formen $(x^n-1)/(x-1)=y^q$. *Norsk. Mat. Tidsskr. I*, H. 25, 17-20 (1943).
- 9 L.J. Mordell: *Diophantine equations*. Academic Press, New York, London 1969.