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## Kleine Mitteilungen

### On the Indirect Product of Subgroups

The following is a well-known property of external direct products<sup>1)</sup>. If  $G$  is the external direct product of groups  $G_1, G_2, \dots, G_n$ , and  $H$  is a group such that  $H = \bigtimes_{i=1}^k G_i$  for some  $k < n$ , then  $G$  is, up to isomorphism, the external direct product of  $H, G_{k+1}, \dots, G_n$ .

Recall that a group  $G$  is said to be the internal direct product of subgroups  $H_1, H_2, \dots, H_n$  if and only if each  $H_i$  is normal and each element  $g$  of  $G$  can be uniquely expressed (excluding order) in the form  $g = h_1 h_2 \dots h_n$ , where  $h_i \in H_i$  for each  $i$ .

It is natural to ask the following question: If  $G$  is the internal direct product of subgroups  $H_1, H_2, \dots, H_n$ , what conditions can be placed on a subgroup  $H$  so that  $G$  is also the internal direct product of  $H, H_{k+1}, \dots, H_n$  for some  $k < n$ ? I have been unable to find a text on group theory which considers this question.

For convenience let us first recall some basic results concerning direct products. A well-known characterization is that a group  $G$  is the internal direct product of subgroups  $H_1, H_2, \dots, H_n$  if and only if

each  $H_i$  is normal, (1)

$G = \prod_{i=1}^n H_i$ , and (2)

$H_j \cap \prod_{i \neq j} H_i = \{e\}$  for each  $j$ . (3)

Another characterization is that  $G$  is the internal direct product if and only if  $\Phi: \bigtimes_{i=1}^n H_i \rightarrow G$ , defined by  $\Phi(h_1, h_2, \dots, h_n) = h_1 h_2 \dots h_n$ , is an isomorphism.

Returning to the question, one might first guess that if  $H \simeq \prod_{i=1}^k H_i$  for some  $k < n$  ( $\prod_{i=1}^k H_i$  is a subgroup since each  $H_i$  is normal and  $H_i \cap \prod_{i \neq j} H_j = \{e\}$ ), then  $G$  will be the internal direct product of  $H, H_{k+1}, \dots, H_n$  since  $G \simeq \bigtimes_{i=1}^n H_i, H \simeq \bigtimes_{i=1}^k H_i$ , and  $G$  will thus be isomorphic to the external direct product of  $H, H_{k+1}, \dots, H_n$ . However, the special map  $\Phi$  may not be an isomorphism. Consider  $Z_2 \times Z_2$ , and let  $H_1 = Z_2 \times \{[0]\}, H_2 = \{[0]\} \times Z_2$ , and  $H = H_2$ . Then  $Z_2 \times Z_2$  is the internal direct product of  $H_1$  and  $H_2$ , and  $H \simeq H_1$ , but even though  $Z_2 \times Z_2 \simeq H \times H_2$ ,  $Z_2 \times Z_2$  is not the internal direct product of  $H$  and  $H_2$  since  $H + H_2 \neq Z_2 \times Z_2$ .

This example also shows that requiring  $H$  to be normal in  $G$ , in addition to

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<sup>1)</sup> External direct product means the same as "ordinary" direct product as indicated by the symbol

$\bigtimes_{i=1}^n G_i$ .

being isomorphic to  $\prod_{i=1}^k H_i$ , does not suffice. But in the example just cited, note that

$H \cap H_2 \neq \{([0], [0])\}$ . Thus, in addition to supposing  $H$  to be isomorphic to  $\prod_{i=1}^k H_i$ , let

us assume that  $H \cap \prod_{i=k+1}^n H_i = \{e\}$ . But consider  $Z_2 \times S_3$ , and let  $H_1 = Z_2 \times \{\varepsilon\}$ ,

$H_2 = \{[0]\} \times S_3$ , and  $H = \{([0], \varepsilon), ([1], a)\}$ , where  $\varepsilon = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$  and  $a = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$ .

Then  $Z_2 \times S_3$  is the internal direct product of  $H_1$  and  $H_2$ ,  $H \simeq H_1$ , and  $H \cap H_2 = \{([0], \varepsilon)\}$ , but  $Z_2 \times S_3$  is not the internal direct product of  $H$  and  $H_2$  since  $H$  is not normal.

If we restrict ourselves to finite groups, we have the following.

**Theorem:** Let  $G$  be a finite group such that  $G$  is the internal direct product of subgroups  $H_1, H_2, \dots, H_n$ . If  $H$  is a normal subgroup such that  $H \simeq \prod_{i=1}^k H_i$  for some

$k < n$  and  $H \cap \prod_{i=k+1}^n H_i = \{e\}$ , then  $G$  is the internal direct product of  $H, H_{k+1}, \dots, H_n$ .

*Proof:* It suffices to show that  $\Phi: H \times H_{k+1} \times \dots \times H_n \rightarrow G$ , defined by  $\Phi(h, h_{k+1}, \dots, h_n) = hh_{k+1} \dots h_n$ , is an isomorphism.

If  $K_1$  and  $K_2$  are normal subgroups and  $K_1 \cap K_2 = \{e\}$ , then the elements of  $K_1$  commute with those of  $K_2$ . Using this, the proof that  $\Phi$  is a homomorphism is straightforward.

If  $\Phi(a, a_{k+1}, \dots, a_n) = \Phi(b, b_{k+1}, \dots, b_n)$ , then  $aa_{k+1} \dots a_n = bb_{k+1} \dots b_n$ . Hence  $b^{-1}a = (b_{k+1} \dots b_n)(a_{k+1} \dots a_n)^{-1}$ . Thus  $b^{-1}a \in H \cap \prod_{i=k+1}^n H_i$ , and  $a = b$ .

Continuing in the same manner,  $a_i = b_i$  for  $i = k+1, \dots, n$ . Thus  $\Phi$  is one-to-one.

Since  $H \simeq \prod_{i=1}^k H_i$ , the orders of  $H \times H_{k+1} \times \dots \times H_n$  and of  $G$  are equal. Thus it follows that  $\Phi$  is also onto. Therefore,  $\Phi$  is an isomorphism.

I have answered the question for finite groups only. If  $G$  is infinite, the same conditions imposed on  $H$  are not sufficient to obtain the same result. Consider  $Z \times Z$ , and let  $H_1 = Z \times \{0\}$ ,  $H_2 = \{0\} \times Z$ , and  $H$  be the subgroup generated by  $(2, 0)$ . Then  $Z \times Z$  is the internal direct product of  $H_1$  and  $H_2$ ,  $H$  is a normal subgroup such that  $H \simeq H_1$  and  $H \cap H_2 = \{(0, 0)\}$ , but  $Z \times Z$  is not the internal direct product of  $H$  and  $H_2$  since  $H + H_2 \neq Z \times Z$ .

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