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## *QR* in Two Dimensions

The *QR* process is one of the most efficient ways of determining the characteristic values of a matrix. It is a unitary analog of the *LR* process of RUTISHAUSER [1]. However even the best proofs available are unfit for beginners' consumption and the later developments of the process are not yet fully understood. We present here a discussion of the two-dimensional case, in its simplest form. The formal description of the process will be given in the  $n$ -dimensional case.

Let  $A$  be a complex  $n \times n$  matrix. It is well-known that  $A$  can be written in the form

$$A = QR \tag{1}$$

where  $Q$  is unitary and  $R$  upper triangular. This is essentially the result of the Gram-Schmidt orthogonalization process. Moreover, if we require the diagonal elements of  $R$  to be positive, then the representation (1) is unique.

The *QR*-algorithm consists in deriving sequences of matrices  $\{A_n\}$ ,  $\{Q_n\}$ ,  $\{R_n\}$  from  $A = A_1$  by repeated use of (1). Given  $A_n = Q_n R_n$  we form the reversed product  $A_{n+1} = R_n Q_n$  and factorize this as  $A_{n+1} = Q_{n+1} R_{n+1}$ . Since

$$A_{n+1} = R_n Q_n = (Q_n^* Q_n) R_n Q_n = Q_n^* (Q_n R_n) Q_n = Q_n^* A_n Q_n \tag{2}$$

the matrices  $\{A_n\}$  are all (unitarily) similar to  $A$  and have the same characteristic values as  $A$ . The basic fact is that, in certain circumstances, the sequence  $\{A_n\}$  converges geometrically to an upper triangular matrix, which has the characteristic values of  $A$  on the diagonal. For discussions of this result see the original papers of H. RUTISHAUSER [1], J. G. F. FRANCIS [2], V. N. KUBLANOWSKAJA [3] and more recent work of B. PARLETT [4, 5, 6], A. S. HOUSEHOLDER [7], G. W. STEWART [8] and J. H. WILKINSON [9].

In practice, appropriate “shifts” are introduced and quadratic convergence can be obtained.

For simplicity we discuss the real two dimensional case. Let

$$A = \begin{bmatrix} a & \beta \\ \gamma & \delta \end{bmatrix}$$

be real and unimodular,  $\det A = 1$ . We compute the  $QR$  decomposition. If, where  $c = \cos \theta$ ,  $s = \sin \theta$ ,

$$A = A_1 = \begin{bmatrix} a & \beta \\ \gamma & \delta \end{bmatrix} = \begin{bmatrix} c & s \\ s & -c \end{bmatrix} \begin{bmatrix} \xi & \eta \\ 0 & \zeta \end{bmatrix}$$

then

$$c = a(a^2 + \gamma^2)^{-1/2}, s = \gamma(a^2 + \gamma^2)^{-1/2}.$$

Next

$$A_2 = \begin{bmatrix} c & s \\ s & -c \end{bmatrix} \begin{bmatrix} a & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} c & s \\ s & -c \end{bmatrix} = \begin{bmatrix} (a + \delta - (a/(a^2 + \gamma^2))) & \gamma - \beta - (\gamma/(a^2 + \gamma^2)) \\ -\gamma/(a^2 + \gamma^2) & a/(a^2 + \gamma^2) \end{bmatrix}.$$

If we write

$$A_n = \begin{bmatrix} a_n & \beta_n \\ \gamma_n & \delta_n \end{bmatrix}$$

then the recurrence relations determining  $a_{n+1}, \gamma_{n+1}$  are

$$\begin{cases} a_{n+1} = (a_1 + \delta_1) - (a_n/(a_n^2 + \gamma_n^2)), \\ \gamma_{n+1} = -\gamma_n/(a_n^2 + \gamma_n^2), \end{cases} \quad n = 1, 2, \dots \quad (3)$$

What we have to prove from (3) is that, in certain circumstances

$$a_n \rightarrow \lambda, \gamma_n \rightarrow 0$$

where  $\lambda$  is an appropriate characteristic value of  $A$ . The solution to non-linear systems of difference equations such as (3) is not usually easy.

We assume that  $A$  has a dominant characteristic value  $\lambda$ . This means that  $A$  has distinct real characteristic values which are reciprocal, since  $A$  is unimodular. We may assume that  $\lambda$  is positive for otherwise we could deal with  $-A$ . Hence

$$\lambda + \lambda^{-1} = a + \delta = k, \text{ say, where } k > 2.$$

We discuss an example first. We normalize the matrix

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

to a unimodular

$$A = A_1 = \begin{bmatrix} 2/\sqrt{3} & -1/\sqrt{3} \\ -1/\sqrt{3} & 2/\sqrt{3} \end{bmatrix},$$

with characteristic roots  $\sqrt{3}, 1/\sqrt{3}$ .

Application of the relations (3) gives

	$n=1$	$n=2$	$n=3$	$n=4$
$a_n/\sqrt{3} =$	2/3	14/15	122/123	1094/1095
$\gamma_n/\sqrt{3} =$	-1/3	1/5	-3/41	9/365

which indicates that  $a_n \rightarrow \sqrt{3}, \gamma_n \rightarrow 0$ . The general form of  $a_n, \gamma_n$  can be conjectured from the above table and established by induction. We find

$$\frac{a_n}{\sqrt{3}} = \frac{3 \times 9^{n-1} + 1}{3(9^{n-1} + 1)}, \quad \frac{(-1)^n \gamma_n}{\sqrt{3}} = \frac{2 \times 3^{n-1}}{3(9^{n-1} + 1)},$$

so that the convergence of  $\{\gamma_n\}$  is ultimately geometric with common ratio  $1/3 = \lambda^{-2}$  while that of  $\{a_n\}$  is ultimately geometric with common ratio  $1/9 = \lambda^{-4}$ .

In the general real  $2 \times 2$  case we can prove that

$$\begin{cases} a_{n+1} = \frac{p(a^2 + \gamma^2 + aq) - (a+q)}{(a+q)^2 + \gamma^2} \\ \gamma_{n+1} = (-1)^n \frac{\gamma(pq+1)}{(a+q)^2 + \gamma^2} \end{cases} \quad (4)$$

where  $v_n = \lambda^n - \lambda^{-n}$ ,  $p = p_n = v_{n+1}/v_n$ ,  $q = q_n = -v_{n-1}/v_n$ . Since  $p_n \sim \lambda$ ,  $q_n \sim -\lambda^{-1}$  this gives  $a_n \rightarrow \lambda, \gamma_n \rightarrow 0$ .

In order to establish (4) we write  $x_n = a_n, y_n = (-1)^n \gamma_n$  in (3) to get

$$x_{n+1} = k - (x_n/(x_n^2 + y_n^2)), y_{n+1} = y_n/(x_n^2 + y_n^2), \quad (5)$$

which we combine as  $z_{n+1} = k - \bar{z}_n / (z_n \bar{z}_n)$ , i.e.,

$$z_{n+1} = k - z_n^{-1}, \quad n = 1, 2, \dots, \quad (6)$$

where  $z_r = x_r + iy_r$ ,  $\bar{z}_r = x_r - iy_r$ .

We have therefore reduced our problem to that of the iteration of the fractional linear transformation

$$w = \frac{kz - 1}{z}. \quad (7)$$

This is a well-known problem. An essentially geometric solution is given, e.g., by T. J. P. A. BROMWICH [10, p. 22, ex. 4]. This depends on the fact that (7) can be represented in the form

$$\frac{w - \lambda}{w - \lambda^{-1}} = \lambda^{-2} \left( \frac{z - \lambda}{z - \lambda^{-1}} \right) \quad (7')$$

which gives

$$\frac{z_{n+1} - \lambda}{z_{n+1} - \lambda^{-1}} = \lambda^{-2n} \left( \frac{z_1 - \lambda}{z_1 - \lambda^{-1}} \right),$$

so that  $z_n \rightarrow \lambda$ , as required. We can derive (7') using the fact that a fractional linear transformation with fixed points at  $0, \infty$  is necessarily linear, or by using the cross-ratio property of fractional linear transformations. For details compare, e.g. CARATHÉODORY [11, p. 14] or KREYSZIG [12, p. 503].

A second method is simply to establish, by induction,

$$z_{n+1} = \frac{pz_1 + 1}{z_1 + q} \quad (8)$$

where  $z_1 = a - iy$  and  $p = p_n$ ,  $q = q_n$  are as defined above. Taking the real and imaginary parts of (8) gives (4).

Our third method, preferable in the matrix context, follows. We begin by recalling that if

$$W = \frac{aw + \beta}{\gamma w + \delta}, \quad w = \frac{az + b}{cz + d}$$

then

$$W = \frac{Az + B}{Cz + D} \quad \text{where} \quad \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} a & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Thus the iteration of a fractional linear transformation is equivalent to the powering of a matrix.

We use the following result.

**Lemma.** If  $M$  is a  $2 \times 2$  matrix with distinct characteristic values  $\lambda, \mu$  then

$$M^n = \begin{bmatrix} ad\lambda^n - bc\mu^n & -ab(\lambda^n - \mu^n) \\ cd(\lambda^n - \mu^n) & -bc\lambda^n + ad\mu^n \end{bmatrix} \quad (9)$$

where

$$T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is a unimodular matrix which diagonalizes  $M$ .

Proof. Either by induction or let  $D = \text{diag} [\lambda, \mu]$  so that  $T^{-1}MT = D$ ,  $M = TDT^{-1}$  and

$$\begin{aligned} M^n &= \{TDT^{-1}\} \{TDT^{-1}\} \dots \{TDT^{-1}\} \\ &= TD^nT^{-1} \\ &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \lambda^n & 0 \\ 0 & \mu^n \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \end{aligned}$$

which gives (9), on multiplying out.

We now observe that when  $A = \begin{bmatrix} k & -1 \\ 1 & 0 \end{bmatrix}$  then

$$T = (\lambda - \lambda^{-1})^{-1/2} \begin{bmatrix} \lambda & \lambda^{-1} \\ 1 & 1 \end{bmatrix},$$

where  $\lambda$  is as already defined. The rest of the discussion is a matter of elementary algebra.

Writing as before,  $v_n = \lambda^n - \lambda^{-n}$ , we find from (9)

$$z_{n+1} \equiv x_{n+1} + iy_{n+1} = \frac{v_{n+1}(x_1 + iy_1) - v_n}{v_n(x_1 + iy_1) - v_{n-1}}.$$

Multiplying above and below on the right by  $v_n(x_1 - iy_1) - v_{n-1}$  and equating real and imaginary parts shows that

$$\begin{cases} x_{n+1} = \{v_n v_{n+1}(x_1^2 + y_1^2) + v_n v_{n-1} - x_1(v_n^2 + v_{n+1}v_{n-1})\}/D, \\ y_{n+1} = (v_n^2 - v_{n+1}v_{n-1})y_1/D, \end{cases}$$

where

$$D = v_n^2(x_1^2 + y_1^2) - 2v_{n-1}v_n x_1 + v_{n-1}^2.$$

This is another form of (4). We find that, as  $n \rightarrow \infty$ ,

$$\begin{cases} x_{n+1} - \lambda = [2(\lambda - \lambda^{-1})\{a^2 + \gamma^2 + 1 - ka\} + O(\lambda^{-2n})]/D, \\ y_{n+1} = (\lambda - \lambda^{-1})^2 \gamma / D, \end{cases} \quad (10)$$

where

$$D = [(a - \lambda^{-1})^2 + \gamma^2] \lambda^{2n} + O(1) = O(\lambda^{2n}).$$

The relations (10) establish the convergence of the *QR*-process.

Note that when the matrix  $A$  is symmetric, as well as unimodular, we have  $a\delta - \gamma^2 = 1$ , i. e.,  $a(k - a) - \gamma^2 = 1$ , i. e.  $a^2 + \gamma^2 + 1 = ka$  so that (10) gives  $x_n - \lambda = O(\lambda^{-4n})$ ,  $y_n = O(\lambda^{-2n})$ , in agreement with the numerical results in the special case.

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