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On Inscribed Circumscribed Conics

There is a well known geometric theorem due to Euler which states the following:

Theorem 1 (Euler). *Given a triangle inscribed in a circle of radius R and circumscribed about a circle of radius r, then*

$$R^2 - d^2 = 2rR, \tag{1}$$

where d is the distance between the circumcenter and the incenter of the triangle.

In this note first we generalize Euler's Theorem as follows:

Theorem 2. *Let $\mathfrak{C}$ be a circle about O with radius R and let $\mathfrak{E}$ be an ellipse contained in $\mathfrak{C}$ with semi-minor axis b and foci F_1, F_2. Set $d_1 = \overline{OF}_1$, $d_2 = \overline{OF}_2$. Then there exists a triangle inscribed in $\mathfrak{C}$ and circumscribed about $\mathfrak{E}$, if and only if*

$$(R^2 - d_1^2)(R^2 - d_2^2) = 4b^2R^2. \tag{2}$$

Proof. Let $\mathfrak{K}$ and $\mathfrak{L}$ be two conics in the projective plane, given in homogeneous coordinates by

$$X^tAX = 0 \quad \text{and} \quad X^tBX = 0 \tag{3}$$

respectively, so that $\mathfrak{K}$ contains $\mathfrak{L}$. It is known (cf. [1] p. 279) that a necessary and sufficient condition for the existence of a triangle inscribed in $\mathfrak{K}$ and circumscribed about $\mathfrak{L}$ is

$$\theta^2 = 4\Delta\theta', \tag{4}$$

where Δ, θ, θ', (and Δ'), are determined by

$$\det(A + \lambda B) \equiv \Delta + \theta\lambda + \theta'\lambda^2 + \Delta'\lambda^3. \tag{5}$$

It can be shown that

$$\Delta = \det A, \quad \theta = \operatorname{tr}[(\operatorname{adj} A)B], \quad \theta' = \operatorname{tr}[(\operatorname{adj} B)A], \quad (\Delta' = \det B), \tag{6}$$

where tr denotes the trace and adj the matrix of cofactors. Thus, the condition in (4) takes the form

$$\operatorname{tr}^2[(\operatorname{adj} A)B] = 4 \det A \cdot \operatorname{tr}[(\operatorname{adj} B)A]. \tag{7}$$

In our case let the ellipse and the circle be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{8}$$

and

$$(x - p)^2 + (y - q)^2 = R^2, \tag{9}$$

respectively. Then the non-singular symmetric matrices in (7) are

$$A = \begin{pmatrix} a^{-2} & 0 & 0 \\ 0 & b^{-2} & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & -p \\ 0 & 1 & -q \\ -p & -q & p^2 + q^2 - R^2 \end{pmatrix}. \tag{10}$$

Therefore, in this particular case, (7) reads

$$(R^2 + a^2 + b^2 - p^2 - q^2)^2 = 4\,(a^2 b^2 - a^2 q^2 - b^2 p^2 + a^2 R^2 + b^2 R^2). \tag{11}$$

Setting $a^2 = b^2 + c^2$, (11) yields

$$(R^2 - q^2 - p^2 - c^2)^2 - 4\,(b^2 R^2 + p^2 c^2) = 0. \tag{12}$$

This is equivalent to the equation

$$\{R^2 - [q^2 + (p + c)^2]\} \cdot \{R^2 - [q^2 + (p - c)^2]\} = 4b^2 R^2, \tag{13}$$

and since

$$d_1^2 = q^2 + (p + c)^2, \quad d_2^2 = q^2 + (p - c)^2, \tag{14}$$

formula (2) holds.

We note that Theorem 2 can also be proved by means of analytic geometry, using Poncelet's porism.

Theorem 2 leads to the following result.

Theorem 3. *Let $\mathfrak{C}$ be a circle about O with radius R and let $\mathfrak{E}$ be an ellipse contained in $\mathfrak{C}$ with semi-minor axis b and foci F_1, F_2. Then, a necessary and sufficient condition for the existence of a triangle which includes $\mathfrak{E}$ and is included in $\mathfrak{C}$ is*

$$(R^2 - d_1^2)\,(R^2 - d_2^2) \geq 4\,b^2 R^2, \tag{15}$$

where $d_1 = \overline{OF_1}$, $d_2 = \overline{OF_2}$.

Proof. Consider the one-parameter family of confocal ellipses

$$\{\mathfrak{E}(t);\quad t \geq 0\}, \tag{16}$$

with semi-minor axis t and fixed foci F_1, F_2. Our ellipse belongs to this family and we have $\mathfrak{E} = \mathfrak{E}(b)$.

Assume now the existence of a triangle $P_1P_2P_3$ which contains $\mathfrak{E}(b)$ and is contained in $\mathfrak{C}$. Then, by a perturbation argument, there exists a triangle with vertices Q_1, $Q_2 = Q_2(b)$ and $Q_3 = Q_3(b)$, which is inscribed in $\mathfrak{C}$ and contains $\mathfrak{E}(b)$ such that two of its sides, $Q_1Q_2(b)$ and $Q_1Q_3(b)$, touch $\mathfrak{E}(b)$, as shown in Figure 1.

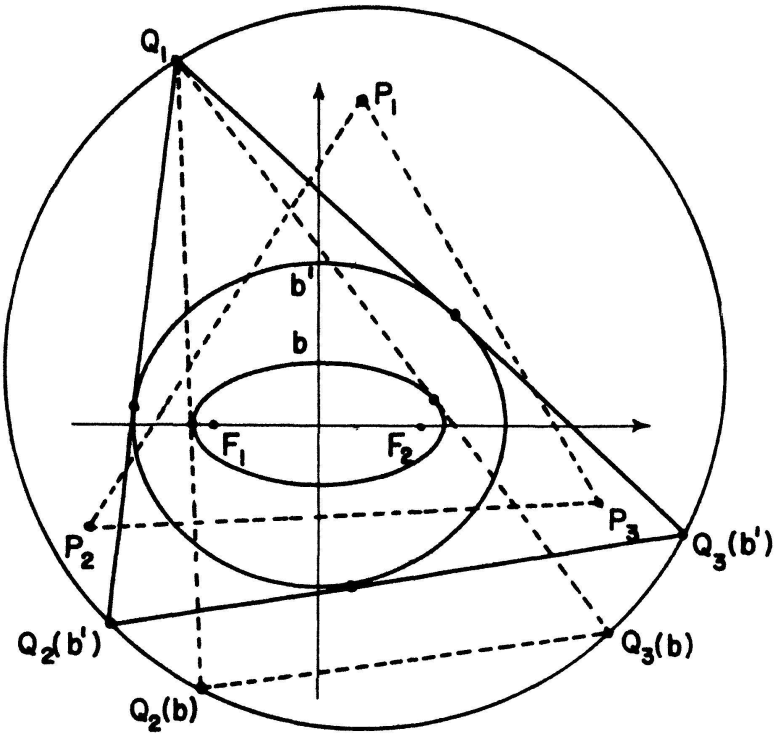

Figure 1

Now, keep Q_1 fixed. Starting with $t = b$, let t increase and let the points $Q_2 = Q_2(t)$ and $Q_3 = Q_3(t)$ move on the circle such that the sides $Q_1Q_2(t)$ and $Q_1Q_3(t)$ touch the ellipse $\mathfrak{E}(t)$. In this continuous process, the side $Q_2(t)Q_3(t)$ approaches $\mathfrak{E}(t)$, and for some $t = b'$ with $b' \geq b$, $Q_2(b')Q_3(b')$ touches $\mathfrak{E}(b')$. Hence, we have obtained a triangle inscribed in $\mathfrak{C}$ and circumscribed about $\mathfrak{E}(b')$; the ellipse $\mathfrak{E}(b')$ being confocal with $\mathfrak{E}(b)$. By Theorem 2

$$(R^2 - d_1^2)\,(R^2 - d_2^2) = 4\,b'^2 R^2, \tag{17}$$

and since $b' \geq b$, inequality (15) holds.

Conversely, assume that (15) holds and that there is no triangle which is included in $\mathfrak{C}$ and includes $\mathfrak{E}(b)$. Then, by a similar argument as above, we decrease t to obtain a confocal ellipse $\mathfrak{E}(b')$ inscribed in a triangle which in turn is inscribed in $\mathfrak{C}$. Therefore, (17) is satisfied, and $b' < b$ implies

$$(R^2 - d_1^2)\,(R^2 - d_2^2) < 4\,b^2 R^2. \tag{18}$$

This contradicts (15) and the theorem follows.

It seems interesting to derive metric relations analogous to those in Theorems 2 and 3, in the more general case of an ellipse within an ellipse.

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