

On Yff's inequality for the brocard angle of a triangle

Autor(en): **Bottema, O.**

Objektyp: **Article**

Zeitschrift: **Elemente der Mathematik**

Band (Jahr): **31 (1976)**

Heft 1

PDF erstellt am: **26.09.2024**

Persistenter Link: <https://doi.org/10.5169/seals-31391>

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.

Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.

Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

On Yff's Inequality for the Brocard Angle of a Triangle

1. Let $\alpha_1, \alpha_2, \alpha_3$ be the angles of a triangle and let ω be its Brocard angle. In 1963 Yff [1] conjectured that ω satisfies the inequality

$$8 \omega^3 \leq \alpha_1 \alpha_2 \alpha_3 . \quad (1)$$

This is a remarkable relation because it contains the angles proper and not, as is usually the case, in geometrical inequalities, their trigonometric representatives. (1) could be called a transcendental relation, while the usual ones are algebraic. There are not many statements of type (1) in elementary geometry.

Many mathematicians have tried in vain to prove (1); the present author knows about it because he was one of them. But now, quite recently, a short, elegant and ingenious proof was published in this journal by Faruk Abi-Khuzam [2]. It depends on the following lemma

$$\sin \alpha_1 \sin \alpha_2 \sin \alpha_3 \leq k \alpha_1 \alpha_2 \alpha_3 , \quad (2)$$

k being the constant $(3 \sqrt{3}/2\pi)^3$; for $\alpha_1 = \alpha_2 = \alpha_3 = \pi/3$ equality holds in (2).

In our opinion the proof, given for (2) is not completely satisfactory. Use is made of the infinite product

$$\sin x = x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{\pi^2 n^2} \right) . \quad (3)$$

After substitution and re-ordering infinitely many factors the lefthand side of (2) is written as *one* infinite product. For further reduction the factors are written as the product of two linear expressions. All this seems rather light-hearted, the more so as it is well-known that the infinite product

$$\prod \left(1 - \frac{x}{\pi n} \right) \left(1 + \frac{x}{\pi n} \right) \quad (4)$$

is not absolutely convergent (see e.g. Whittaker-Watson, *Modern Analysis*, p. 33–34).

Here follows a more elementary proof of (2).

2. For $0 < \alpha_i < \pi$, $\alpha_1 + \alpha_2 + \alpha_3 = \pi$ we consider the function

$$F(\alpha_1, \alpha_2, \alpha_3) = \frac{\sin \alpha_1 \sin \alpha_2 \sin \alpha_3}{\alpha_1 \alpha_2 \alpha_3} \quad (5)$$

F is defined on the region G in the $(\alpha_1, \alpha_2, \alpha_3)$ -space consisting of the points *inside* the triangle with the vertices $P_1(\pi, 0, 0)$, $P_2(0, \pi, 0)$, $P_3(0, 0, \pi)$. We define F on the perimeter of the triangle by

$$F(0, \alpha_2, \alpha_3) = \frac{\sin \alpha_2 \sin \alpha_3}{\alpha_2 \alpha_3}, \quad \alpha_2 \neq 0, \alpha_3 \neq 0, \alpha_2 + \alpha_3 = \pi$$

and analogously $F(\alpha_1, 0, \alpha_3)$ and $F(\alpha_1, \alpha_2, 0)$; furthermore $F(\pi, 0, 0) = F(0, \pi, 0) = F(0, 0, \pi) = 0$. F is now defined on a closed region \bar{G} ; it is continuous and derivable

on \bar{G} ; moreover as $0 \leq F < 1$ there is (at least) one point in \bar{G} where F has its maximum value. By the usual procedure, in view of $\alpha_1 + \alpha_2 + \alpha_3 = \pi$, a maximum satisfies

$$\frac{\partial F}{\partial \alpha_1} = \frac{\partial F}{\partial \alpha_2} = \frac{\partial F}{\partial \alpha_3} (= \lambda) . \quad (6)$$

In G we have

$$\frac{\partial F}{\partial \alpha_1} = \frac{\sin \alpha_2 \sin \alpha_3}{\alpha_2 \alpha_3} \cdot \frac{\alpha_1 \cos \alpha_1 - \sin \alpha_1}{\alpha_1^2} = F (\cot \alpha_1 - \alpha_1^{-1}) ,$$

and, as $F \neq 0$, (6) implies

$$\cot \alpha_1 - \alpha_1^{-1} = \cot \alpha_2 - \alpha_2^{-1} = \cot \alpha_3 - \alpha_3^{-1} . \quad (7)$$

For $f = \cot \alpha - \alpha^{-1}$ we obtain $f' = -\sin^{-2} \alpha + \alpha^{-2} < 0$; f is therefore a decreasing function of α (we have $0 > f > -\infty$); hence (7) implies

$$\alpha_1 = \alpha_2 = \alpha_3 (= \pi/3) . \quad (8)$$

in this point we have $F = k$.

We must verify whether larger values appear on the boundary of \bar{G} . Between P_2 and P_3 yields

$$F = \frac{\sin \alpha_2 \sin \alpha_3}{\alpha_2 \alpha_3} , \quad \alpha_2 + \alpha_3 = \pi$$

and by an argumentation analogous to the former, but now with two factors instead of three, it follows that for the maximum on P_2P_3 we have $\alpha_2 = \alpha_3 = \pi/2$ and $F = 4/\pi^2$, that is less than k . Hence $F \leq k$ on \bar{G} , which concludes the proof.

O. Bottema, Delft

REFERENCES

- [1] P. YFF, *An analogue of the Brocard points*, Amer. Math. Monthly 70, 500 (1963).
- [2] FARUK ABI-KHUZAM, *Proof of Yff's Conjecture on the Brocard Angle of a Triangle*, El. Math. 29 141-142 (1974).

Aufgaben

Aufgabe 733. Let n be a positive integer ≥ 2 . Let L be a line which intersects the $(n-1)$ -dimensional hyperplanes containing the $(n-1)$ -dimensional faces of a given n -dimensional simplex of vertices A_i ($i=1, \dots, n+1$) in the uniquely determined points B_i . Prove that the n -dimensional volume of the convex hull of the midpoints of $\overline{A_i B_i}$ is zero. This extends the known results for $n=2, 3$ for which the midpoints are collinear and coplanar, respectively.

M.S. Klamkin, Dearborn, Michigan, USA