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On Yff's Inequality for the Brocard Angle of a Triangle

1. Let $\alpha_1, \alpha_2, \alpha_3$ be the angles of a triangle and let ω be its Brocard angle. In 1963 Yff [1] conjectured that ω satisfies the inequality

$$8\,\omega^3 \leq \alpha_1\,\alpha_2\,\alpha_3\,. \tag{1}$$

This is a remarkable relation because it contains the angles proper and not, as is usually the case, in geometrical inequalities, their trigonometric representatives. (1) could be called a transcendental relation, while the usual ones are algebraic. There are not many statements of type (1) in elementary geometry.

Many mathematicians have tried in vain to prove (1); the present author knows about it because he was one of them. But now, quite recently, a short, elegant and ingenious proof was published in this journal by Faruk Abi-Khuzam [2]. It depends on the following lemma

$$\sin\alpha_1 \sin\alpha_2 \sin\alpha_3 \leq k\alpha_1\,\alpha_2\,\alpha_3\,, \tag{2}$$

k being the constant $(3\sqrt{3}/2\pi)^3$; for $\alpha_1 = \alpha_2 = \alpha_3 = \pi/3$ equality holds in (2).

In our opinion the proof, given for (2) is not completely satisfactory. Use is made of the infinite product

$$\sin x = x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{\pi^2\,n^2}\right). \tag{3}$$

After substitution and re-ordering infinitely many factors the lefthand side of (2) is written as *one* infinite product. For further reduction the factors are written as the product of two linear expressions. All this seems rather light-hearted, the more so as it is well-known that the infinite product

$$\prod \left(1 - \frac{x}{\pi n}\right) \left(1 + \frac{x}{\pi n}\right) \tag{4}$$

is not absolutely convergent (see e.g. Whittaker-Watson, Modern Analysis, p. 33–34).

Here follows a more elementary proof of (2).

2. For $0 < \alpha_i < \pi$, $\alpha_1 + \alpha_2 + \alpha_3 = \pi$ we consider the function

$$F\,(\alpha_1, \alpha_2, \alpha_3) = \frac{\sin\alpha_1\,\sin\alpha_2\,\sin\alpha_3}{\alpha_1\,\alpha_2\,\alpha_3} \tag{5}$$

F is defined on the region G in the $(\alpha_1, \alpha_2, \alpha_3)$-space consisting of the points *inside* the triangle with the vertices $P_1\,(\pi, 0, 0)$, $P_2\,(0, \pi, 0)$, $P_3\,(0, 0, \pi)$. We define F on the perimeter of the triangle by

$$F\,(0, \alpha_2, \alpha_3) = \frac{\sin\alpha_2\,\sin\alpha_3}{\alpha_2\,\alpha_3}\,,\ \alpha_2 \neq 0,\ \alpha_3 \neq 0,\ \alpha_2 + \alpha_3 = \pi$$

and analogously $F\,(\alpha_1, 0, \alpha_3)$ and $F\,(\alpha_1, \alpha_2, 0)$; furthermore $F\,(\pi, 0, 0) = F\,(0, \pi, 0) = F\,(0, 0, \pi) = 0$. F is now defined on a closed region $\overline{G}$; it is continuous and derivable on $\bar{G}$; moreover as $0 \leq F < 1$ there is (at least) one point in $\bar{G}$ where F has its maximum value. By the usual procedure, in view of $\alpha_1 + \alpha_2 + \alpha_3 = \pi$, a maximum satisfies

$$\frac{\partial F}{\partial \alpha_1} = \frac{\partial F}{\partial \alpha_2} = \frac{\partial F}{\partial \alpha_3} (= \lambda) . \tag{6}$$

In G we have

$$\frac{\partial F}{\partial \alpha_1} = \frac{\sin\alpha_2 \sin\alpha_3}{\alpha_2 \alpha_3} \cdot \frac{\alpha_1 \cos\alpha_1 - \sin\alpha_1}{\alpha_1^2} = F (\cot \alpha_1 - \alpha_1^{-1}) ,$$

and, as $F \neq 0$, (6) implies

$$\cot\alpha_1 - \alpha_1^{-1} = \cot\alpha_2 - \alpha_2^{-1} = \cot\alpha_3 - \alpha_3^{-1} . \tag{7}$$

For $f = \cot\alpha - \alpha^{-1}$ we obtain $f' = -\sin^{-2}\alpha + \alpha^{-2} < 0$; f is therefore a decreasing function of α (we have $0 > f > -\infty$); hence (7) implies

$$\alpha_1 = \alpha_2 = \alpha_3 (= \pi/3) . \tag{8}$$

in this point we have $F = k$.

We must verify whether larger values appear on the boundary of $\bar{G}$. Between P_2 and P_3 yields

$$F = \frac{\sin\alpha_2 \sin\alpha_3}{\alpha_2 \alpha_3} , \quad \alpha_2 + \alpha_3 = \pi$$

and by an argumentation analogous to the former, but now with two factors instead of three, it follows that for the maximum on $\overline{P_2 P_3}$ we have $\alpha_2 = \alpha_3 = \pi/2$ and $F = 4/\pi^2$, that is less than k. Hence $F \leq k$ on $\bar{G}$, which concludes the proof.

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REFERENCES

[1] P. YFF, *An analogue of the Brocard points*, Amer. Math. Monthly *70*, 500 (1963).
[2] FARUK ABI-KHUZAM, *Proof of Yff's Conjecture on the Brocard Angle of a Triangle*, El. Math. *29* 141–142 (1974).

Aufgaben

Aufgabe 733. Let n be a positive integer ≥ 2. Let L be a line which intersects the $(n-1)$-dimensional hyperplanes containing the $(n-1)$-dimensional faces of a given n-dimensional simplex of vertices A_i $(i = 1, \ldots, n+1)$ in the uniquely determined points B_i. Prove that the n-dimensional volume of the convex hull of the midpoints of $\overline{A_i B_i}$ is zero. This extends the known results for $n = 2, 3$ for which the midpoints are collinear and coplanar, respectively.

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