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On Yff's Inequality for the Brocard Angle of a Triangle

1. Let $\alpha_1, \alpha_2, \alpha_3$ be the angles of a triangle and let ω be its Brocard angle. In 1963 Yff [1] conjectured that ω satisfies the inequality

$$8 \omega^3 \leq \alpha_1 \alpha_2 \alpha_3. \quad (1)$$

This is a remarkable relation because it contains the angles proper and not, as is usually the case, in geometrical inequalities, their trigonometric representatives. (1) could be called a transcendental relation, while the usual ones are algebraic. There are not many statements of type (1) in elementary geometry.

Many mathematicians have tried in vain to prove (1); the present author knows about it because he was one of them. But now, quite recently, a short, elegant and ingenious proof was published in this journal by Faruk Abi-Khuzam [2]. It depends on the following lemma

$$\sin \alpha_1 \sin \alpha_2 \sin \alpha_3 \leq k \alpha_1 \alpha_2 \alpha_3, \quad (2)$$

k being the constant $(3 \sqrt{3}/2\pi)^3$; for $\alpha_1 = \alpha_2 = \alpha_3 = \pi/3$ equality holds in (2).

In our opinion the proof, given for (2) is not completely satisfactory. Use is made of the infinite product

$$\sin x = x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{\pi^2 n^2} \right). \quad (3)$$

After substitution and re-ordering infinitely many factors the lefthand side of (2) is written as *one* infinite product. For further reduction the factors are written as the product of two linear expressions. All this seems rather light-hearted, the more so as it is well-known that the infinite product

$$\prod \left(1 - \frac{x}{\pi n} \right) \left(1 + \frac{x}{\pi n} \right) \quad (4)$$

is not absolutely convergent (see e.g. Whittaker-Watson, *Modern Analysis*, p. 33–34).

Here follows a more elementary proof of (2).

2. For $0 < \alpha_i < \pi$, $\alpha_1 + \alpha_2 + \alpha_3 = \pi$ we consider the function

$$F(\alpha_1, \alpha_2, \alpha_3) = \frac{\sin \alpha_1 \sin \alpha_2 \sin \alpha_3}{\alpha_1 \alpha_2 \alpha_3} \quad (5)$$

F is defined on the region G in the $(\alpha_1, \alpha_2, \alpha_3)$ -space consisting of the points *inside* the triangle with the vertices $P_1(\pi, 0, 0)$, $P_2(0, \pi, 0)$, $P_3(0, 0, \pi)$. We define F on the perimeter of the triangle by

$$F(0, \alpha_2, \alpha_3) = \frac{\sin \alpha_2 \sin \alpha_3}{\alpha_2 \alpha_3}, \quad \alpha_2 \neq 0, \alpha_3 \neq 0, \alpha_2 + \alpha_3 = \pi$$

and analogously $F(\alpha_1, 0, \alpha_3)$ and $F(\alpha_1, \alpha_2, 0)$; furthermore $F(\pi, 0, 0) = F(0, \pi, 0) = F(0, 0, \pi) = 0$. F is now defined on a closed region \bar{G} ; it is continuous and derivable

on \bar{G} ; moreover as $0 \leq F < 1$ there is (at least) one point in \bar{G} where F has its maximum value. By the usual procedure, in view of $\alpha_1 + \alpha_2 + \alpha_3 = \pi$, a maximum satisfies

$$\frac{\partial F}{\partial \alpha_1} = \frac{\partial F}{\partial \alpha_2} = \frac{\partial F}{\partial \alpha_3} (= \lambda) . \quad (6)$$

In G we have

$$\frac{\partial F}{\partial \alpha_1} = \frac{\sin \alpha_2 \sin \alpha_3}{\alpha_2 \alpha_3} \cdot \frac{\alpha_1 \cos \alpha_1 - \sin \alpha_1}{\alpha_1^2} = F (\cot \alpha_1 - \alpha_1^{-1}) ,$$

and, as $F \neq 0$, (6) implies

$$\cot \alpha_1 - \alpha_1^{-1} = \cot \alpha_2 - \alpha_2^{-1} = \cot \alpha_3 - \alpha_3^{-1} . \quad (7)$$

For $f = \cot \alpha - \alpha^{-1}$ we obtain $f' = -\sin^{-2} \alpha + \alpha^{-2} < 0$; f is therefore a decreasing function of α (we have $0 > f > -\infty$); hence (7) implies

$$\alpha_1 = \alpha_2 = \alpha_3 (= \pi/3) . \quad (8)$$

in this point we have $F = k$.

We must verify whether larger values appear on the boundary of \bar{G} . Between P_2 and P_3 yields

$$F = \frac{\sin \alpha_2 \sin \alpha_3}{\alpha_2 \alpha_3} , \quad \alpha_2 + \alpha_3 = \pi$$

and by an argumentation analogous to the former, but now with two factors instead of three, it follows that for the maximum on P_2P_3 we have $\alpha_2 = \alpha_3 = \pi/2$ and $F = 4/\pi^2$, that is less than k . Hence $F \leq k$ on \bar{G} , which concludes the proof.

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- [1] P. YFF, *An analogue of the Brocard points*, Amer. Math. Monthly 70, 500 (1963).
- [2] FARUK ABI-KHUZAM, *Proof of Yff's Conjecture on the Brocard Angle of a Triangle*, El. Math. 29 141-142 (1974).

Aufgaben

Aufgabe 733. Let n be a positive integer ≥ 2 . Let L be a line which intersects the $(n-1)$ -dimensional hyperplanes containing the $(n-1)$ -dimensional faces of a given n -dimensional simplex of vertices A_i ($i=1, \dots, n+1$) in the uniquely determined points B_i . Prove that the n -dimensional volume of the convex hull of the midpoints of $\overline{A_i B_i}$ is zero. This extends the known results for $n=2, 3$ for which the midpoints are collinear and coplanar, respectively.

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