

**Zeitschrift:** Elemente der Mathematik  
**Herausgeber:** Schweizerische Mathematische Gesellschaft  
**Band:** 31 (1976)  
**Heft:** 1

**Artikel:** On Yff's inequality for the brocard angle of a triangle  
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**DOI:** <https://doi.org/10.5169/seals-31391>

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## On Yff's Inequality for the Brocard Angle of a Triangle

1. Let  $\alpha_1, \alpha_2, \alpha_3$  be the angles of a triangle and let  $\omega$  be its Brocard angle. In 1963 Yff [1] conjectured that  $\omega$  satisfies the inequality

$$8 \omega^3 \leq \alpha_1 \alpha_2 \alpha_3. \quad (1)$$

This is a remarkable relation because it contains the angles proper and not, as is usually the case, in geometrical inequalities, their trigonometric representatives. (1) could be called a transcendental relation, while the usual ones are algebraic. There are not many statements of type (1) in elementary geometry.

Many mathematicians have tried in vain to prove (1); the present author knows about it because he was one of them. But now, quite recently, a short, elegant and ingenious proof was published in this journal by Faruk Abi-Khuzam [2]. It depends on the following lemma

$$\sin \alpha_1 \sin \alpha_2 \sin \alpha_3 \leq k \alpha_1 \alpha_2 \alpha_3, \quad (2)$$

$k$  being the constant  $(3 \sqrt{3}/2\pi)^3$ ; for  $\alpha_1 = \alpha_2 = \alpha_3 = \pi/3$  equality holds in (2).

In our opinion the proof, given for (2) is not completely satisfactory. Use is made of the infinite product

$$\sin x = x \prod_{n=1}^{\infty} \left( 1 - \frac{x^2}{\pi^2 n^2} \right). \quad (3)$$

After substitution and re-ordering infinitely many factors the lefthand side of (2) is written as *one* infinite product. For further reduction the factors are written as the product of two linear expressions. All this seems rather light-hearted, the more so as it is well-known that the infinite product

$$\prod \left( 1 - \frac{x}{\pi n} \right) \left( 1 + \frac{x}{\pi n} \right) \quad (4)$$

is not absolutely convergent (see e.g. Whittaker-Watson, *Modern Analysis*, p. 33–34).

Here follows a more elementary proof of (2).

2. For  $0 < \alpha_i < \pi$ ,  $\alpha_1 + \alpha_2 + \alpha_3 = \pi$  we consider the function

$$F(\alpha_1, \alpha_2, \alpha_3) = \frac{\sin \alpha_1 \sin \alpha_2 \sin \alpha_3}{\alpha_1 \alpha_2 \alpha_3} \quad (5)$$

$F$  is defined on the region  $G$  in the  $(\alpha_1, \alpha_2, \alpha_3)$ -space consisting of the points *inside* the triangle with the vertices  $P_1(\pi, 0, 0)$ ,  $P_2(0, \pi, 0)$ ,  $P_3(0, 0, \pi)$ . We define  $F$  on the perimeter of the triangle by

$$F(0, \alpha_2, \alpha_3) = \frac{\sin \alpha_2 \sin \alpha_3}{\alpha_2 \alpha_3}, \quad \alpha_2 \neq 0, \alpha_3 \neq 0, \alpha_2 + \alpha_3 = \pi$$

and analogously  $F(\alpha_1, 0, \alpha_3)$  and  $F(\alpha_1, \alpha_2, 0)$ ; furthermore  $F(\pi, 0, 0) = F(0, \pi, 0) = F(0, 0, \pi) = 0$ .  $F$  is now defined on a closed region  $\bar{G}$ ; it is continuous and derivable

on  $\bar{G}$ ; moreover as  $0 \leq F < 1$  there is (at least) one point in  $\bar{G}$  where  $F$  has its maximum value. By the usual procedure, in view of  $\alpha_1 + \alpha_2 + \alpha_3 = \pi$ , a maximum satisfies

$$\frac{\partial F}{\partial \alpha_1} = \frac{\partial F}{\partial \alpha_2} = \frac{\partial F}{\partial \alpha_3} (= \lambda) . \quad (6)$$

In  $G$  we have

$$\frac{\partial F}{\partial \alpha_1} = \frac{\sin \alpha_2 \sin \alpha_3}{\alpha_2 \alpha_3} \cdot \frac{\alpha_1 \cos \alpha_1 - \sin \alpha_1}{\alpha_1^2} = F (\cot \alpha_1 - \alpha_1^{-1}) ,$$

and, as  $F \neq 0$ , (6) implies

$$\cot \alpha_1 - \alpha_1^{-1} = \cot \alpha_2 - \alpha_2^{-1} = \cot \alpha_3 - \alpha_3^{-1} . \quad (7)$$

For  $f = \cot \alpha - \alpha^{-1}$  we obtain  $f' = -\sin^{-2} \alpha + \alpha^{-2} < 0$ ;  $f$  is therefore a decreasing function of  $\alpha$  (we have  $0 > f > -\infty$ ); hence (7) implies

$$\alpha_1 = \alpha_2 = \alpha_3 (= \pi/3) . \quad (8)$$

in this point we have  $F = k$ .

We must verify whether larger values appear on the boundary of  $\bar{G}$ . Between  $P_2$  and  $P_3$  yields

$$F = \frac{\sin \alpha_2 \sin \alpha_3}{\alpha_2 \alpha_3} , \quad \alpha_2 + \alpha_3 = \pi$$

and by an argumentation analogous to the former, but now with two factors instead of three, it follows that for the maximum on  $P_2P_3$  we have  $\alpha_2 = \alpha_3 = \pi/2$  and  $F = 4/\pi^2$ , that is less than  $k$ . Hence  $F \leq k$  on  $\bar{G}$ , which concludes the proof.

O. Bottema, Delft

## REFERENCES

- [1] P. YFF, *An analogue of the Brocard points*, Amer. Math. Monthly 70, 500 (1963).
- [2] FARUK ABI-KHUZAM, *Proof of Yff's Conjecture on the Brocard Angle of a Triangle*, El. Math. 29 141-142 (1974).

## Aufgaben

**Aufgabe 733.** Let  $n$  be a positive integer  $\geq 2$ . Let  $L$  be a line which intersects the  $(n-1)$ -dimensional hyperplanes containing the  $(n-1)$ -dimensional faces of a given  $n$ -dimensional simplex of vertices  $A_i$  ( $i = 1, \dots, n+1$ ) in the uniquely determined points  $B_i$ . Prove that the  $n$ -dimensional volume of the convex hull of the midpoints of  $\overline{A_i B_i}$  is zero. This extends the known results for  $n = 2, 3$  for which the midpoints are collinear and coplanar, respectively.

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