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P_3' and P_3'' respectively denote the $x - v$ and $v - z$ subpaths of P_3 . Then, P_3'' has the same length as P . Hence, the paths P_1 , P_3' , and $\langle\{w, x\}\rangle$ constitute a $u - w$ walk of at most $e(z)$. Since this is impossible, it must be the case that $u \in Z(G)$. As such, the theorem follows.

As a special case of the preceding theorem, we have the following corollary.

Corollary 10. If a graph G is randomly eulerian from any vertex, then the center $Z(G)$ induces a connected subgraph.

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Kleine Mitteilungen

When is the divisibility relation in a monoid a partial ordering?

1. Let $\langle M, \cdot, e \rangle$ be a monoid, i.e., a semigroup $\langle M, \cdot \rangle$ with an identity element e . We define the *divisibility relation* \leq in M by

$$x, y \in M; x \leq y \quad :\leftrightarrow \quad xu = y \quad \text{for some } u \in M.$$

By a non-trivial group we mean a group consisting of two or more elements. For $x \in M$, we denote the principal right ideal $\{xu; u \in M\}$ by xM . It is easily seen that, for arbitrary $x, y \in M$,

$$x \leq y \quad \leftrightarrow \quad yM \subset xM \quad \leftrightarrow \quad y \in xM \tag{1}$$

and that \leq is reflexive and transitive. SHWU-YENG T. LIN [5] raised the problem to find a necessary and sufficient condition on M for \leq to be a partial ordering. In this note we present an answer to this question and several remarks about it.

2. Criterion 1: For a monoid $\langle M, \cdot, e \rangle$, the following statements are equivalent:

$$(*) \quad x, u, v \in M; xuv = x \rightarrow xu = x,$$

$$(*') \quad x, y \in M; xM = yM \rightarrow x = y,$$

(**") the divisibility relation \leq in M is a partial ordering.

Proof: $(*) \rightarrow (*')$: Assume that $xM = yM$. Then $x = xe \in xM = yM$ and, analogously, $y \in xM$. Therefore there exist $u, v \in M$ such that $y = xu$, $x = yv$, hence $xuv = x$, and $(*)$ implies $xu = x$, i.e., $x = y$. $(*') \rightarrow (**")$: Suppose that $x \leq y$ and $y \leq x$. From (1) we conclude $xM = yM$, and by virtue of $(*)$ we get $x = y$. $(**") \rightarrow (*)$: Let be $xuv = x$. Then $xu \leq x$ and $x \leq xu$, and antisymmetry yields $xu = x$.

Corollary 1: *The following conditions are necessary for \leq to be a partial ordering:*

M has no non-trivial subgroups. (2)

$u, v \in M; uv = e \rightarrow u = v = e$. (3)

The subgroup U of invertible elements of M is $\{e\}$. (4)

$\langle M, \cdot, e \rangle$ is not a non-trivial group. (5)

Proof: (2): Assume that M has a non-trivial subgroup L with identity element e' (notice that e' need not be equal to e ; cf. Remark 6 below). Let be $u \in L$, $u \neq e'$. Then $e'uu^{-1} = e'$, but $e'u = u \neq e'$, a contradiction to (*). – (3): Let be $uv = e$, hence $eu v = e$. (*) implies $eu = e$, i.e., $u = e$. Now $uv = e$ yields $v = e$. – (4) follows from (3) and (5) from (4). For (5) in this connection see [1], p. 176, Theorem 6.

On the other hand, it is quite natural to ask for what familiar classes of monoids condition (*) does hold. $\langle M, \cdot, e \rangle$ is called

left cancellative if, for any $x, y, z \in M$, $xy = xz$ implies $y = z$, (6)

idempotent if $xx = x$ for any $x \in M$. (7)

Corollary 2: *Each of the following two conditions a), b) is sufficient for \leq to be a partial ordering:*

a) (4) and (6) (see also [6], p. 123),

b) (7) and commutativity of \cdot .

Proof: a) $xuv = x$ implies $(xu)vu = xu$, and (6) leads to $uv = e$, $vu = e$, i.e., by virtue of (4), to $v = e$. Therefore $xu = xuv = x$. Thus, (*) holds. – b) $xuv = x$ and (7) imply $xux = xu(xuv) = (xuxu)v = (xu)v = xuv = x$. With the additional help of commutativity we get $xu = xxu = xux = x$.

3. Remark 1: Any semilattice (with or without identity element), i.e., any commutative idempotent semigroup, satisfies (*) (see [2], p. 22, Lemma 2); reflexivity is ensured by (7). For idempotent semigroups the divisibility relation equals the so-called natural partial ordering \leq_n defined by $x \leq_n y: \leftrightarrow xy = y$ ([3], p. 23–24).

Remark 2: Condition a) in Corollary 2 is not necessary for (*): Let E be a non-empty set and $\mathfrak{P}(E)$ the collection of all subsets of E . Then the monoid $\langle \mathfrak{P}(E), \cup, \phi \rangle$ is a semilattice, so satisfies (*). But it is not left cancellative. Here \leq is set-theoretical inclusion.

Remark 3: Condition b) in Corollary 2 is not necessary for (*): Let \mathbf{N} denote the set of positive integers. Then $\langle \mathbf{N}, \cdot, 1 \rangle$ satisfies (4) and (6), thus (*) holds. Here \leq turns out to be the usual divisibility relation in \mathbf{N} . Since $\langle \mathbf{N}, \cdot, 1 \rangle$ is not idempotent, we have a negative answer to an additional question in [5].

Remark 4: Let be $E = \{a, b\}$ ($a \neq b$). Let $e, \underline{a}, \underline{b}$ mappings from E into E , namely: e identical, $\underline{a}(E) = \{a\}$, $\underline{b}(E) = \{b\}$. If $M = \{e, \underline{a}, \underline{b}\}$, and if we define \cdot in M by $xy := y \circ x$, then $\langle M, \cdot, e \rangle$ is an idempotent noncommutative monoid; it is not left cancellative, but (2) and (3) hold: The subgroups of M are $\{e\}$, $\{\underline{a}\}$, and $\{\underline{b}\}$. The equations $\underline{a}\underline{b}\underline{a} = \underline{a}$, $\underline{a}\underline{b} = \underline{b}$ violate (*). Therefore none of the conditions (2), (3), (7) and of their conjunctions is sufficient for (*). However, in the commutative case, (7) is (Corollary 2b)).

Remark 5: (6) does not imply (*): Consider a non-trivial group M and notice that (*) implies (5). Here \leq is the universal relation on M : $x \leq y$ for any $x, y \in M$.

Remark 6: Even for commutative monoids, (3) is not sufficient for (*): For any real number a denote the matrix $\begin{pmatrix} a & a \\ a & a \end{pmatrix}$ by m_a . For $e := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $M := \{e, m_a; a \in \mathbf{R}\}$ and the usual matrix multiplication, $\langle M, \cdot, e \rangle$ is a commutative monoid satisfying (3). But $\{m_a; a \in \mathbf{R} \setminus \{0\}\}$ forms a non-trivial subgroup of M , its identity element being $m_{1/2}$. So M does not satisfy (2), a fortiori not (*).

4. Finally we discuss the question about necessary and sufficient conditions on the monoid $\langle M, \cdot, e \rangle$ for \leq to have other important properties. The proofs of the following three criteria are left to the reader.

Criterion 2: *The following statements are equivalent:*

$$x, y \in M \rightarrow xM \cap yM \neq \phi, \text{ i.e., } M \text{ is left reversible ([4], p. 194),} \quad (8)$$

$$x, y \in M \rightarrow x \leq z, y \leq z \text{ for some } z \in M \text{ (Moore-Smith property).} \quad (8')$$

Commutativity of \cdot is sufficient for (8) but by no means necessary: Consider a non-abelian group.

Criterion 3: *The following statements are equivalent:*

$$x, y \in M \rightarrow \{x, y\} \cap xM \cap yM \neq \phi, \quad (9)$$

$$x, y \in M \rightarrow x \leq y \text{ and/or } y \leq x. \quad (9')$$

Criterion 4: *The following statements are equivalent:*

$$x, y \in M \rightarrow xM \cap yM = zM \text{ for some } z \in M, \quad (10)$$

$$x, y \in M \rightarrow x \leq z \text{ and } y \leq z \text{ for some } z \in M, \text{ and} \\ w \in M, x \leq w, y \leq w \text{ imply } z \leq w. \quad (10')$$

Evidently, (9) implies (10), and (10) implies (8). (9), (10), (8) correspond to the situations in totally ordered, semilattice-ordered and directed sets, respectively.

Corollary 3: *Let $\langle M, \cdot, e \rangle$ be a monoid for which (*) and (10) hold. Then there exists a binary operation $*$ on M such that*

a) $\langle M, *, e \rangle$ is a commutative idempotent monoid,

b) the divisibility relations \leq' and \leq in $\langle M, *, e \rangle$ and $\langle M, \cdot, e \rangle$ are the same.

Proof: a) Let x, y be arbitrary elements of M . By (10), there exists $z \in M$ such that $xM \cap yM = zM$. By (*), z is uniquely determined by x and y . Thus $*$ is well-defined by

$$x * y = z \text{ where } xM \cap yM = zM.$$

Therefore, the mapping $f : M \rightarrow \{xM; x \in M\}$ defined by $f(x) = xM$ for every $x \in M$ is bijective and has the property $f(x * y) = f(x) \cap f(y)$ ($\langle M, *, e \rangle$ is the 'transplant' of $\langle \{xM; x \in M\}, \cap, M \rangle$ under f^{-1} ; see [7], p. 43/44). Since $\langle \{xM; x \in M\}, \cap, M \rangle$ is a semilattice with identity, so is $\langle M, *, e \rangle$ (For a different proof cf. [2], p. 10, Corollary). – b) For arbitrary elements $x, y \in M$ we have $x \leq' y \leftrightarrow x * u = y$ for some $u \in M \leftrightarrow xM \cap uM = yM$ for some $u \in M \leftrightarrow yM \subset xM \leftrightarrow x \leq y$, the last step being ensured by (1).

Remark 7: Of course, $x * y = \sup \{x, y\}$ with respect to \leq . For $\langle \mathbf{N}, \cdot, 1 \rangle$ (\mathbf{N} the set of positive integers), $*$ is the least common multiple operation, and both operations \cdot and $*$ lead to the usual divisibility relation. J. Rätz, Bern

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A formula for the least prime greater than a given integer

We shall prove the following theorem, which gives an apparently new formula for the n th prime p_n .

Theorem. Let m be any natural number ≥ 2 . Put

$$d = ((m!)^{m!} - 1, (2m)!),$$

$$t = \frac{d^d}{(d^d, d!)}$$

and define α by the condition $d^\alpha \parallel t$. Then the integer

$$p = \frac{d}{(t/d^\alpha, d)}$$

is the least prime greater than m .

In particular, taking $m = p_{n-1}$, we see that $p = p_n$. So, in principle, this formula enables us to compute p_n once p_{n-1} is known. Of course the result is purely of theoretical interest, since even computing the number d (by the Euclidean algorithm) is, in general, completely impractical.

Some years ago Gandhi (see [1]) proved the following result: if Q is the product of the first $n - 1$ primes, then the inequality

$$1 < 2^{p_n} \left[-\frac{1}{2} + \sum_{d|Q} \left(\frac{\mu(d)}{2^d - 1} \right) \right] < 2$$

(μ denotes the Möbius function) gives p_n explicitly in terms of p_1, \dots, p_{n-1} . There are also older formulas for p_n (see [2, p. 344], [3]).

Lemma. The representation of d as a product of primes is

$$d = q_1 q_2 \cdots q_n \quad (n \geq 1),$$

where $m < q_1 < q_2 < \dots < q_n < 2m$ and q_1 is the least prime $p > m$.

Proof. Obviously d is square-free, and each of its prime factors is between m and $2m$. Thus we have only to show that d is divisible by p , the smallest prime which

exceeds m . By Bertrand's theorem [2, p. 343] $m < p < 2m$. Therefore, if $m > 3$, $\varphi(p) = 2[(p-1)/2] |m|$, so that Euler's theorem gives $(m!)^{m!} \equiv 1 \pmod{p}$. It is easy to see that this congruence also holds for $m = 2, 3$. Since $p \mid (2m)!$ the proof of the lemma is complete.

Proof of the theorem. For $i = 1, \dots, n$, the exponent e_i of the highest power of q_i which divides $d!$, is

$$e_i = \sum_{j=1}^{\infty} \left[\frac{d}{q_i^j} \right] < \sum_{j=1}^{\infty} \frac{d}{q_i^j} < d$$

[2, p. 342]. Since for $1 \leq i \leq n-1$

$$\left[\frac{d}{q_{i+1}} \right] = \frac{d}{q_{i+1}} < \frac{d}{q_i} = \left[\frac{d}{q_i} \right],$$

$$\left[\frac{d}{q_{i+1}^j} \right] \leq \left[\frac{d}{q_i^j} \right] \quad (j = 2, 3, \dots),$$

we have $d > e_1 > e_2 > \dots > e_n$. Hence $t = q_1^{\alpha(1)} q_2^{\alpha(2)} \dots q_n^{\alpha(n)}$, where $0 < \alpha(1) < \alpha(2) < \dots < \alpha(n)$. This completes the proof.

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Aufgaben

Aufgabe 709. It is well known (cf. e.g. H. Hadwiger, H. Debrunner, V. Klee, *Combinatorial Geometry in the Plane*, New York 1964, p. 4, Problem 5) that no three distinct points of a square lattice can be the vertices of an equilateral triangle. Show that no four distinct points of an equilateral triangular lattice can be the vertices of a square.

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Erste Lösung: Wir beweisen allgemeiner den folgenden Satz:

Ein rechtwinkliges Dreieck kann genau dann in ein reguläres Dreiecksgitter eingelagert werden (d.h., die Ecken sind Gitterpunkte), wenn das Verhältnis seiner Katheten die Form $r\sqrt{3}$ mit einer positiven rationalen Zahl r hat.

Daraus folgt unmittelbar die Behauptung der Aufgabe 709, aber z.B. auch der Satz: Kein pythagoräisches Dreieck kann in ein reguläres Dreiecksgitter eingelagert werden.

Beweis: I. \mathbf{a} und \mathbf{b} seien zwei Einheitsvektoren, die das Gitter aufspannen, $\mathbf{a} \cdot \mathbf{b} = 1/2$. A , B und C seien drei verschiedene Gitterpunkte, die ein bei C rechtwinkliges Dreieck bilden. Ist dann etwa $\mathbf{BC} = x\mathbf{a} + y\mathbf{b}$, $\mathbf{CA} = u\mathbf{a} + v\mathbf{b}$ mit ganzen Zahlen x, y, u, v , so ist $\mathbf{BC} \cdot \mathbf{CA} = 0$, folglich $x(2u+v) + y(u+2v) = 0$. Es gibt also