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Vertex Cyclic Graphs

§ 1. Definitions

In the following, we consider graphs which are finite, undirected, loop-free, and without multiple edges.

Let u and v be vertices of a graph G. A $u-v$ *walk* in G is an alterating sequence of vertices and edges beginning with u, ending with v, and such that each edge is incident with the vertices immediately preceeding and succeeding it. A $u-v$ walk is *open* if $u \neq v$ and *closed* if $u = v$. A *trail* is a walk without repeated edges and a *path* is a trail without repeated vertices. A *circuit* is a closed trail and a *cycle* is a circuit in which the intermediate vertices are not repeated.

A graph is *connected* if there is a walk joining every pair of vertices. A *component* of a graph G is a connected subgraph not properly contained in any other connected subgraph of G. A vertex v of a graph G is a *cut-vertex* of G if $G-v$ has more components than does G. A graph G is a *block* if it is connected and has no cut-vertex. A *block of a graph* G is a subgraph of G which is maximal with respect to being a block.

Let $V(G)$ and $E(G)$ denote respectively the vertex and edge sets of a graph G. For vertices u and v of G, let the *distance* $d_G(u, v)$ between u and v be the length of a shortest $u-v$ path. The *eccentricity* $e(v)$ for $v \in V(G)$ is $e(v) = \max\{d_G(u, v) : u \in V(G)\}$ and the *radius* rad G of G is rad $G = \min\{e(v) : v \in V(G)\}$. The *center* $Z(G)$ of G is $Z(G) = \{v \in V(G) : e(v) = \text{rad } G\}$.

In general, we will follow the conventions of Behzad and Chartrand [2].

§ 2. Randomly Eulerian graphs

Although we will consider 'randomly eulerian' graphs only to the extent that they exist in a larger class of graphs, they are introduced here for perspective and to illustrate the property we will investigate.

Let G be a connected graph. An *eulerian trail* in G is an open trail of G containing all edges of G and an *eulerian circuit* of G is a circuit of G which contains all edges of G. The graph G is *eulerian* if it has an eulerian circuit. Also, G is *randomly eulerian* from a vertex v if each trail with initial vertex v can be extended to an eulerian $v-v$ circuit of G.

Euler [3] characterized eulerian graphs and Ore [4] characterized graphs which are randomly eulerian from a vertex. In particular, if the degree $\deg_G v$ of $v \in V(G)$ is the number of edges in G incident with the vertex v, then we have the following well-known propositions.

Proposition 1. A connected graph is eulerian if and only if each vertex has even degree.

Proposition 2. A connected graph has an eulerian trail if and only if it has exactly two vertices of odd degree.

Proposition 3. An eulerian graph is randomly eulerian from a vertex v if and only if v belongs to every cycle of G.

It is a property inherent in the third proposition in which we are most interested and will pursue in the next section.

§ 3. Vertex cyclic graphs

A connected graph G with only cyclic blocks is *vertex cyclic* if it has a vertex which belongs to every cycle of G. In particular, a vertex cyclic graph G is *v-cyclic* if v is a vertex belonging to each cycle of G. To see that non-eulerian vertex cyclic graphs exist, it suffices to consider the complete bipartite graph $K(2, 3)$.

Noting that a (p, q)-graph is a graph with p vertices and q edges, we have the following result.

Theorem 1. If G is a *v-cyclic* (p, q)-graph, then $q \leq 2p - 3$.

Proof: The graph $G - v$ is a forest with $p - 1$ vertices and at most $p - 2$ edges. Since v can be adjacent to at most $p - 1$ vertices, G can have at most $2p - 3$ edges.

For a graph G, let $\Delta(G)$ and $\delta(G)$ respectively denote the maximum and minimum degree among the vertices of G. Another consequence following from the proof of Theorem 1 is presented below.

Corollary 2. If G is vertex cyclic, then $\delta(G) = 2$.

In [1], Bäbler showed for a graph G randomly eulerian from a vertex v that $\deg_G v = \Delta(G)$. We now generalize this result by showing this is a property of vertex cyclic graphs.

Theorem 3. If G is a v-cyclic graph, then $\deg_G v = \Delta(G)$.

Proof: Since $H = G - v$ is a forest, we have that $\Delta(H)$ does not exceed the number n of end-vertices of H. In G, the vertex v is adjacent to each end-vertex of H, thus, $\Delta(H) \leq n \leq \deg_G v$. Furthermore, for $u \in V(H)$, $\deg_G u = \deg_H u$ if $uv \notin E(G)$ and $\deg_G u = 1 + \deg_H u$ if $uv \in E(G)$. In any event, $\deg_G u \leq \deg_G v$ for all $u \in V(H)$ since the only edges in G which are not in H, are those edges joining v to some vertex in H.

We may now obtain the following result.

Theorem 4. If G is a v-cyclic graph and $\deg_G w = \Delta(G)$ for some $w \in V(G) - \{v\}$, then G is also w-cyclic and $\deg_G u = \delta(G)$ for all $u \in V(G) - \{v, w\}$.

Proof: If G is a cycle, then the theorem follows. So, suppose G is not a cycle. Let n be the number of end-vertices of the forest $H = G - v$. Then, $\deg_G w = \deg_G v \geq n$.

We now show that $\deg_H w = n$. Since H is acyclic, we have that $\deg_H w \leq n$. So, suppose $\deg_H w < n$. Then the edge vw must be in $E(G)$ and we have that $n \geq 1 + \deg_H w = \deg_G w = \deg_G v \geq n$. Thus, w is an end-vertex of H. Hence, $\deg_H w = 1$ which implies that $\Delta(G) = \deg_G v = \deg_G w = 2$. As such, G must be a cycle and this is a contradiction. Thus, $\deg_H w = n$.

Since $\deg_H w = n$, H is a tree. Also, $\deg_G w = n$ implies all vertices of H different from w have degree at most two in H. As such, every path joining two distinct end-vertices of H must contain u. Furthermore, $\deg_G w = \deg_G v$ implies that v is adjacent to only end-vertices of H and possibly w. Consequently, every vertex of G different from v and w has degree $\delta(G) = 2$ and w lies on every cycle of G.

As an immediate consequence of the preceding two results, we have the following.

Corollary 5. A graph is vertex cyclic from at least three vertices if and only if it is a cycle.

A property which is inherent in the eulerian situation, but not for vertex cyclic graphs in general, is presented below.

Lemma 6. If G is randomly eulerian from a vertex v and T is any trail with initial vertex v, then $G - E(T)$ has at most one nontrivial component.

Proof: If T is a circuit, then each nontrivial component of $G - E(T)$ is eulerian and, as such, contains a cycle which in turn contains v. Hence, $G - E(T)$ has at most one nontrivial component and it contains v. If T is not a circuit, then we can extend T by a path P to yield a circuit T'. Let H_v be the component of $G - E(T')$ containing v. Then, any other component of $G - E(T')$ is trivial. Also, $G - E(T)$ is $G - E(T')$ together with the path P. Hence, given any component of G $- E(T)$ not containing v, it must be trivial. Thus, the lemma follows.

A vertex cyclic graph G

A circuit T of G

Figure 1

To see that the result in Lemma 6 does not generalize to all vertex cyclic graphs, it suffices to consider the vertex cyclic graph G and the circuit T of G in Figure 1. Then, $G - E(T)$ has two nontrivial components, neither of which contain v. However, there do exist noneulerian vertex cyclic graphs with this property. In fact, the following theorem characterizes all such vertex cyclic graphs.

Theorem 7. Let G be a v-cyclic graph. Then, $G - E(T)$ has at most one nontrivial component for each trail T with intial vertex v if and only if:

a) G is vertex cyclic from exactly two vertices; or

b) G is eulerian.

Proof: The sufficiency of a) or b) follows from Theorem 4 and Lemma 6 respectively. To show the necessity of a) or b), we show that if G is noneulerian and vertex cyclic from only v, then G has a trail T with initial vertex v such that $G - E(T)$ has at least two nontrivial components. We now consider the following two cases.

Case 1. *Suppose G has a block B with at least two vertices different from v and both of odd degree.* Then, there exist vertices u and w in B of odd degree together with a $u - w$ path P containing neither v nor any other odd vertex.

For each edge e in $G - E(P)$ incident with a vertex x in P, there is an $x - v$ path P_e in $G - E(P)$. Also, for each pair of edges e_1, and e_2 in $G - E(P)$ incident with a vertex x in P, the paths P_{e_1} and P_{e_2} have only x and v in common. Since each vertex x of P has even degree in $G - E(P)$, we may pair them to form cycles, the union of which is a $v - v$ circuit C_x which exhausts the edges in $G - E(P)$ incident with x. Also, if P_{e_1} and P_{e_2} correspond to edges e_1 and e_2 incident with distinct vertices x_1 and x_2 respectively, then P_{e_1} and P_{e_2} have only v in common. Consequently, the $v - v$ circuits C_{x_1} and C_{x_2} have only v in common if $x_1 \neq x_2$. Thus, the union T of all the circuits C_x, $x \in V(P)$, is a $v - v$ circuit in $G - E(P)$ exhausting all the edges in $G - E(P)$ incident with vertices of P.

Let C be a cycle in T containing w. Then C has an edge xv incident with v but not with w. Then $T - xv$ has a $v - x$ trail T' exhausting the edges in $G - E(P)$ incident with vertices of P. Thus, the paths P and x, v must be in different components of $G - E(T')$.

Case 2. *Suppose G has a block B with exactly one vertex w different from v and of odd degree.* Necessarily, the vertex v must also be of odd degree in B.

Suppose G is not a block. Let u be a vertex of B different from v and adjacent to w. Then, $B - uw$ is connected and has u and v as its only vertices of odd degree. By Proposition 2, B $- uw$ has an eulerian $u - v$ trail T. Let B' be any block of G different from B. Then the path u, w and the block B' lie in different components of $G - E(T)$.

Conversely, suppose $G = B$. Since G is not w-cyclic, there is a cycle C in G not containing w. Since w can be adjacent to atmost one vertex of $C - v$, there is a vertex x in $G - V(C)$ adjacent to w. Note that $G - E(C)$ has only one nontrivial component H and $H - xw$ has only two vertices of odd degree; in particular, x and v are of odd degree. By Proposition 2, $H - xw$ has an eulerian $v - x$ trail T. Since T exhausts the edges in $G - xw$ incident with x and w, the path x, y and the cycle C lie in different components of $G - E(T)$.

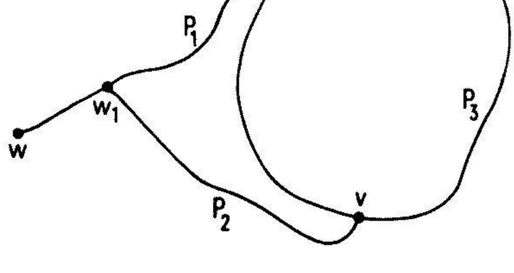
We now consider the center of a vertex cyclic graph and show that it must contain any vertex for which the graph is vertex cyclic.

Theorem 8. If G is a v-cyclic graph, then $v \in Z(G)$.

Proof: Let $u \in V(G)$ be such that $d_G(u, v) = e_G(v)$ and suppose u is in block B of G. If $B \neq G$, then for each $w \in V(G) - V(B)$ we have that $e_G(w) \geq d_G(w, u) = d_G(w, v) + d_G(v, u) > d_G(v, u) = e(v)$ since v can be the only cut-vertex of G. In any event, $Z(G) \subseteq V(B)$ since $e(z) \leq e(v)$ for all $z \in Z(G)$.

Since u and v are in a block, there exists a cycle containing u and v. Let C be a smallest such cycle. Given any two vertices of C and a diagonal path joining them, the path must contain v. Since C is a smallest cycle, we have rad $C = e_G(v)$. Thus, $e_C(x) = \text{rad } C$ for each $x \in V(C)$. Since there are no shorter paths in G joining any two vertices of C, we also have $e_G(x) \geq e_C(x)$ for all $x \in V(C)$.

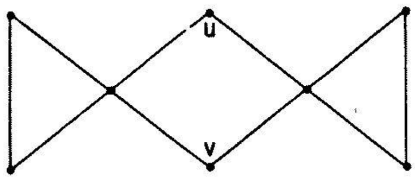
Figure 2



If $B = C$, we are done. So, suppose $B \neq C$. Let $w \in V(B - C)$. Then there is exactly one path P not containing v but joining w to C. Suppose P joins C at the vertex x. Let $H = G - E(C)$ and let w_1 be the vertex on $P - x$ closest to x which minimizes $d_P(w, w_1) + d_H(w_1, v)$. Let P_1 be the $w_1 - x$ subpath of P and let P_2 be a shortest $w_1 - v$ path in H. Clearly, P_1 and P_2 have only w_1 in common. Let P_3 be a shortest $x - v$ subpath of C containing u. Then P_1, P_2 and P_3 form a cycle C_1 (cf. Figure 2) and rad $C_1 \geq$ rad C. As such, there is a vertex $s \in V(P_3)$ such that $d_{C_1}(w_1, s) \geq$ rad C. By our choice of w_1, there is no shorter $s - w_1$ path in G and we have that $e(w) \geq d(w, s) \geq d(w_1, s) \geq$ rad $C \geq e(v)$. Hence, it follows that $v \in Z(G)$.

Given a set V of vertices of a graph G, the *induced subgraph* $\langle V \rangle$ *of* G has vertex set V and edge set $E = \{uv \in E(G) : u, v \in V\}$. It is well known that the center need not induce a connected subgraph. This is also the case for eulerian graphs. In particular, the graph in Figure 3 is eulerian, has center $\{u, v\}$, and $\langle\{u, v\}\rangle$ is not connected. However, this is not the case for vertex cyclic graphs.

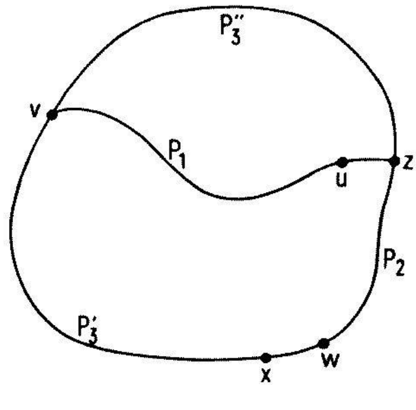
Figure 3



Theorem 9. If G is a vertex cyclic graph, then $Z(G)$ is connected.

Proof: Suppose G is v-cyclic. If $Z(G) = \{v\}$ or $d_G(v, z) \leq 1$ for all $z \in Z(G)$, then the result follows. So, suppose there is a $z \in Z(G)$ such that $d_G(v, z) \geq 2$ and let P be any shortest $v - z$ path. It suffices to show $V(P) \subseteq Z(G)$. To show this, it suffices to prove that the vertex u adjacent in P to z is also in $Z(G)$. Let P_1 be the $v - u$ subpath of P. This is shown in Figure 4, the remainder of which we will construct in the following.

Figure 4



Suppose $u \notin Z(G)$. Since $z \in Z(G)$ and $uz \in E(G)$, we have that $e(u) = e(z) + 1$. Let $w \in V(G)$ be such that $d(u, w) = e(u)$. Then, $v \neq w \neq z$ and $d(w, z) = e(z)$.

Let P_2 be a shortest $z - w$ path. Since G is v-cyclic, v is the only vertex which P_1 and P_2 can have in common. In this case, the $z - v$ subpath P_2' of P_2 is of the same length as P. Hence, P_1 together with the $v - w$ subpath of P_2 is a $u - w$ walk of length $e(z)$. Since this is impossible, the paths P_1 and P_2 are disjoint.

Since $\deg_G w \geq 2$, there is a vertex x adjacent to w but not on P_2. Then, $d_G(x, z) \leq e(z)$. Let P_3 be a shortest $x - z$ path. Clearly, $w \notin V(P_3)$ and v must be on P_3. Let

P_3' and P_3'' respectively denote the $x - v$ and $v - z$ subpaths of P_3. Then, P_3'' has the same length as P. Hence, the paths P_1, P_3', and $\langle\{w, x\}\rangle$ constitute a $u - w$ walk of at most $e(z)$. Since this is impossible, it must be the case that $u \in Z(G)$. As such, the theorem follows.

As a special case of the preceding theorem, we have the following corollary.

Corollary 10. If a graph G is randomly eulerian from any vertex, then the center $Z(G)$ induces a connected subgraph.

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Kleine Mitteilungen

When is the divisibility relation in a monoid a partial ordering?

1. Let $\langle M, \cdot, e\rangle$ be a monoid, i.e., a semigroup $\langle M, \cdot\rangle$ with an identity element e. We define the *divisibility relation* $\leq$ in M by

$$x, y \in M;\, x \leq y \ :\leftrightarrow\ xu = y \quad \textit{for some} \quad u \in M\,.$$

By a non-trivial group we mean a group consisting of two or more elements. For $x \in M$, we denote the principal right ideal $\{xu;\, u \in M\}$ by xM. It is easily seen that, for arbitrary $x, y \in M$,

$$x \leq y \ \leftrightarrow\ yM \subset xM \ \leftrightarrow\ y \in xM \tag{1}$$

and that $\leq$ is reflexive and transitive. SHWU-YENG T. LIN [5] raised the problem to find a necessary and sufficient condition on M for $\leq$ to be a partial ordering. In this note we present an answer to this question and several remarks about it.

2. Criterion 1: *For a monoid $\langle M, \cdot, e\rangle$, the following statements are equivalent:*

(*) $x, u, v \in M;\, xuv = x \rightarrow xu = x$,

(*') $x, y \in M;\, xM = yM \rightarrow x = y$,

(*") *the divisibility relation $\leq$ in M is a partial ordering.*

Proof: (*) $\rightarrow$ (*'): Assume that $xM = yM$. Then $x = xe \in xM = yM$ and, analogously, $y \in xM$. Therefore there exist $u, v \in M$ such that $y = xu$, $x = yv$, hence $xuv = x$, and (*) implies $xu = x$, i.e., $x = y$. – (*') $\rightarrow$ (*"): Suppose that $x \leq y$ and $y \leq x$. From (1) we conclude $xM = yM$, and by virtue of (*') we get $x = y$. – (*") $\rightarrow$ (*): Let be $xuv = x$. Then $xu \leq x$ and $x \leq xu$, and antisymmetry yields $xu = x$.