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# Vertex Cyclic Graphs

## § 1. Definitions

In the following, we consider graphs which are finite, undirected, loop-free, and without multiple edges.

Let  $u$  and  $v$  be vertices of a graph  $G$ . A  $u - v$  *walk* in  $G$  is an alternating sequence of vertices and edges beginning with  $u$ , ending with  $v$ , and such that each edge is incident with the vertices immediately preceeding and succeeding it. A  $u - v$  walk is *open* if  $u \neq v$  and *closed* if  $u = v$ . A *trail* is a walk without repeated edges and a *path* is a trail without repeated vertices. A *circuit* is a closed trail and a *cycle* is a circuit in which the intermediate vertices are not repeated.

A graph is *connected* if there is a walk joining every pair of vertices. A *component* of a graph  $G$  is a connected subgraph not properly contained in any other connected subgraph of  $G$ . A vertex  $v$  of a graph  $G$  is a *cut-vertex* of  $G$  if  $G - v$  has more components than does  $G$ . A graph  $G$  is a *block* if it is connected and has no cut-vertex. A *block of a graph*  $G$  is a subgraph of  $G$  which is maximal with respect to being a block.

Let  $V(G)$  and  $E(G)$  denote respectively the vertex and edge sets of a graph  $G$ . For vertices  $u$  and  $v$  of  $G$ , let the *distance*  $d_G(u, v)$  between  $u$  and  $v$  be the length of a shortest  $u - v$  path. The *eccentricity*  $e(v)$  for  $v \in V(G)$  is  $e(v) = \max \{d_G(u, v) : u \in V(G)\}$  and the *radius*  $\text{rad } G$  of  $G$  is  $\text{rad } G = \min \{e(v) : v \in V(G)\}$ . The *center*  $Z(G)$  of  $G$  is  $Z(G) = \{v \in V(G) : e(v) = \text{rad } G\}$ .

In general, we will follow the conventions of Behzad and Chartrand [2].

## § 2. Randomly Eulerian graphs

Although we will consider ‘randomly eulerian’ graphs only to the extent that they exist in a larger class of graphs, they are introduced here for perspective and to illustrate the property we will investigate.

Let  $G$  be a connected graph. An *eulerian trail* in  $G$  is an open trail of  $G$  containing all edges of  $G$  and an *eulerian circuit* of  $G$  is a circuit of  $G$  which contains all edges of  $G$ . The graph  $G$  is *eulerian* if it has an eulerian circuit. Also,  $G$  is *randomly eulerian* from a vertex  $v$  if each trail with initial vertex  $v$  can be extended to an eulerian  $v - v$  circuit of  $G$ .

Euler [3] characterized eulerian graphs and Ore [4] characterized graphs which are randomly eulerian from a vertex. In particular, if the degree  $\deg_G v$  of  $v \in V(G)$  is the number of edges in  $G$  incident with the vertex  $v$ , then we have the following well-known propositions.

*Proposition 1.* A connected graph is eulerian if and only if each vertex has even degree.

*Proposition 2.* A connected graph has an eulerian trail if and only if it has exactly two vertices of odd degree.

*Proposition 3.* An eulerian graph is randomly eulerian from a vertex  $v$  if and only if  $v$  belongs to every cycle of  $G$ .

It is a property inherent in the third proposition in which we are most interested and will pursue in the next section.

### § 3. Vertex cyclic graphs

A connected graph  $G$  with only cyclic blocks is *vertex cyclic* if it has a vertex which belongs to every cycle of  $G$ . In particular, a vertex cyclic graph  $G$  is *v-cyclic* if  $v$  is a vertex belonging to each cycle of  $G$ . To see that non-eulerian vertex cyclic graphs exist, it suffices to consider the complete bipartite graph  $K(2, 3)$ .

Noting that a  $(p, q)$ -graph is a graph with  $p$  vertices and  $q$  edges, we have the following result.

*Theorem 1.* If  $G$  is a *v-cyclic*  $(p, q)$ -graph, then  $q \leq 2p - 3$ .

*Proof:* The graph  $G - v$  is a forest with  $p - 1$  vertices and at most  $p - 2$  edges. Since  $v$  can be adjacent to at most  $p - 1$  vertices,  $G$  can have at most  $2p - 3$  edges.

For a graph  $G$ , let  $\Delta(G)$  and  $\delta(G)$  respectively denote the maximum and minimum degree among the vertices of  $G$ . Another consequence following from the proof of Theorem 1 is presented below.

*Corollary 2.* If  $G$  is vertex cyclic, then  $\delta(G) = 2$ .

In [1], Babler showed for a graph  $G$  randomly eulerian from a vertex  $v$  that  $\deg_G v = \Delta(G)$ . We now generalize this result by showing this is a property of vertex cyclic graphs.

*Theorem 3.* If  $G$  is a *v-cyclic* graph, then  $\deg_G v = \Delta(G)$ .

*Proof:* Since  $H = G - v$  is a forest, we have that  $\Delta(H)$  does not exceed the number  $n$  of end-vertices of  $H$ . In  $G$ , the vertex  $v$  is adjacent to each end-vertex of  $H$ , thus,  $\Delta(H) \leq n \leq \deg_G v$ . Furthermore, for  $u \in V(H)$ ,  $\deg_G u = \deg_H u$  if  $uv \notin E(G)$  and  $\deg_G u = 1 + \deg_H u$  if  $uv \in E(G)$ . In any event,  $\deg_G u \leq \deg_G v$  for all  $u \in V(H)$  since the only edges in  $G$  which are not in  $H$ , are those edges joining  $v$  to some vertex in  $H$ .

We may now obtain the following result.

*Theorem 4.* If  $G$  is a *v-cyclic* graph and  $\deg_G w = \Delta(G)$  for some  $w \in V(G) - \{v\}$ , then  $G$  is also *w-cyclic* and  $\deg_G u = \delta(G)$  for all  $u \in V(G) - \{v, w\}$ .

*Proof:* If  $G$  is a cycle, then the theorem follows. So, suppose  $G$  is not a cycle. Let  $n$  be the number of end-vertices of the forest  $H = G - v$ . Then,  $\deg_G w = \deg_G v \geq n$ .

We now show that  $\deg_H w = n$ . Since  $H$  is acyclic, we have that  $\deg_H w \leq n$ . So, suppose  $\deg_H w < n$ . Then the edge  $vw$  must be in  $E(G)$  and we have that  $n \geq 1 + \deg_H w = \deg_G w = \deg_G v \geq n$ . Thus,  $w$  is an end-vertex of  $H$ . Hence,  $\deg_H w = 1$  which implies that  $\Delta(G) = \deg_G v = \deg_G w = 2$ . As such,  $G$  must be a cycle and this is a contradiction. Thus,  $\deg_H w = n$ .

Since  $\deg_H w = n$ ,  $H$  is a tree. Also,  $\deg_G w = n$  implies all vertices of  $H$  different from  $w$  have degree at most two in  $H$ . As such, every path joining two distinct end-vertices of  $H$  must contain  $w$ . Furthermore,  $\deg_G w = \deg_G v$  implies that  $v$  is adjacent to only end-vertices of  $H$  and possibly  $w$ . Consequently, every vertex of  $G$  different from  $v$  and  $w$  has degree  $\delta(G) = 2$  and  $w$  lies on every cycle of  $G$ .

As an immediate consequence of the preceding two results, we have the following.

*Corollary 5.* A graph is vertex cyclic from at least three vertices if and only if it is a cycle.

A property which is inherent in the eulerian situation, but not for vertex cyclic graphs in general, is presented below.

*Lemma 6.* If  $G$  is randomly eulerian from a vertex  $v$  and  $T$  is any trail with initial vertex  $v$ , then  $G - E(T)$  has at most one nontrivial component.

*Proof:* If  $T$  is a circuit, then each nontrivial component of  $G - E(T)$  is eulerian and, as such, contains a cycle which in turn contains  $v$ . Hence,  $G - E(T)$  has at most one nontrivial component and it contains  $v$ . If  $T$  is not a circuit, then we can extend  $T$  by a path  $P$  to yield a circuit  $T'$ . Let  $H_v$  be the component of  $G - E(T')$  containing  $v$ . Then, any other component of  $G - E(T')$  is trivial. Also,  $G - E(T)$  is  $G - E(T')$  together with the path  $P$ . Hence, given any component of  $G - E(T)$  not containing  $v$ , it must be trivial. Thus, the lemma follows.



Figure 1

To see that the result in Lemma 6 does not generalize to all vertex cyclic graphs, it suffices to consider the vertex cyclic graph  $G$  and the circuit  $T$  of  $G$  in Figure 1. Then,  $G - E(T)$  has two nontrivial components, neither of which contain  $v$ . However, there do exist noneulerian vertex cyclic graphs with this property. In fact, the following theorem characterizes all such vertex cyclic graphs.

*Theorem 7.* Let  $G$  be a  $v$ -cyclic graph. Then,  $G - E(T)$  has at most one nontrivial component for each trail  $T$  with initial vertex  $v$  if and only if:

- a)  $G$  is vertex cyclic from exactly two vertices; or
- b)  $G$  is eulerian.

*Proof:* The sufficiency of a) or b) follows from Theorem 4 and Lemma 6 respectively. To show the necessity of a) or b), we show that if  $G$  is noneulerian and vertex cyclic from only  $v$ , then  $G$  has a trail  $T$  with initial vertex  $v$  such that  $G - E(T)$  has at least two nontrivial components. We now consider the following two cases.

*Case 1.* Suppose  $G$  has a block  $B$  with at least two vertices different from  $v$  and both of odd degree. Then, there exist vertices  $u$  and  $w$  in  $B$  of odd degree together with a  $u - w$  path  $P$  containing neither  $v$  nor any other odd vertex.

For each edge  $e$  in  $G - E(P)$  incident with a vertex  $x$  in  $P$ , there is an  $x - v$  path  $P_e$  in  $G - E(P)$ . Also, for each pair of edges  $e_1$  and  $e_2$  in  $G - E(P)$  incident with a vertex  $x$  in  $P$ , the paths  $P_{e_1}$  and  $P_{e_2}$  have only  $x$  and  $v$  in common. Since each vertex  $x$  of  $P$  has even degree in  $G - E(P)$ , we may pair them to form cycles, the union of which is a  $v - v$  circuit  $C_x$  which exhausts the edges in  $G - E(P)$  incident

with  $x$ . Also, if  $P_{e_1}$  and  $P_{e_2}$  correspond to edges  $e_1$  and  $e_2$  incident with distinct vertices  $x_1$  and  $x_2$  respectively, then  $P_{e_1}$  and  $P_{e_2}$  have only  $v$  in common. Consequently, the  $v - v$  circuits  $C_{x_1}$  and  $C_{x_2}$  have only  $v$  in common if  $x_1 \neq x_2$ . Thus, the union  $T$  of all the circuits  $C_x$ ,  $x \in V(P)$ , is a  $v - v$  circuit in  $G - E(P)$  exhausting all the edges in  $G - E(P)$  incident with vertices of  $P$ .

Let  $C$  be a cycle in  $T$  containing  $w$ . Then  $C$  has an edge  $xv$  incident with  $v$  but not with  $w$ . Then  $T - xv$  has a  $v - x$  trail  $T'$  exhausting the edges in  $G - E(P)$  incident with vertices of  $P$ . Thus, the paths  $P$  and  $x, v$  must be in different components of  $G - E(T')$ .

*Case 2. Suppose  $G$  has a block  $B$  with exactly one vertex  $w$  different from  $v$  and of odd degree. Necessarily, the vertex  $v$  must also be of odd degree in  $B$ .*

Suppose  $G$  is not a block. Let  $u$  be a vertex of  $B$  different from  $v$  and adjacent to  $w$ . Then,  $B - uw$  is connected and has  $u$  and  $v$  as its only vertices of odd degree. By Proposition 2,  $B - uw$  has an eulerian  $u - v$  trail  $T$ . Let  $B'$  be any block of  $G$  different from  $B$ . Then the path  $u, w$  and the block  $B'$  lie in different components of  $G - E(T)$ .

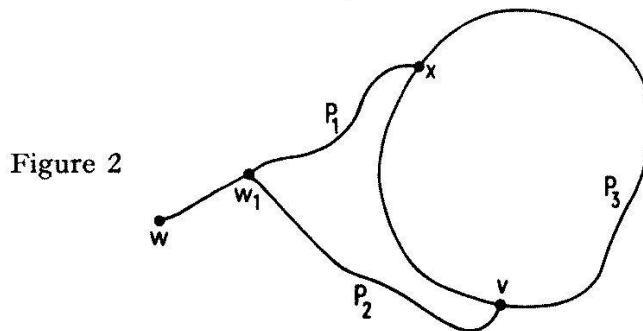
Conversely, suppose  $G = B$ . Since  $G$  is not  $w$ -cyclic, there is a cycle  $C$  in  $G$  not containing  $w$ . Since  $w$  can be adjacent to at most one vertex of  $C - v$ , there is a vertex  $x$  in  $G - V(C)$  adjacent to  $w$ . Note that  $G - E(C)$  has only one nontrivial component  $H$  and  $H - xw$  has only two vertices of odd degree; in particular,  $x$  and  $v$  are of odd degree. By Proposition 2,  $H - xw$  has an eulerian  $v - x$  trail  $T$ . Since  $T$  exhausts the edges in  $G - xw$  incident with  $x$  and  $w$ , the path  $x, y$  and the cycle  $C$  lie in different components of  $G - E(T)$ .

We now consider the center of a vertex cyclic graph and show that it must contain any vertex for which the graph is vertex cyclic.

**Theorem 8.** If  $G$  is a  $v$ -cyclic graph, then  $v \in Z(G)$ .

*Proof:* Let  $u \in V(G)$  be such that  $d_G(u, v) = e_G(v)$  and suppose  $u$  is in block  $B$  of  $G$ . If  $B \neq G$ , then for each  $w \in V(G) - V(B)$  we have that  $e_G(w) \geq d_G(w, u) = d_G(w, v) + d_G(v, u) > d_G(v, u) = e(v)$  since  $v$  can be the only cut-vertex of  $G$ . In any event,  $Z(G) \subseteq V(B)$  since  $e(z) \leq e(v)$  for all  $z \in Z(G)$ .

Since  $u$  and  $v$  are in a block, there exists a cycle containing  $u$  and  $v$ . Let  $C$  be a smallest such cycle. Given any two vertices of  $C$  and a diagonal path joining them, the path must contain  $v$ . Since  $C$  is a smallest cycle, we have  $\text{rad } C = e_G(v)$ . Thus,  $e_C(x) = \text{rad } C$  for each  $x \in V(C)$ . Since there are no shorter paths in  $G$  joining any two vertices of  $C$ , we also have  $e_G(x) \geq e_C(x)$  for all  $x \in V(C)$ .



If  $B = C$ , we are done. So, suppose  $B \neq C$ . Let  $w \in V(B - C)$ . Then there is exactly one path  $P$  not containing  $v$  but joining  $w$  to  $C$ . Suppose  $P$  joins  $C$  at the vertex  $x$ . Let  $H = G - E(C)$  and let  $w_1$  be the vertex on  $P - x$  closest to  $x$  which minimizes  $d_P(w, w_1) + d_H(w_1, v)$ . Let  $P_1$  be the  $w_1 - x$  subpath of  $P$  and let  $P_2$  be a shortest  $w_1 - v$  path in  $H$ . Clearly,  $P_1$  and  $P_2$  have only  $w_1$  in common. Let  $P_3$  be a shortest  $x - v$  subpath of  $C$  containing  $u$ . Then  $P_1, P_2$  and  $P_3$  form a cycle  $C_1$  (cf. Figure 2) and  $\text{rad } C_1 \geq \text{rad } C$ . As such, there is a vertex  $s \in V(P_3)$  such that  $d_{C_1}(w_1, s) \geq \text{rad } C$ . By our choice of  $w_1$ , there is no shorter  $s - w_1$  path in  $G$  and we have that  $e(w) \geq d(w, s) \geq d(w_1, s) \geq \text{rad } C \geq e(v)$ . Hence, it follows that  $v \in Z(G)$ .

Given a set  $V$  of vertices of a graph  $G$ , the *induced subgraph*  $\langle V \rangle$  of  $G$  has vertex set  $V$  and edge set  $E = \{uv \in E(G) : u, v \in V\}$ . It is well known that the center need not induce a connected subgraph. This is also the case for eulerian graphs. In particular, the graph in Figure 3 is eulerian, has center  $\{u, v\}$ , and  $\langle \{u, v\} \rangle$  is not connected. However, this is not the case for vertex cyclic graphs.

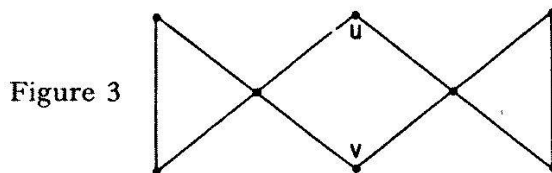


Figure 3

**Theorem 9.** If  $G$  is a vertex cyclic graph, then  $Z(G)$  is connected.

*Proof:* Suppose  $G$  is  $v$ -cyclic. If  $Z(G) = \{v\}$  or  $d_G(v, z) \leq 1$  for all  $z \in Z(G)$ , then the result follows. So, suppose there is a  $z \in Z(G)$  such that  $d_G(v, z) \geq 2$  and let  $P$  be any shortest  $v - z$  path. It suffices to show  $V(P) \subseteq Z(G)$ . To show this, it suffices to prove that the vertex  $u$  adjacent in  $P$  to  $z$  is also in  $Z(G)$ . Let  $P_1$  be the  $v - u$  subpath of  $P$ . This is shown in Figure 4, the remainder of which we will construct in the following.

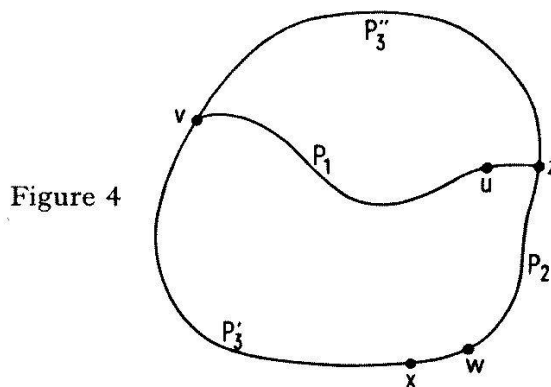


Figure 4

Suppose  $u \notin Z(G)$ . Since  $z \in Z(G)$  and  $uz \in E(G)$ , we have that  $e(u) = e(z) + 1$ . Let  $w \in V(G)$  be such that  $d(u, w) = e(u)$ . Then,  $v \neq w \neq z$  and  $d(w, z) = e(z)$ .

Let  $P_2$  be a shortest  $z - w$  path. Since  $G$  is  $v$ -cyclic,  $v$  is the only vertex which  $P_1$  and  $P_2$  can have in common. In this case, the  $z - v$  subpath  $P_2'$  of  $P_2$  is of the same length as  $P$ . Hence,  $P_1$  together with the  $v - w$  subpath of  $P_2$  is a  $u - w$  walk of length  $e(z)$ . Since this is impossible, the paths  $P_1$  and  $P_2$  are disjoint.

Since  $\deg_G w \geq 2$ , there is a vertex  $x$  adjacent to  $w$  but not on  $P_2$ . Then,  $d_G(x, z) \leq e(z)$ . Let  $P_3$  be a shortest  $x - z$  path. Clearly,  $w \notin V(P_3)$  and  $v$  must be on  $P_3$ . Let



$P_3'$  and  $P_3''$  respectively denote the  $x - v$  and  $v - z$  subpaths of  $P_3$ . Then,  $P_3''$  has the same length as  $P$ . Hence, the paths  $P_1$ ,  $P_3'$ , and  $\langle\{w, x\}\rangle$  constitute a  $u - w$  walk of at most  $e(z)$ . Since this is impossible, it must be the case that  $u \in Z(G)$ . As such, the theorem follows.

As a special case of the preceding theorem, we have the following corollary.

*Corollary 10.* If a graph  $G$  is randomly eulerian from any vertex, then the center  $Z(G)$  induces a connected subgraph.

John Roberts, Western Michigan Univ., Kalamazoo, USA

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## Kleine Mitteilungen

### When is the divisibility relation in a monoid a partial ordering?

1. Let  $\langle M, \cdot, e \rangle$  be a monoid, i.e., a semigroup  $\langle M, \cdot \rangle$  with an identity element  $e$ . We define the *divisibility relation*  $\leq$  in  $M$  by

$$x, y \in M; x \leq y \iff xu = y \text{ for some } u \in M.$$

By a non-trivial group we mean a group consisting of two or more elements. For  $x \in M$ , we denote the principal right ideal  $\{xu; u \in M\}$  by  $xM$ . It is easily seen that, for arbitrary  $x, y \in M$ ,

$$x \leq y \iff yM \subset xM \iff y \in xM \tag{1}$$

and that  $\leq$  is reflexive and transitive. SHWU-YENG T. LIN [5] raised the problem to find a necessary and sufficient condition on  $M$  for  $\leq$  to be a partial ordering. In this note we present an answer to this question and several remarks about it.

**2. Criterion 1:** For a monoid  $\langle M, \cdot, e \rangle$ , the following statements are equivalent:

$$(*) \quad x, u, v \in M; xuv = x \rightarrow xu = x,$$

$$(*') \quad x, y \in M; xM = yM \rightarrow x = y,$$

$$(**) \quad \text{the divisibility relation } \leq \text{ in } M \text{ is a partial ordering.}$$

*Proof:*  $(*) \rightarrow (*)'$ : Assume that  $xM = yM$ . Then  $x = xe \in xM = yM$  and, analogously,  $y \in xM$ . Therefore there exist  $u, v \in M$  such that  $y = xu$ ,  $x = yv$ , hence  $xuv = x$ , and  $(*)$  implies  $xu = x$ , i.e.,  $x = y$ . –  $(*)' \rightarrow (**)$ : Suppose that  $x \leq y$  and  $y \leq x$ . From (1) we conclude  $xM = yM$ , and by virtue of  $(*)'$  we get  $x = y$ . –  $(**) \rightarrow (*)$ : Let be  $xuv = x$ . Then  $xu \leq x$  and  $x \leq xu$ , and antisymmetry yields  $xu = x$ .