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Vertex Cyclic Graphs

§ 1. Definitions

In the following, we consider graphs which are finite, undirected, loop-free, and without multiple edges.

Let u and v be vertices of a graph G . A $u - v$ *walk* in G is an alternating sequence of vertices and edges beginning with u , ending with v , and such that each edge is incident with the vertices immediately preceeding and succeeding it. A $u - v$ walk is *open* if $u \neq v$ and *closed* if $u = v$. A *trail* is a walk without repeated edges and a *path* is a trail without repeated vertices. A *circuit* is a closed trail and a *cycle* is a circuit in which the intermediate vertices are not repeated.

A graph is *connected* if there is a walk joining every pair of vertices. A *component* of a graph G is a connected subgraph not properly contained in any other connected subgraph of G . A vertex v of a graph G is a *cut-vertex* of G if $G - v$ has more components than does G . A graph G is a *block* if it is connected and has no cut-vertex. A *block of a graph* G is a subgraph of G which is maximal with respect to being a block.

Let $V(G)$ and $E(G)$ denote respectively the vertex and edge sets of a graph G . For vertices u and v of G , let the *distance* $d_G(u, v)$ between u and v be the length of a shortest $u - v$ path. The *eccentricity* $e(v)$ for $v \in V(G)$ is $e(v) = \max \{d_G(u, v) : u \in V(G)\}$ and the *radius* $\text{rad } G$ of G is $\text{rad } G = \min \{e(v) : v \in V(G)\}$. The *center* $Z(G)$ of G is $Z(G) = \{v \in V(G) : e(v) = \text{rad } G\}$.

In general, we will follow the conventions of Behzad and Chartrand [2].

§ 2. Randomly Eulerian graphs

Although we will consider ‘randomly eulerian’ graphs only to the extent that they exist in a larger class of graphs, they are introduced here for perspective and to illustrate the property we will investigate.

Let G be a connected graph. An *eulerian trail* in G is an open trail of G containing all edges of G and an *eulerian circuit* of G is a circuit of G which contains all edges of G . The graph G is *eulerian* if it has an eulerian circuit. Also, G is *randomly eulerian* from a vertex v if each trail with initial vertex v can be extended to an eulerian $v - v$ circuit of G .

Euler [3] characterized eulerian graphs and Ore [4] characterized graphs which are randomly eulerian from a vertex. In particular, if the degree $\deg_G v$ of $v \in V(G)$ is the number of edges in G incident with the vertex v , then we have the following well-known propositions.

Proposition 1. A connected graph is eulerian if and only if each vertex has even degree.

Proposition 2. A connected graph has an eulerian trail if and only if it has exactly two vertices of odd degree.

Proposition 3. An eulerian graph is randomly eulerian from a vertex v if and only if v belongs to every cycle of G .

It is a property inherent in the third proposition in which we are most interested and will pursue in the next section.

§ 3. Vertex cyclic graphs

A connected graph G with only cyclic blocks is *vertex cyclic* if it has a vertex which belongs to every cycle of G . In particular, a vertex cyclic graph G is *v-cyclic* if v is a vertex belonging to each cycle of G . To see that non-eulerian vertex cyclic graphs exist, it suffices to consider the complete bipartite graph $K(2, 3)$.

Noting that a (p, q) -graph is a graph with p vertices and q edges, we have the following result.

Theorem 1. If G is a *v-cyclic* (p, q) -graph, then $q \leq 2p - 3$.

Proof: The graph $G - v$ is a forest with $p - 1$ vertices and at most $p - 2$ edges. Since v can be adjacent to at most $p - 1$ vertices, G can have at most $2p - 3$ edges.

For a graph G , let $\Delta(G)$ and $\delta(G)$ respectively denote the maximum and minimum degree among the vertices of G . Another consequence following from the proof of Theorem 1 is presented below.

Corollary 2. If G is vertex cyclic, then $\delta(G) = 2$.

In [1], Babler showed for a graph G randomly eulerian from a vertex v that $\deg_G v = \Delta(G)$. We now generalize this result by showing this is a property of vertex cyclic graphs.

Theorem 3. If G is a *v-cyclic* graph, then $\deg_G v = \Delta(G)$.

Proof: Since $H = G - v$ is a forest, we have that $\Delta(H)$ does not exceed the number n of end-vertices of H . In G , the vertex v is adjacent to each end-vertex of H , thus, $\Delta(H) \leq n \leq \deg_G v$. Furthermore, for $u \in V(H)$, $\deg_G u = \deg_H u$ if $uv \notin E(G)$ and $\deg_G u = 1 + \deg_H u$ if $uv \in E(G)$. In any event, $\deg_G u \leq \deg_G v$ for all $u \in V(H)$ since the only edges in G which are not in H , are those edges joining v to some vertex in H .

We may now obtain the following result.

Theorem 4. If G is a *v-cyclic* graph and $\deg_G w = \Delta(G)$ for some $w \in V(G) - \{v\}$, then G is also *w-cyclic* and $\deg_G u = \delta(G)$ for all $u \in V(G) - \{v, w\}$.

Proof: If G is a cycle, then the theorem follows. So, suppose G is not a cycle. Let n be the number of end-vertices of the forest $H = G - v$. Then, $\deg_G w = \deg_G v \geq n$.

We now show that $\deg_H w = n$. Since H is acyclic, we have that $\deg_H w \leq n$. So, suppose $\deg_H w < n$. Then the edge vw must be in $E(G)$ and we have that $n \geq 1 + \deg_H w = \deg_G w = \deg_G v \geq n$. Thus, w is an end-vertex of H . Hence, $\deg_H w = 1$ which implies that $\Delta(G) = \deg_G v = \deg_G w = 2$. As such, G must be a cycle and this is a contradiction. Thus, $\deg_H w = n$.

Since $\deg_H w = n$, H is a tree. Also, $\deg_G w = n$ implies all vertices of H different from w have degree at most two in H . As such, every path joining two distinct end-vertices of H must contain w . Furthermore, $\deg_G w = \deg_G v$ implies that v is adjacent to only end-vertices of H and possibly w . Consequently, every vertex of G different from v and w has degree $\delta(G) = 2$ and w lies on every cycle of G .

As an immediate consequence of the preceding two results, we have the following.

Corollary 5. A graph is vertex cyclic from at least three vertices if and only if it is a cycle.

A property which is inherent in the eulerian situation, but not for vertex cyclic graphs in general, is presented below.

Lemma 6. If G is randomly eulerian from a vertex v and T is any trail with initial vertex v , then $G - E(T)$ has at most one nontrivial component.

Proof: If T is a circuit, then each nontrivial component of $G - E(T)$ is eulerian and, as such, contains a cycle which in turn contains v . Hence, $G - E(T)$ has at most one nontrivial component and it contains v . If T is not a circuit, then we can extend T by a path P to yield a circuit T' . Let H_v be the component of $G - E(T')$ containing v . Then, any other component of $G - E(T')$ is trivial. Also, $G - E(T)$ is $G - E(T')$ together with the path P . Hence, given any component of $G - E(T)$ not containing v , it must be trivial. Thus, the lemma follows.



Figure 1

To see that the result in Lemma 6 does not generalize to all vertex cyclic graphs, it suffices to consider the vertex cyclic graph G and the circuit T of G in Figure 1. Then, $G - E(T)$ has two nontrivial components, neither of which contain v . However, there do exist noneulerian vertex cyclic graphs with this property. In fact, the following theorem characterizes all such vertex cyclic graphs.

Theorem 7. Let G be a v -cyclic graph. Then, $G - E(T)$ has at most one nontrivial component for each trail T with initial vertex v if and only if:

- a) G is vertex cyclic from exactly two vertices; or
- b) G is eulerian.

Proof: The sufficiency of a) or b) follows from Theorem 4 and Lemma 6 respectively. To show the necessity of a) or b), we show that if G is noneulerian and vertex cyclic from only v , then G has a trail T with initial vertex v such that $G - E(T)$ has at least two nontrivial components. We now consider the following two cases.

Case 1. Suppose G has a block B with at least two vertices different from v and both of odd degree. Then, there exist vertices u and w in B of odd degree together with a $u - w$ path P containing neither v nor any other odd vertex.

For each edge e in $G - E(P)$ incident with a vertex x in P , there is an $x - v$ path P_e in $G - E(P)$. Also, for each pair of edges e_1 and e_2 in $G - E(P)$ incident with a vertex x in P , the paths P_{e_1} and P_{e_2} have only x and v in common. Since each vertex x of P has even degree in $G - E(P)$, we may pair them to form cycles, the union of which is a $v - v$ circuit C_x which exhausts the edges in $G - E(P)$ incident

with x . Also, if P_{e_1} and P_{e_2} correspond to edges e_1 and e_2 incident with distinct vertices x_1 and x_2 respectively, then P_{e_1} and P_{e_2} have only v in common. Consequently, the $v - v$ circuits C_{x_1} and C_{x_2} have only v in common if $x_1 \neq x_2$. Thus, the union T of all the circuits C_x , $x \in V(P)$, is a $v - v$ circuit in $G - E(P)$ exhausting all the edges in $G - E(P)$ incident with vertices of P .

Let C be a cycle in T containing w . Then C has an edge xv incident with v but not with w . Then $T - xv$ has a $v - x$ trail T' exhausting the edges in $G - E(P)$ incident with vertices of P . Thus, the paths P and x, v must be in different components of $G - E(T')$.

Case 2. Suppose G has a block B with exactly one vertex w different from v and of odd degree. Necessarily, the vertex v must also be of odd degree in B .

Suppose G is not a block. Let u be a vertex of B different from v and adjacent to w . Then, $B - uw$ is connected and has u and v as its only vertices of odd degree. By Proposition 2, $B - uw$ has an eulerian $u - v$ trail T . Let B' be any block of G different from B . Then the path u, w and the block B' lie in different components of $G - E(T)$.

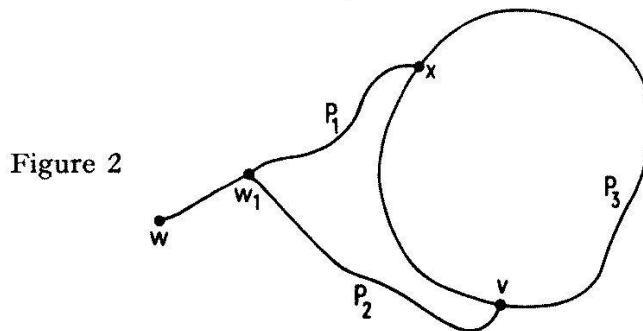
Conversely, suppose $G = B$. Since G is not w -cyclic, there is a cycle C in G not containing w . Since w can be adjacent to at most one vertex of $C - v$, there is a vertex x in $G - V(C)$ adjacent to w . Note that $G - E(C)$ has only one nontrivial component H and $H - xw$ has only two vertices of odd degree; in particular, x and v are of odd degree. By Proposition 2, $H - xw$ has an eulerian $v - x$ trail T . Since T exhausts the edges in $G - xw$ incident with x and w , the path x, y and the cycle C lie in different components of $G - E(T)$.

We now consider the center of a vertex cyclic graph and show that it must contain any vertex for which the graph is vertex cyclic.

Theorem 8. If G is a v -cyclic graph, then $v \in Z(G)$.

Proof: Let $u \in V(G)$ be such that $d_G(u, v) = e_G(v)$ and suppose u is in block B of G . If $B \neq G$, then for each $w \in V(G) - V(B)$ we have that $e_G(w) \geq d_G(w, u) = d_G(w, v) + d_G(v, u) > d_G(v, u) = e(v)$ since v can be the only cut-vertex of G . In any event, $Z(G) \subseteq V(B)$ since $e(z) \leq e(v)$ for all $z \in Z(G)$.

Since u and v are in a block, there exists a cycle containing u and v . Let C be a smallest such cycle. Given any two vertices of C and a diagonal path joining them, the path must contain v . Since C is a smallest cycle, we have $\text{rad } C = e_G(v)$. Thus, $e_C(x) = \text{rad } C$ for each $x \in V(C)$. Since there are no shorter paths in G joining any two vertices of C , we also have $e_G(x) \geq e_C(x)$ for all $x \in V(C)$.



If $B = C$, we are done. So, suppose $B \neq C$. Let $w \in V(B - C)$. Then there is exactly one path P not containing v but joining w to C . Suppose P joins C at the vertex x . Let $H = G - E(C)$ and let w_1 be the vertex on $P - x$ closest to x which minimizes $d_P(w, w_1) + d_H(w_1, v)$. Let P_1 be the $w_1 - x$ subpath of P and let P_2 be a shortest $w_1 - v$ path in H . Clearly, P_1 and P_2 have only w_1 in common. Let P_3 be a shortest $x - v$ subpath of C containing u . Then P_1, P_2 and P_3 form a cycle C_1 (cf. Figure 2) and $\text{rad } C_1 \geq \text{rad } C$. As such, there is a vertex $s \in V(P_3)$ such that $d_{C_1}(w_1, s) \geq \text{rad } C$. By our choice of w_1 , there is no shorter $s - w_1$ path in G and we have that $e(w) \geq d(w, s) \geq d(w_1, s) \geq \text{rad } C \geq e(v)$. Hence, it follows that $v \in Z(G)$.

Given a set V of vertices of a graph G , the *induced subgraph* $\langle V \rangle$ of G has vertex set V and edge set $E = \{uv \in E(G) : u, v \in V\}$. It is well known that the center need not induce a connected subgraph. This is also the case for eulerian graphs. In particular, the graph in Figure 3 is eulerian, has center $\{u, v\}$, and $\langle \{u, v\} \rangle$ is not connected. However, this is not the case for vertex cyclic graphs.

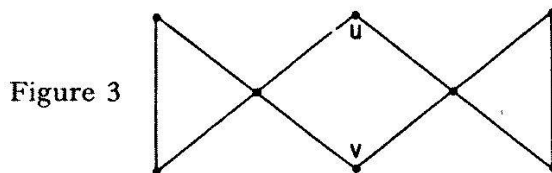


Figure 3

Theorem 9. If G is a vertex cyclic graph, then $Z(G)$ is connected.

Proof: Suppose G is v -cyclic. If $Z(G) = \{v\}$ or $d_G(v, z) \leq 1$ for all $z \in Z(G)$, then the result follows. So, suppose there is a $z \in Z(G)$ such that $d_G(v, z) \geq 2$ and let P be any shortest $v - z$ path. It suffices to show $V(P) \subseteq Z(G)$. To show this, it suffices to prove that the vertex u adjacent in P to z is also in $Z(G)$. Let P_1 be the $v - u$ subpath of P . This is shown in Figure 4, the remainder of which we will construct in the following.

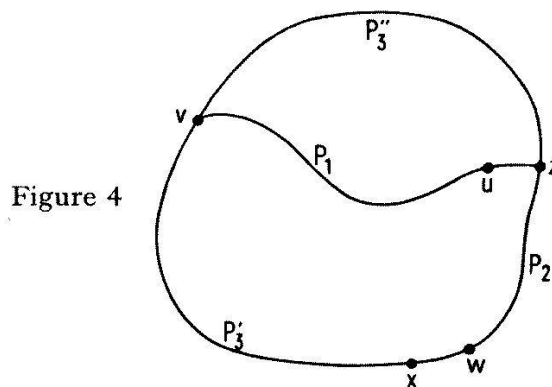


Figure 4

Suppose $u \notin Z(G)$. Since $z \in Z(G)$ and $uz \in E(G)$, we have that $e(u) = e(z) + 1$. Let $w \in V(G)$ be such that $d(u, w) = e(u)$. Then, $v \neq w \neq z$ and $d(w, z) = e(z)$.

Let P_2 be a shortest $z - w$ path. Since G is v -cyclic, v is the only vertex which P_1 and P_2 can have in common. In this case, the $z - v$ subpath P_2' of P_2 is of the same length as P . Hence, P_1 together with the $v - w$ subpath of P_2 is a $u - w$ walk of length $e(z)$. Since this is impossible, the paths P_1 and P_2 are disjoint.

Since $\deg_G w \geq 2$, there is a vertex x adjacent to w but not on P_2 . Then, $d_G(x, z) \leq e(z)$. Let P_3 be a shortest $x - z$ path. Clearly, $w \notin V(P_3)$ and v must be on P_3 . Let

P_3' and P_3'' respectively denote the $x - v$ and $v - z$ subpaths of P_3 . Then, P_3'' has the same length as P . Hence, the paths P_1 , P_3' , and $\langle\{w, x\}\rangle$ constitute a $u - w$ walk of at most $e(z)$. Since this is impossible, it must be the case that $u \in Z(G)$. As such, the theorem follows.

As a special case of the preceding theorem, we have the following corollary.

Corollary 10. If a graph G is randomly eulerian from any vertex, then the center $Z(G)$ induces a connected subgraph.

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Kleine Mitteilungen

When is the divisibility relation in a monoid a partial ordering?

1. Let $\langle M, \cdot, e \rangle$ be a monoid, i.e., a semigroup $\langle M, \cdot \rangle$ with an identity element e . We define the *divisibility relation* \leq in M by

$$x, y \in M; x \leq y \iff xu = y \text{ for some } u \in M.$$

By a non-trivial group we mean a group consisting of two or more elements. For $x \in M$, we denote the principal right ideal $\{xu; u \in M\}$ by xM . It is easily seen that, for arbitrary $x, y \in M$,

$$x \leq y \iff yM \subset xM \iff y \in xM \tag{1}$$

and that \leq is reflexive and transitive. SHWU-YENG T. LIN [5] raised the problem to find a necessary and sufficient condition on M for \leq to be a partial ordering. In this note we present an answer to this question and several remarks about it.

2. Criterion 1: For a monoid $\langle M, \cdot, e \rangle$, the following statements are equivalent:

$$(*) \quad x, u, v \in M; xuv = x \rightarrow xu = x,$$

$$(*)' \quad x, y \in M; xM = yM \rightarrow x = y,$$

$$(*)'' \quad \text{the divisibility relation } \leq \text{ in } M \text{ is a partial ordering.}$$

Proof: $(*) \rightarrow (*)'$: Assume that $xM = yM$. Then $x = xe \in xM = yM$ and, analogously, $y \in xM$. Therefore there exist $u, v \in M$ such that $y = xu$, $x = yv$, hence $xuv = x$, and $(*)$ implies $xu = x$, i.e., $x = y$. – $(*)' \rightarrow (*)''$: Suppose that $x \leq y$ and $y \leq x$. From (1) we conclude $xM = yM$, and by virtue of $(*)'$ we get $x = y$. – $(*)'' \rightarrow (*)$: Let be $xuv = x$. Then $xu \leq x$ and $x \leq xu$, and antisymmetry yields $xu = x$.