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# Vertex Cyclic Graphs

## § 1. Definitions

In the following, we consider graphs which are finite, undirected, loop-free, and without multiple edges.

Let u and v be vertices of a graph G. A u - v walk in G is an alterating sequence of vertices and edges beginning with u, ending with v, and such that each edge is incident with the vertices immediately preceeding and succeeding it. A u - v walk is open if  $u \neq v$  and closed if u = v. A trail is a walk without repeated edges and a path is a trail without repeated vertices. A circuit is a closed trail and a cycle is a circuit in which the intermediate vertices are not repeated.

A graph is connected if there is a walk joining every pair of vertices. A component of a graph G is a connected subgraph not properly contained in any other connected subgraph of G. A vertex v of a graph G is a cut-vertex of G if G - v has more components than does G. A graph G is a block if it is connected and has no cut-vertex. A block of a graph G is a subgraph of G which is maximal with respect to being a block.

Let V(G) and E(G) denote respectively the vertex and edge sets of a graph G. For vertices u and v of G, let the *distance*  $d_G(u, v)$  between u and v be the length of a shortest u - v path. The eccentricity e(v) for  $v \in V(G)$  is  $e(v) = \max\{d_G(u, v) : u \in V(G)\}$  and the radius rad G of G is rad  $G = \min\{e(v) : v \in V(G)\}$ . The center Z(G) of G is  $Z(G) = \{v \in V(G) : e(v) = \operatorname{rad} G\}$ .

In general, we will follow the conventions of Behzad and Chartrand [2].

### § 2. Randomly Eulerian graphs

Although we will consider 'randomly eulerian' graphs only to the extent that they exist in a larger class of graphs, they are introduced here for perspective and to illustrate the property we will investigate.

Let G be a connected graph. An *eulerian trail* in G is an open trail of G containing all edges of G and an *eulerian circuit* of G is a circuit of G which contains all edges of G. The graph G is *eulerian* if it has an eulerian circuit. Also, G is *randomly eulerian* from a vertex v if each trail with initial vertex v can be extended to an eulerian v - vcircuit of G.

Euler [3] characterized eulerian graphs and Ore [4] characterized graphs which are randomly eulerian from a vertex. In particular, if the degree  $\deg_G v$  of  $v \in V(G)$  is the number of edges in G incident with the vertex v, then we have the following well-known propositions.

*Proposition* 1. A connected graph is eulerian if and only if each vertex has even degree.

*Proposition 2.* A connected graph has an eulerian trail if and only if it has exactly two vertices of odd degree.

Proposition 3. An eulerian graph is randomly eulerian from a vertex v if and only if v belongs to every cycle of G.

It is a property inherent in the third proposition in which we are most interested and will pursue in the next section.

## § 3. Vertex cyclic graphs

A connected graph G with only cyclic blocks is *vertex cyclic* if it has a vertex which belongs to every cycle of G. In particular, a vertex cyclic graph G is *v-cyclic* if v is a vertex belonging to each cycle of G. To see that non-eulerian vertex cyclic graphs exist, it suffices to consider the complete bipartite graph K(2, 3).

Noting that a (p, q)-graph is a graph with p vertices and q edges, we have the following result.

Theorem 1. If G is a v-cyclic (p, q)-graph, then  $q \leq 2p - 3$ .

*Proof*: The graph G - v is a forest with p - 1 vertices and at most p - 2 edges. Since v can be adjacent to at most p - 1 vertices, G can have at most 2p - 3 edges.

For a graph G, let  $\Delta(G)$  and  $\delta(G)$  respectively denote the maximum and minimum degree among the vertices of G. Another consequence following from the proof of Theorem 1 is presented below.

Corollary 2. If G is vertex cyclic, then  $\delta(G) = 2$ .

In [1], Bäbler showed for a graph G randomly eulerian from a vertex v that  $\deg_G v = \varDelta(G)$ . We now generalize this result by showing this is a property of vertex cyclic graphs.

Theorem 3. If G is a v-cyclic graph, then  $\deg_G v = \Delta(G)$ .

*Proof*: Since H = G - v is a forest, we have that  $\Delta(H)$  does not exceed the number n of end-vertices of H. In G, the vertex v is adjacent to each end-vertex of H, thus,  $\Delta(H) \leq n \leq \deg_G v$ . Furthermore, for  $u \in V(H)$ ,  $\deg_G u = \deg_H u$  if  $uv \notin E(G)$  and  $\deg_G u = 1 + \deg_H u$  if  $uv \in E(G)$ . In any event,  $\deg_G u \leq \deg_G v$  for all  $u \in V(H)$  since the only edges in G which are not in H, are those edges joining v to some vertex in H.

We may now obtain the following result.

Theorem 4. If G is a v-cyclic graph and  $\deg_G w = \Delta(G)$  for some  $w \in V(G) - \{v\}$ , then G is also w-cyclic and  $\deg_G u = \delta(G)$  for all  $u \in V(G) - \{v, w\}$ .

*Proof*: If G is a cycle, then the theorem follows. So, suppose G is not a cycle. Let n be the number of end-vertices of the forest H = G - v. Then,  $\deg_G w = \deg_G v \ge n$ .

We now show that  $\deg_H w = n$ . Since *H* is acyclic, we have that  $\deg_H w \leq n$ . So, suppose  $\deg_H w < n$ . Then the edge vw must be in E(G) and we have that  $n \geq 1 + \deg_H w = \deg_G w = \deg_G v \geq n$ . Thus, *w* is an end-vertex of *H*. Hence,  $\deg_H w = 1$  which implies that  $\Delta(G) = \deg_G v = \deg_G w = 2$ . As such, *G* must be a cycle and this is a contradiction. Thus,  $\deg_H w = n$ .

Since  $\deg_H w = n$ , *H* is a tree. Also,  $\deg_G w = n$  implies all vertices of *H* different from w have degree at most two in *H*. As such, every path joining two distinct endvertices of *H* must contain *u*. Furthermore,  $\deg_G w = \deg_G v$  implies that *v* is adjacent to only end-vertices of *H* and possibly w. Consequently, every vertex of *G* different from *v* and *w* has degree  $\delta(G) = 2$  and *w* lies on every cycle of *G*.

As an immediate consequence of the preceding two results, we have the following.

Corollary 5. A graph is vertex cyclic from at least three vertices if and only if it is a cycle.

A property which is inherent in the eulerian situation, but not for vertex cyclic graphs in general, is presented below.

Lemma 6. If G is randomly eulerian from a vertex v and T is any trail with initial vertex v, then G - E(T) has at most one nontrivial component.

**Proof:** If T is a circuit, then each nontrivial component of G - E(T) is eulerian and, as such, contains a cycle which in turn contains v. Hence, G - E(T) has at most one nontrivial component and it contains v. If T is not a circuit, then we can extend T by a path P to yield a circuit T'. Let  $H_v$  be the component of G - E(T') containing v. Then, any other component of G - E(T') is trivial. Also, G - E(T) is G - E(T')together with the path P. Hence, given any component of G - E(T) not containing v, it must be trivial. Thus, the lemma follows.



Figure 1

To see that the result in Lemma 6 does not generalize to all vertex cyclic graphs, it suffices to consider the vertex cyclic graph G and the circuit T of G in Figure 1. Then, G - E(T) has two nontrivial components, neither of which contain v. However, there do exist noneulerian vertex cyclic graphs with this property. In fact, the following theorem characterizes all such vertex cyclic graphs.

Theorem 7. Let G be a v-cyclic graph. Then, G - E(T) has at most one nontrivial component for each trail T with initial vertex v if and only if:

- a) G is vertex cyclic from exactly two vertices; or
- b) G is eulerian.

**Proof:** The sufficiency of a) or b) follows from Theorem 4 and Lemma 6 respectively. To show the necessity of a) or b), we show that if G is noneulerian and vertex cyclic from only v, then G has a trail T with initial vertex v such that G - E(T) has at least two nontrivial components. We now consider the following two cases.

Case 1. Suppose G has a block B with at least two vertices different from v and both of odd degree. Then, there exist vertices u and w in B of odd degree together with a u - w path P containing neither v nor any other odd vertex.

For each edge e in G - E(P) incident with a vertex x in P, there is an x - vpath  $P_e$  in G - E(P). Also, for each pair of edges  $e_1$ , and  $e_2$  in G - E(P) incident with a vertex x in P, the paths  $P_{e_1}$  and  $P_{e_2}$  have only x and v in common. Since each vertex x of P has even degree in G - E(P), we may pair them to form cycles, the union of which is a v - v circuit  $C_x$  which exhausts the edges in G - E(P) incident with x. Also, if  $P_{e_1}$  and  $P_{e_2}$  correspond to edges  $e_1$  and  $e_2$  incident with distinct vertices  $x_1$  and  $x_2$  respectively, then  $P_{e_1}$  and  $P_{e_2}$  have only v in common. Consequently, the v - v circuits  $C_{x_1}$  and  $C_{x_2}$  have only v in common if  $x_1 \neq x_2$ . Thus, the union T of all the circuits  $C_x$ ,  $x \in V(P)$ , is a v - v circuit in G - E(P) exhausting all the edges in G - E(P) incident with vertices of P.

Let C be a cycle in T containing w. Then C has an edge xv incident with v but not with w. Then T - xv has a v - x trail T' exhausting the edges in G - E(P)incident with vertices of P. Thus, the paths P and x, v must be in different components of G - E(T').

Case 2. Suppose G has a block B with exactly one vertex w different from v and of odd degree. Necessarily, the vertex v must also be of odd degree in B.

Suppose G is not a block. Let u be a vertex of B different from v and adjacent to w. Then, B - uw is connected and has u and v as its only vertices of odd degree. By Proposition 2, B - uw has an eulerian u - v trail T. Let B' be any block of G different from B. Then the path u, w and the block B' lie in different components of G - E(T).

Conversely, suppose G = B. Since G is not w-cyclic, there is a cycle C in G not containing w. Since w can be adjacent to atmost one vertex of C - v, there is a vertex x in G - V(C) adjacent to w. Note that G - E(C) has only one nontrivial component H and H - xw has only two vertices of odd degree; in particular, x and v are of odd degree. By Proposition 2, H - xw has an eulerian v - x trail T. Since T exhausts the edges in G - xw incident with x and w, the path x, y and the cycle C lie in different components of G - E(T).

We now consider the center of a vertex cyclic graph and show that it must contain any vertex for which the graph is vertex cyclic.

Theorem 8. If G is a v-cyclic graph, then  $v \in Z(G)$ .

*Proof*: Let  $u \in V(G)$  be such that  $d_G(u, v) = e_G(v)$  and suppose u is in block B of G. If  $B \neq G$ , then for each  $w \in V(G) - V(B)$  we have that  $e_G(w) \geq d_G(w, u) = d_G(w, v) + d_G(v, u) > d_G(v, u) = e(v)$  since v can be the only cut-vertex of G. In any event,  $Z(G) \subseteq V(B)$  since  $e(z) \leq e(v)$  for all  $z \in Z(G)$ .

Since u and v are in a block, there exists a cycle containing u and v. Let C be a smallest such cycle. Given any two vertices of C and a diagonal path joining them, the path must contain v. Since C is a smallest cycle, we have rad  $C = e_G(v)$ . Thus,  $e_C(x) = \operatorname{rad} C$  for each  $x \in V(C)$ . Since there are no shorter paths in G joining any two vertices of C, we also have  $e_G(x) \ge e_C(x)$  for all  $x \in V(C)$ .



If B = C, we are done. So, suppose  $B \neq C$ . Let  $w \in V(B - C)$ . Then there is exactly one path P not containing v but joining w to C. Suppose P joins C at the vertex x. Let H = G - E(C) and let  $w_1$  be the vertex on P - x closest to x which minimizes  $d_P(w, w_1) + d_H(w_1, v)$ . Let  $P_1$  be the  $w_1 - x$  subpath of P and let  $P_2$  be a shortest  $w_1 - v$  path in H. Clearly,  $P_1$  and  $P_2$  have only  $w_1$  in common. Let  $P_3$  be a shortest x - v subpath of C containing u. Then  $P_1$ ,  $P_2$  and  $P_3$  form a cycle  $C_1$  (cf. Figure 2) and rad  $C_1 \ge rad C$ . As such, there is a vertex  $s \in V(P_3)$  such that  $d_{C_1}(w_1, s)$  $\ge$  rad C. By our choice of  $w_1$ , there is no shorter  $s - w_1$  path in G and we have that  $e(w) \ge d(w, s) \ge d(w_1, s) \ge rad C \ge e(v)$ . Hence, it follows that  $v \in Z(G)$ .

Given a set V of vertices of a graph G, the *induced subgraph*  $\langle V \rangle$  of G has vertex set V and edge set  $E = \{uv \in E(G) : u, v \in V\}$ . It is well known that the center need not induce a connected subgraph. This is also the case for eulerian graphs. In particular, the graph in Figure 3 is eulerian, has center  $\{u, v\}$ , and  $\langle \{u, v\} \rangle$  is not connected. However, this is not the case for vertex cyclic graphs.



Theorem 9. If G is a vertex cyclic graph, then Z(G) is connected.

**Proof:** Suppose G is v-cyclic. If  $Z(G) = \{v\}$  or  $d_G(v, z) \leq 1$  for all  $z \in Z(G)$ , then the result follows. So, suppose there is a  $z \in Z(G)$  such that  $d_G(v, z) \geq 2$  and let P be any shortest v - z path. It suffices to show  $V(P) \subseteq Z(G)$ . To show this, it suffices to prove that the vertex u adjacent in P to z is also in Z(G). Let  $P_1$  be the v - usubpath of P. This is shown in Figure 4, the remainder of which we will construct in the following.



Suppose  $u \notin Z(G)$ . Since  $z \in Z(G)$  and  $uz \in E(G)$ , we have that e(u) = e(z) + 1. Let  $w \in V(G)$  be such that d(u, w) = e(u). Then,  $v \neq w \neq z$  and d(w, z) = e(z).

Let  $P_2$  be a shortest z - w path. Since G is v-cyclic, v is the only vertex which  $P_1$  and  $P_2$  can have in common. In this case, the z - v subpath  $P'_2$  of  $P_2$  is of the same length as P. Hence,  $P_1$  together with the v - w subpath of  $P_2$  is a u - w walk of length e(z). Since this is impossible, the paths  $P_1$  and  $P_2$  are disjoint.

Since deg<sub>G</sub> $w \ge 2$ , there is a vertex x adjacent to w but not on  $P_2$ . Then,  $d_G(x, z) \le e(z)$ . Let  $P_3$  be a shortest x - z path. Clearly,  $w \notin V(P_3)$  and v must be on  $P_3$ . Let

 $P'_{\mathbf{s}}$  and  $P''_{\mathbf{s}}$  respectively denote the x - v and v - z subpaths of  $P_{\mathbf{s}}$ . Then,  $P''_{\mathbf{s}}$  has the same length as P. Hence, the paths  $P_1$ ,  $P'_{\mathbf{s}}$ , and  $\langle \{w, x\} \rangle$  constitute a u - w walk of at most e(z). Since this is impossible, it must be the case that  $u \in Z(G)$ . As such, the theorem follows.

As a special case of the preceding theorem, we have the following corollary.

Corollary 10. If a graph G is randomly eulerian from any vertex, then the center Z(G) induces a connected subgraph.

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## Kleine Mitteilungen

## When is the divisibility relation in a monoid a partial ordering?

**1.** Let  $\langle M, \cdot, e \rangle$  be a monoid, i.e., a semigroup  $\langle M, \cdot \rangle$  with an identity element *e*. We define the *divisibility relation*  $\leq$  in *M* by

 $x, y \in M; x \leq y : \leftrightarrow xu = y$  for some  $u \in M$ .

By a non-trivial group we mean a group consisting of two or more elements. For  $x \in M$ , we denote the principal right ideal  $\{xu; u \in M\}$  by xM. It is easily seen that, for arbitrary  $x, y \in M$ ,

$$x \le y \iff yM \subseteq xM \iff y \in xM \tag{1}$$

and that  $\leq$  is reflexive and transitive. SHWU-YENG T. LIN [5] raised the problem to find a necessary and sufficient condition on M for  $\leq$  to be a partial ordering. In this note we present an answer to this question and several remarks about it.

**2.** Criterion 1: For a monoid  $\langle M, \cdot, e \rangle$ , the following statements are equivalent:

(\*)  $x, u, v \in M$ ;  $xuv = x \rightarrow xu = x$ ,

(\*')  $x, y \in M; xM = yM \rightarrow x = y$ ,

(\*") the divisibility relation  $\leq$  in M is a partial ordering.

*Proof*: (\*)  $\rightarrow$  (\*'): Assume that xM = yM. Then  $x = xe \in xM = yM$  and, analogously,  $y \in xM$ . Therefore there exist  $u, v \in M$  such that y = xu, x = yv, hence xuv = x, and (\*) implies xu = x, i.e., x = y.  $-(*') \rightarrow (*'')$ : Suppose that  $x \leq y$  and  $y \leq x$ . From (1) we conclude xM = yM, and by virtue of (\*') we get x = y.  $-(*'') \rightarrow$  (\*): Let be xuv = x. Then  $xu \leq x$  and  $x \leq xu$ , and antisymmetry yields xu = x.