Zeitschrift: Elemente der Mathematik
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 29 (1974)
Heft: 6

Artikel: A triangle transformation
Autor: Bottema, O.
DOI: https://doi.org/10.5169/seals-29907

Nutzungsbedingungen
Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Mehr erfahren

Conditions d'utilisation
L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. En savoir plus

Terms of use
The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. Find out more

Download PDF: 18.02.2026

ETH-Bibliothek Zürich, E-Periodica, https://www.e-periodica.ch

stehende Polytop ist wiederum ein reguläres 16-Zell mit der Kantenlänge $\sqrt{2}$, also den beiden anderen kongruent.

Beide Beispiele stehen nach H. Groemer [3] in engem Zusammenhang damit, dass es Parkettierungen des E_4 mit lauter regulären 24-Zellen (alle in gleicher Drehlage) bzw. mit lauter regulären 16-Zellen (in drei verschiedenen Drehlagen) gibt (vgl. H. S. M. Coxeter [1], S. 296). Da solche Parkettierungen mit regulären 120-Zellen nicht existieren, ist es vermutlich schwierig, die Zerlegungsgleichheit des 120-Zells mit einem Hyperwürfel explizit zu realisieren.

P. Mürner, Gymnasium Interlaken

LITERATURVERZEICHNIS

[1] H. S. M. COXETER, *Regular Polytopes*, (Methuen, London 1948).
[2] M. DEHN, *Über den Rauminhalt*, Math. Ann. *55*, 465–478 (1901).
[3] H. GROEMER, *Über Würfel- und Raumzerlegungen*, El. Math. *19*, 25–27 (1964).
[4] H. HADWIGER, *Translationsinvariante, additive und schwachstetige Polyederfunktionale*, Arch. Math. *3*, 387–394 (1952).
[5] H. HADWIGER, *Vorlesungen über Inhalt, Oberfläche und Isoperimetrie*, Springer-Verlag 1957.
[6] B. JESSEN, *Zur Algebra der Polytope*, Nachr. Akad. Wiss. Göttingen II 1972, 47–53.
[7] B. JESSEN, und A. THORUP, *The Algebra of Polytopes in Affine Spaces* (1973, noch unpubliziert).
[8] H. LEBESGUE, *Sur les subdivisions des polyèdres réguliers en polyèdres réguliers*, Publ. math. Univ. Belgrade *6-7*, 183–188 (1938).
[9] H. LEBESGUE, *Sur l'équivalence des polyèdres, en particulier des polèdres réguliers, et sur la dissection des polyèdres réguliers en polyèdres réguliers*, Ann. Soc. Polon. Math. *17*, 193–226 (1939).

A Triangle Transformation

1. The configuration of a triangle on the sides of which polygons of a certain kind are described is a much studied theme in elementary geometry. The following variant does not seem to be well-known.

On the sides of a given triangle ABC similar isosceles triangles BCA, CAB, ABC, are constructed (Fig. 1), with bases BC, CA, AB, all outward or all inward, the base angle φ being taken positive or negative respectively ($-\pi/2 < \varphi < \pi/2$). The operation thus defined which transforms the triangle $\Delta = ABC$ into $\Delta_1 = A_1B_1C_1$ will be denoted by $T(\varphi)$.

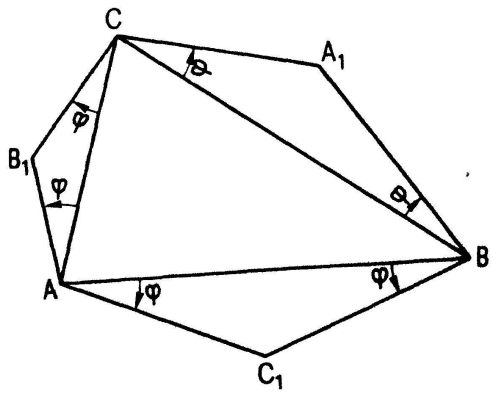

Fig. 1

The sides of Δ, Δ_1 are a, b, c and a_1, b_1, c_1, the oriented areas F and F_1, $S = a^2 + b^2 + c^2$, $S_1 = a_1^2 + b_1^2 + c_1^2$. As $A_1 B = A_1 C = (1/2)\, a \cos^{-1} \varphi$ etc., the cosine rule gives

$$\begin{aligned} 4a_1^2 &= \cos^{-2}\varphi\,\{b^2 + c^2 - 2\,b\,c\,\cos(\alpha + 2\,\varphi)\} \\ &= \cos^{-2}\varphi\,\{b^2 + c^2 - (b^2 + c^2 - a^2)\,(1 - 2\sin^2\varphi) + 2\,b\,c\sin\alpha\,\sin 2\varphi\} \\ &= (2\,b^2 + 2\,c^2 - a^2)\,t^2 + a^2 + 8\,F\,t\,, \end{aligned} \tag{1.1}$$

with $t = \tan\varphi$. For $4b_1^2$ and $4c_1^2$ we obtain analogous formulas. Hence

$$S_1 = \frac{1}{4}\,(3t^2 + 1)\,S + 6\,t\,F\,. \tag{1.2}$$

For the area of Δ_1 we have

$$F_1 = F + \sum(\text{area } BA_1C) - \sum(\text{area } B_1AC_1)\,. \tag{1.3}$$

Obviously

$$\text{area } BA_1C = \frac{1}{4}\,t\,a^2\,, \tag{1.4}$$

and

$$\begin{aligned} \text{area } B_1AC_1 &= \frac{1}{8}\cos^{-2}\varphi\cdot bc\,\sin(\alpha + 2\varphi) \\ &= \frac{1}{8}\cos^{-2}\varphi\,\{2F\,(\cos^2\varphi - \sin^2\varphi) + (b^2 + c^2 - a^2)\,\sin\varphi\cos\varphi\} \\ &= \frac{1}{4}\,(1 - t^2)\,F + \frac{1}{8}\,t\,(b^2 + c^2 - a^2)\,. \end{aligned} \tag{1.5}$$

Hence from 1.4 and 1.5

$$\text{area } BA_1C - \text{area } B_1AC_1 = \frac{1}{4}\,(t^2 - 1)\,F + \frac{1}{8}\,t\,(3a^2 - b^2 - c^2)\,,$$

and 1.3 gives us

$$F_1 = \frac{1}{4}\,(3\,t^2 + 1)\,F + \frac{1}{8}\,t\,S \tag{1.6}$$

The formulas (1.2) and (1.6) express F_1 and S_1 as linear functions of F and S. They have been derived recently in a different context, for $\varphi > 0$ only and with a less simple proof of (1.6) by Toscano [1]. They are valid for all values of φ; if F_1 and F have different signs the orientation of Δ_1 is the opposite of that of Δ.

2. F and S are related to the well-known Brocard's angle ω of the triangle [2]. We have, accepting a negative value of ω for a triangle with negative orientation

$$\cot\omega = S/4\,F \tag{2.1}$$

One always has $\cot^2\omega \geqq 3$, or $-\pi/6 \leqq \omega \leqq \pi/6$, with equality only for equilateral triangles.

From (1.2) and (1.6) it follows

$$\cot\,\omega_1 = \frac{(3\,t^2 + 1)\cot\,\omega + 6\,t}{2\,t\cot\omega + (3\,t^2 + 1)}\,, \tag{2.2}$$

which means that ω_1 depends only on t and ω. *If two triangles are equibrocardian their transforms by $T(\varphi)$ are equibrocardian as well.*

The right-hand side of (2.2) is a linear function of $\cot\omega$ the determinant of which is $(3t^2-1)^2$. Hence it is singular only if $t = \pm\sqrt{1/3}$; then for any ω we have $\cot\omega_1 = \pm\sqrt{3}$. Therefore *by $T(\pm\pi/6)$ any triangle is tranformed into an equilateral triangle,* a well-known theorem.

We divide the set of all triangles into classes $K(\omega)$, the elements of a class being the triangles with a given ω. The classes $K(\pm\pi/6)$ contain the equilateral triangles, $K(0)$ is the class of degenerated triangles. The conclusion is: any $T(\varphi)$ permutes the classes $K(\omega)$.

(2.2) may be written as follows

$$(3t^2+1)\ (\cot\omega_1 - \cot\omega) + 2t\ \cot\omega_1\ \cot\omega - 6t = 0 \tag{2.3}$$

$\omega_1 = \omega$ implies

$$t\,(\cot^2\omega - 3) = 0 \tag{2.4}$$

Hence the two classes of equilateral triangles are invariant for any $T(\varphi)$ and any class $K(\omega)$ is invariant for $T(0)$. Both properties are obvious: $T(0)$ transforms a triangle into that of the midpoints of its sides.

If ω and ω_1 are given, (2.3) is a quadratic equation for t:

$$3t^2\,(\cot\omega_1 - \cot\omega) + 2t\,(\cot\omega_1\,\cot\omega - 3) + (\cot\omega_1 - \cot\omega) = 0 \tag{2.5}$$

Its discriminant d satisfies

$$d = (\cot^2\omega_1 - 3)\ (\cot^2\omega - 3) \gtreqqless 0 \tag{2.6}$$

Hence 2.5 has real roots t_1, t_2; *in general, two transformations $T(\varphi_1)$, $T(\varphi_2)$ exist which transform a given class $K(\omega)$ into a given class $K(\omega_1)$.* One always has $t_1 t_2 = 1/3$.

If t_0 is a root of (2.5) then $-t_0$ is one of the equation with ω and ω_1 interchanged. *If $T(t)$ transforms $K(\omega)$ into $K(\omega_1)$ the two transforming $K(\omega_1)$ into $K(\omega)$ are $T(-t)$ and $T(-1/3\,t)$.* In particular: Δ and $T(-t)\{T(t)\cdot\Delta\}$ are equibrocardian.

If $T(t_1)\cdot\Delta = \Delta_1$ and $T(t_2)\cdot\Delta_1 = \Delta_2$, we have

$$\cot\omega_2 = \frac{\{3\,(t_1+t_2)^2 + (3\,t_1 t_2 + 1)^2\}\cos\omega + 6\,(t_1+t_2)\ (3\,t_1 t_2 + 1)}{2\,(t_1+t_2)\,(3\,t_1 t_2 + 1)\cos\omega + \{3(t_1+t_2)^2 + (3\,t_1 t_2 + 1)^2\}}$$

or, if $3\,t_1 t_2 + 1 \neq 0$,

$$\cos\omega_2 = \frac{(3\,t_{12}^2 + 1)\cos\omega + 6\,t_{12}}{2\,t_{12}\cos\omega + (3\,t_{12}^2 + 1)}, \tag{2.7}$$

with

$$t_{12} = \frac{t_1 + t_2}{3\,t_1 t_2 + 1}, \tag{2.8}$$

giving *the multiplication rule* for two transformations of the set: $T(t_2)\cdot\{T(t_1)\cdot\Delta\}$ and $T(t_{12})\cdot\Delta$ are equibrocardian triangles. The multiplication is *commutative.* If Δ is a given triangle with Brocard angle ω there are two transformations T such that $\omega_1 = 0$ and hence $\cot\omega_1 = \infty$. From (2.5) it follows that they are $T(t_i)$, $i = 1,2$, such that t_i are the roots of

$$3\,t^2 + 2\,t\cot\omega + 1 = 0\,, \tag{2.9}$$

that is

$$t_i = \frac{1}{3}\{-\cot\omega \pm (\cot^2\omega - 3)^{1/2}\}\,. \tag{2.10}$$

If on the sides of a triangle we construct isosceles triangles with $\tan\varphi = t_1$ or t_2 the points A_1, B_1, C_1 are collinear. As $t_1 t_2 = 1/3$, $t_1 + t_2 = (-2/3)\cot\omega$ we have $\varphi_1 + \varphi_2 = \omega - \pi/2$.

3. Properties of the transformation $T(t)$ may also be found by vector algebra. We denote the point P of the plane by the vector $\overline{P}$ from the origin O to P; the unit vector perpendicular to the plane by $\bar{e}$.

We obtain

$$2\overline{A}_1 = (\overline{B} + \overline{C}) + t\bar{e}\times(\overline{B} - \overline{C}),\ 2\overline{B}_1 = (\overline{C} + \overline{A}) + t\,\bar{e}\times(\overline{C} - \overline{A}),$$
$$2\overline{C}_1 = (\overline{A} + \overline{B}) + t\,\bar{e}\times(\overline{A} - \overline{B}) \tag{3.1}$$

As $\overline{A}_1 + \overline{B}_1 + \overline{C}_1 = \overline{A} + \overline{B} + \overline{C}$, we conclude: *the triangles Δ and $T\cdot\Delta$ have the same centroid.*

Furthermore if $\Delta_1 = T(t_1)\cdot\Delta$ amd $\Delta_2 = T(t_2)\cdot\Delta_1$ we have

$$2\overline{A}_2 = (\overline{B}_1 + \overline{C}_1) + t_2\bar{e}\times(\overline{B}_1 - \overline{C}_1)$$

or

$$4\overline{A}_2 = (2\overline{A} + \overline{B} + \overline{C}) + (t_1 + t_2)\,\bar{e}\times(\overline{C} - \overline{B}) + t_1 t_2\,(2\overline{A} - \overline{B} - \overline{C})\,, \tag{3.2}$$

and analogously for $4\,B_2$ and $4\,\overline{C}_2$. If we take the origin O at the centroid G of ABC we obtain

$$4\overline{A}_2 = (1 + 3\,t_1 t_2)\,\overline{A} + (t_1 + t_2)\,\bar{e}\times(\overline{C} - \overline{B})\,. \tag{3.3}$$

Two special cases are of interest. If $t_1 + t_2 = 0$, $-t_2 = +t_1 = t$ one has

$$4\overline{A}_2 = (1 - 3\,t^2)\,\overline{A},\ 4\overline{B}_2 = (1 - 3\,t^2)\,\overline{B},\ 4\overline{C}_2 = (1 - 3\,t^2)\,\overline{C}\,. \tag{3.4}$$

Therefore: *If on the sides of ABC we describe outward (inward) isosceles triangles with base angle φ and then on those of $A_1B_1C_1$ inward (outward) isosceles triangles with the same angle φ then $A_2B_2C_2$ and ABC are homothetic with respect to the centroid G, with the scale factor* $(1/4)\,(1 - 3\tan^2\varphi)$. The two triangles are congruent if $\tan\varphi = \pm\sqrt{5/3}$.

If $1 + 3\,t_1 t_2 = 0$, $t_1 + t_2 = t$ we obtain

$$4\overline{A}_2 = t\,\bar{e}\times(\overline{C} - \overline{B}),\quad 4\overline{B}_2 = t\bar{e}\times(\overline{A} - \overline{C}),\quad 4\overline{C}_2 = t\bar{e}\times(\overline{B} - \overline{A})\,. \tag{3.5}$$

Hence the medians of $A_2B_2C_2$ are perpendicular to the sides of ABC and equal to $(3/8)\,ta$, $(3/8)\,tb$, $(3/8)\,tc$; from this is follows that the sides of $A_2B_2C_2$ are proportional to the medians of ABC with ratio $t/2$.

O. Bottema, Delft

REFERENCES

[1] L. TOSCANO, *Sur deux suites récurrentes du second ordre, linéaires et homogènes*, Математически Весник, *9(24)*, СТР. 131–140 (1972).

[2] Cf. T. LALESCO, *La géométrie du triangle*, (Paris, 1937), 48–55, 117–120; R. A. JOHNSON, *Advanced Euclidean Geometry* (Dover Publ. New York, 1960), Ch. XVI, The Brocard configuration, 263–286, Ch. XVII, Equibrocardal triangles, 287–301.