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stehende Polytop ist wiederum ein reguläres 16-Zell mit der Kantenlänge $\sqrt{2}$, also den beiden anderen kongruent.

Beide Beispiele stehen nach H. Groemer [3] in engem Zusammenhang damit, dass es Parkettierungen des E_4 mit lauter regulären 24-Zellen (alle in gleicher Drehlage) bzw. mit lauter regulären 16-Zellen (in drei verschiedenen Drehlagen) gibt (vgl. H. S. M. Coxeter [1], S. 296). Da solche Parkettierungen mit regulären 120-Zellen nicht existieren, ist es vermutlich schwierig, die Zerlegungsgleichheit des 120-Zells mit einem Hyperwürfel explizit zu realisieren.

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A Triangle Transformation

1. The configuration of a triangle on the sides of which polygons of a certain kind are described is a much studied theme in elementary geometry. The following variant does not seem to be well-known.

On the sides of a given triangle ABC similar isosceles triangles BCA_1 , CAB_1 , ABC_1 are constructed (Fig. 1), with bases BC , CA , AB , all outward or all inward, the base angle φ being taken positive or negative respectively ($-\pi/2 < \varphi < \pi/2$). The operation thus defined which transforms the triangle $\Delta = ABC$ into $\Delta_1 = A_1B_1C_1$ will be denoted by $T(\varphi)$.

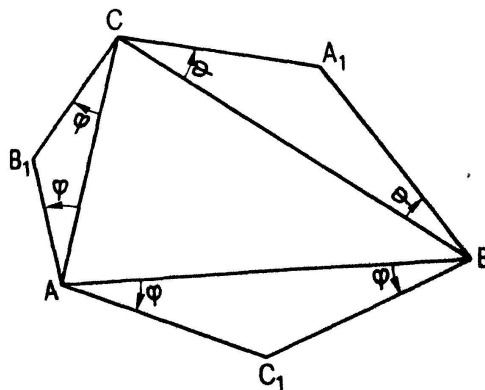


Fig. 1

The sides of Δ , Δ_1 are a, b, c and a_1, b_1, c_1 , the oriented areas F and F_1 , $S = a^2 + b^2 + c^2$, $S_1 = a_1^2 + b_1^2 + c_1^2$. As $A_1B = A_1C = (1/2) a \cos^{-1} \varphi$ etc., the cosine rule gives

$$\begin{aligned} 4a_1^2 &= \cos^{-2} \varphi \{b^2 + c^2 - 2bc \cos(\alpha + 2\varphi)\} \\ &= \cos^{-2} \varphi \{b^2 + c^2 - (b^2 + c^2 - a^2)(1 - 2\sin^2 \varphi) + 2bc \sin \alpha \sin 2\varphi\} \\ &= (2b^2 + 2c^2 - a^2)t^2 + a^2 + 8Ft, \end{aligned} \quad (1.1)$$

with $t = \tan \varphi$. For $4b_1^2$ and $4c_1^2$ we obtain analogous formulas. Hence

$$S_1 = \frac{1}{4} (3t^2 + 1) S + 6tF. \quad (1.2)$$

For the area of Δ_1 we have

$$F_1 = F + \sum (\text{area } BA_1C) - \sum (\text{area } B_1AC_1). \quad (1.3)$$

Obviously

$$\text{area } BA_1C = \frac{1}{4} t a^2, \quad (1.4)$$

and

$$\begin{aligned} \text{area } B_1AC_1 &= \frac{1}{8} \cos^{-2} \varphi \cdot bc \sin(\alpha + 2\varphi) \\ &= \frac{1}{8} \cos^{-2} \varphi \{2F(\cos^2 \varphi - \sin^2 \varphi) + (b^2 + c^2 - a^2) \sin \varphi \cos \varphi\} \\ &= \frac{1}{4} (1 - t^2) F + \frac{1}{8} t (b^2 + c^2 - a^2). \end{aligned} \quad (1.5)$$

Hence from 1.4 and 1.5

$$\text{area } BA_1C - \text{area } B_1AC_1 = \frac{1}{4} (t^2 - 1) F + \frac{1}{8} t (3a^2 - b^2 - c^2),$$

and 1.3 gives us

$$F_1 = \frac{1}{4} (3t^2 + 1) F + \frac{1}{8} t S \quad (1.6)$$

The formulas (1.2) and (1.6) express F_1 and S_1 as linear functions of F and S . They have been derived recently in a different context, for $\varphi > 0$ only and with a less simple proof of (1.6) by Toscano [1]. They are valid for all values of φ ; if F_1 and F have different signs the orientation of Δ_1 is the opposite of that of Δ .

2. F and S are related to the well-known Brocard's angle ω of the triangle [2]. We have, accepting a negative value of ω for a triangle with negative orientation

$$\cot \omega = S/4F \quad (2.1)$$

One always has $\cot^2 \omega \geq 3$, or $-\pi/6 \leq \omega \leq \pi/6$, with equality only for equilateral triangles.

From (1.2) and (1.6) it follows

$$\cot \omega_1 = \frac{(3t^2 + 1) \cot \omega + 6t}{2t \cot \omega + (3t^2 + 1)}, \quad (2.2)$$

which means that ω_1 depends only on t and ω . *If two triangles are equibrocardian their transforms by $T(\varphi)$ are equibrocardian as well.*

The right-hand side of (2.2) is a linear function of $\cot \omega$ the determinant of which is $(3t^2 - 1)^2$. Hence it is singular only if $t = \pm \sqrt{1/3}$; then for any ω we have $\cot \omega_1 = \pm \sqrt{3}$. Therefore by $T(\pm \pi/6)$ any triangle is transformed into an equilateral triangle, a well-known theorem.

We divide the set of all triangles into classes $K(\omega)$, the elements of a class being the triangles with a given ω . The classes $K(\pm \pi/6)$ contain the equilateral triangles, $K(0)$ is the class of degenerated triangles. The conclusion is: any $T(\varphi)$ permutes the classes $K(\omega)$.

(2.2) may be written as follows

$$(3t^2 + 1)(\cot \omega_1 - \cot \omega) + 2t \cot \omega_1 \cot \omega - 6t = 0 \quad (2.3)$$

$\omega_1 = \omega$ implies

$$t(\cot^2 \omega - 3) = 0 \quad (2.4)$$

Hence the two classes of equilateral triangles are invariant for any $T(\varphi)$ and any class $K(\omega)$ is invariant for $T(0)$. Both properties are obvious: $T(0)$ transforms a triangle into that of the midpoints of its sides.

If ω and ω_1 are given, (2.3) is a quadratic equation for t :

$$3t^2(\cot \omega_1 - \cot \omega) + 2t(\cot \omega_1 \cot \omega - 3) + (\cot \omega_1 - \cot \omega) = 0 \quad (2.5)$$

Its discriminant d satisfies

$$d = (\cot^2 \omega_1 - 3)(\cot^2 \omega - 3) \geq 0 \quad (2.6)$$

Hence 2.5 has real roots t_1, t_2 ; in general, two transformations $T(\varphi_1), T(\varphi_2)$ exist which transform a given class $K(\omega)$ into a given class $K(\omega_1)$. One always has $t_1 t_2 = 1/3$.

If t_0 is a root of (2.5) then $-t_0$ is one of the equation with ω and ω_1 interchanged. If $T(t)$ transforms $K(\omega)$ into $K(\omega_1)$ the two transforming $K(\omega_1)$ into $K(\omega)$ are $T(-t)$ and $T(-1/3t)$. In particular: Δ and $T(-t)\{\Delta\}$ are equibrocardian.

If $T(t_1)\Delta = \Delta_1$ and $T(t_2)\Delta_1 = \Delta_2$, we have

$$\cot \omega_2 = \frac{\{3(t_1 + t_2)^2 + (3t_1 t_2 + 1)^2\} \cos \omega + 6(t_1 + t_2)(3t_1 t_2 + 1)}{2(t_1 + t_2)(3t_1 t_2 + 1) \cos \omega + \{3(t_1 + t_2)^2 + (3t_1 t_2 + 1)^2\}}$$

or, if $3t_1 t_2 + 1 \neq 0$,

$$\cos \omega_2 = \frac{(3t_{12}^2 + 1) \cos \omega + 6t_{12}}{2t_{12} \cos \omega + (3t_{12}^2 + 1)}, \quad (2.7)$$

with

$$t_{12} = \frac{t_1 + t_2}{3t_1 t_2 + 1}, \quad (2.8)$$

giving the multiplication rule for two transformations of the set: $T(t_2) \cdot \{T(t_1) \cdot \Delta\}$ and $T(t_{12}) \cdot \Delta$ are equibrocardian triangles. The multiplication is commutative. If Δ is a given triangle with Brocard angle ω there are two transformations T such that $\omega_1 = 0$ and hence $\cot \omega_1 = \infty$. From (2.5) it follows that they are $T(t_i)$, $i = 1, 2$, such that t_i are the roots of

$$3t^2 + 2t \cot \omega + 1 = 0, \quad (2.9)$$

that is

$$t_i = \frac{1}{3} \{-\cot\omega \pm (\cot^2\omega - 3)^{1/2}\}. \quad (2.10)$$

If on the sides of a triangle we construct isosceles triangles with $\tan\varphi = t_1$ or t_2 the points A_1, B_1, C_1 are collinear. As $t_1 t_2 = 1/3$, $t_1 + t_2 = (-2/3)\cot\omega$ we have $\varphi_1 + \varphi_2 = \omega - \pi/2$.

3. Properties of the transformation $T(t)$ may also be found by vector algebra. We denote the point P of the plane by the vector \bar{P} from the origin O to P ; the unit vector perpendicular to the plane by \bar{e} .

We obtain

$$\begin{aligned} 2\bar{A}_1 &= (\bar{B} + \bar{C}) + t\bar{e} \times (\bar{B} - \bar{C}), & 2\bar{B}_1 &= (\bar{C} + \bar{A}) + t\bar{e} \times (\bar{C} - \bar{A}), \\ 2\bar{C}_1 &= (\bar{A} + \bar{B}) + t\bar{e} \times (\bar{A} - \bar{B}) \end{aligned} \quad (3.1)$$

As $\bar{A}_1 + \bar{B}_1 + \bar{C}_1 = \bar{A} + \bar{B} + \bar{C}$, we conclude: *the triangles Δ and $T \cdot \Delta$ have the same centroid.*

Furthermore if $\Delta_1 = T(t_1) \cdot \Delta$ and $\Delta_2 = T(t_2) \cdot \Delta_1$ we have

$$2\bar{A}_2 = (\bar{B}_1 + \bar{C}_1) + t_2 \bar{e} \times (\bar{B}_1 - \bar{C}_1)$$

or

$$4\bar{A}_2 = (2\bar{A} + \bar{B} + \bar{C}) + (t_1 + t_2) \bar{e} \times (\bar{C} - \bar{B}) + t_1 t_2 (2\bar{A} - \bar{B} - \bar{C}), \quad (3.2)$$

and analogously for $4\bar{B}_2$ and $4\bar{C}_2$. If we take the origin O at the centroid G of ABC we obtain

$$4\bar{A}_2 = (1 + 3t_1 t_2) \bar{A} + (t_1 + t_2) \bar{e} \times (\bar{C} - \bar{B}). \quad (3.3)$$

Two special cases are of interest. If $t_1 + t_2 = 0$, $-t_2 = +t_1 = t$ one has

$$4\bar{A}_2 = (1 - 3t^2) \bar{A}, \quad 4\bar{B}_2 = (1 - 3t^2) \bar{B}, \quad 4\bar{C}_2 = (1 - 3t^2) \bar{C}. \quad (3.4)$$

Therefore: *If on the sides of ABC we describe outward (inward) isosceles triangles with base angle φ and then on those of $A_1 B_1 C_1$ inward (outward) isosceles triangles with the same angle φ then $A_2 B_2 C_2$ and ABC are homothetic with respect to the centroid G , with the scale factor $(1/4)(1 - 3\tan^2\varphi)$. The two triangles are congruent if $\tan\varphi = \pm\sqrt{5/3}$.*

If $1 + 3t_1 t_2 = 0$, $t_1 + t_2 = t$ we obtain

$$4\bar{A}_2 = t\bar{e} \times (\bar{C} - \bar{B}), \quad 4\bar{B}_2 = t\bar{e} \times (\bar{A} - \bar{C}), \quad 4\bar{C}_2 = t\bar{e} \times (\bar{B} - \bar{A}). \quad (3.5)$$

Hence the medians of $A_2 B_2 C_2$ are perpendicular to the sides of ABC and equal to $(3/8)ta$, $(3/8)tb$, $(3/8)tc$; from this it follows that the sides of $A_2 B_2 C_2$ are proportional to the medians of ABC with ratio $t/2$.

O. Bottema, Delft

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