

# On 1-factorability and edge-colorability of cartesian products of graphs

Autor(en): **Himmelwright, P.E. / Williamson, J.E.**

Objektyp: **Article**

Zeitschrift: **Elemente der Mathematik**

Band (Jahr): **29 (1974)**

Heft 3

PDF erstellt am: **21.09.2024**

Persistenter Link: <https://doi.org/10.5169/seals-29897>

## **Nutzungsbedingungen**

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.

Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.

Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

## **Haftungsausschluss**

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

Hence  $G_0 \neq G$ ,  $G_{\{0, \infty\}} \neq G_0$ . It is also clear that  $(G_{\{0, \infty\}}, k - \{0\}, *)$  is transitive, for if  $x \neq 0$  and  $y \neq 0$ , then

$$\begin{bmatrix} y & 0 \\ 0 & x \end{bmatrix} * x = y.$$

Hence by Theorem 3,  $(G, X, *)$  is 3-fold transitive. We note that  $(G, X, *)$  is not 4-fold transitive, for then  $(G_{\{0, \infty\}}, k - \{0\}, *)$  would be 2-fold transitive.

David P. Sumner, University of South Carolina, USA

### *Acknowledgement*

I would like to thank D. J. Foulis and L. N. Mann for conducting the seminar on transformation groups during which this note was first written.

### REFERENCE

- [1] H. WIELANDT, *Finite Permutation Groups*, trans. by R. Bercov, Academic Press, New York, 1964.

## On 1-Factorability and Edge-Colorability of Cartesian Products of Graphs

There is no characterization of 1-factorable graphs. Thus, it is natural that many of the results on this topic have been the determination of classes of 1-factorable graphs. The object of this paper is to present a sufficient condition for the 1-factorability of the cartesian product of two graphs. We begin with some notation and definitions.

The vertex set of a graph  $G$  will be denoted by  $V(G)$  and its edge set by  $E(G)$ . In this paper we consider only finite, undirected graphs without loops or multiple edges. Let  $G$  and  $H$  be two nonempty graphs for which  $V(G) = V(H)$  and  $E(G) \cap E(H) = \Phi$ ; then the graph  $G'$  is the *sum* of  $G$  and  $H$ , written  $G' = G + H$ , if  $V(G') = V(G)$  and  $E(G') = E(G) \cup E(H)$ . A *1-factor* of a graph  $G$  is a spanning 1-regular subgraph of  $G$ . A graph is *1-factorable* if it can be expressed as a sum of edge-disjoint 1-factors. The *cartesian product* (or *product*) of the graph  $G$  with the graph  $H$ , denoted by  $G \times H$ , is defined by:  $V(G \times H) = V(G) \times V(H)$ ;  $E(G \times H) = \{[(u_1, v_1), (u_2, v_2)] \mid u_1 = u_2 \text{ and } v_1 v_2 \in E(H), \text{ or } v_1 = v_2 \text{ and } u_1 u_2 \in E(G)\}$ .

An assignment of  $n$  colors to the edges of a nonempty graph  $G$  so that adjacent edges are colored differently is an  *$n$ -edge-coloring* of  $G$ . The minimum  $n$  for which a graph  $G$  is  $n$ -edge-colorable is its *edge-chromatic number*  $\chi_1(G)$ . By a theorem of Vizing [2], the edge-chromatic number  $\chi_1(G)$  of a graph  $G$  is bounded by:  $\Delta(G) \leq \chi_1(G) \leq \Delta(G) + 1$ , where  $\Delta(G)$  is the maximum degree of  $G$ . If  $G$  is regular, then  $G$  is 1-factorable if and only if  $\chi_1(G) = \Delta(G)$ . Hence any theorem concerning the 1-factorability of regular graphs has as an immediate corollary a result concerning edge-colorability, which is useful since there is also no characterization of those graphs which are  $\Delta(G)$ -edge-colorable. For other notations and definitions, we follow [1].

If  $K_2$  denotes the complete graph on two vertices, then  $K_2 \times H$ , where  $H$  is any regular graph, is shown to be 1-factorable in the following lemma.

**Lemma:** If  $H$  is a regular graph, then  $K_2 \times H$  is 1-factorable.

*Proof.* If  $H$  is 1-factorable, then the result follows immediately. Hence we consider the case that  $H$  is not 1-factorable. If  $H$  is an  $r$ -regular graph, then by a previous remark,  $\chi_1(H) = r + 1$ . Let an  $(r + 1)$ -edge-coloring of  $H$  be given and let  $C_1, C_2, \dots, C_{r+1}$  be the edge-color classes of  $E(H)$ . Now  $K_2 \times H$  contains two disjoint copies of  $H$ . Let the  $(r + 1)$ -edge-coloring of  $H$  be applied to these disjoint copies, and assign to each edge  $[(u_1, v), (u_2, v)]$  of  $K_2 \times H$  the only color among the  $r + 1$  colors which was assigned to no edge of  $H$  incident with  $v$ . Hence  $K_2 \times H$  may be  $(r + 1)$ -edge-colored. But  $K_2 \times H$  is  $(r + 1)$ -regular. Hence  $\chi_1(K_2 \times H) = r + 1$ , and  $K_2 \times H$  is 1-factorable.

We now state and prove the main result.

**Theorem:** If  $G$  is a 1-factorable graph and  $H$  is a regular graph, then  $G \times H$  is a 1-factorable graph.

*Proof:* Let  $G$  be a 1-factorable,  $r$ -regular graph of order  $p_1$  with 1-factors  $G_1, G_2, \dots, G_r$ , and let  $H$  be an  $s$ -regular graph of order  $p_2$ . Then consider the subgraphs  $G_1 \times H, G_2 \times \bar{K}_{p_2}, \dots, G_r \times \bar{K}_{p_2}$  of  $G \times H$ , where  $\bar{K}_{p_2}$  denotes the graph consisting of  $p_2$  isolated vertices. Note that these subgraphs are mutually edge-disjoint subgraphs spanning  $G \times H$ , and  $G \times H = G_1 \times H + \sum_{i=2}^r G_i \times \bar{K}_{p_2}$ . Moreover, the subgraphs  $G_2 \times \bar{K}_{p_2}, \dots, G_r \times \bar{K}_{p_2}$  are 1-regular and thus are 1-factors of  $G \times H$ . Hence if  $G_1 \times H$  is 1-factorable,  $G \times H$  is 1-factorable. Now  $G_1 \times H$  is a spanning  $(s + 1)$ -regular subgraph of  $G \times H$  consisting of  $p_1/2$  components each of which is isomorphic to  $K_2 \times H$ . By the Lemma,  $K_2 \times H$  is 1-factorable and of regularity  $s + 1$ . Let the 1-factors of  $K_2 \times H$  be  $F_1, F_2, \dots, F_{s+1}$  in a 1-factorization of  $K_2 \times H$ . Select in every component of  $G_1 \times H$ , the same 1-factor  $F_k$ , where  $1 \leq k \leq s + 1$ , and designate the resultant subgraph of  $G_1 \times H$  by  $F'_k$ . Then by the choice of  $F'_k$  it follows that  $F'_k$  is a spanning 1-regular subgraph of  $G_1 \times H$ , and hence a 1-factor of  $G_1 \times H$ . In a like manner mutually edge-disjoint 1-factors  $F'_1, F'_2, \dots, F'_{s+1}$  of  $G_1 \times H$  can be obtained from each of  $F_1, F_2, \dots, F_{s+1}$ , respectively. Therefore  $G_1 \times H$  is 1-factorable, which implies that  $G \times H$  is also 1-factorable as previously indicated.

*Corollary:* If  $G$  and  $H$  are regular graphs, and  $\chi_1(G) = \Delta(G)$ , then  $\chi_1(G \times H) = \Delta(G) + \Delta(H)$ .

We remark that the theorem gives a sufficient condition for 1-factorability which is, however, not a necessary condition, since 1-factorable products of two non-1-factorable graphs are known. An example of this is the cartesian product of the Petersen graph with a triangle.

P. E. Himelwright and J. E. Williamson,  
Grand Valley State College, and Southern Illinois University, USA

#### REFERENCES

- [1] M. BEHZAD and G. CHARTRAND, *Introduction of the Theory of Graphs*, Allyn and Bacon, Inc. 1972.
- [2] V. G. VIZING, *On an estimate of the chromatic class of a p-graph* (Russian), *Diskret. Analiz.* 3, 25-30 (1964).