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## A Criterion for $n$ -Fold Transitivity of Transformation Groups

Let  $G$  be a group and let  $X$  be a nonempty set. An *action*  $*$  on  $X$  is a function  $*$ :  $G \times X \rightarrow X$  such that for every  $g, h \in G$  and  $x \in X$ , (i)  $(gh) * x = g * (h * x)$  and (ii)  $1 * x = x$ .

A triple  $(G, X, *)$  where  $*$  is an action of  $G$  on  $X$  is called a *transformation group*. For  $S \subseteq X$  the stability subgroup of  $S$  is  $G_S = \{g \in G \mid g * s = s \text{ for every } s \in S\}$ . (We will write  $G_x$  instead of  $G_{\{x\}}$ .)

If  $n$  is a positive integer, we say that  $G$  is  *$n$ -fold transitive* whenever for every two sequences  $x_1, x_2, \dots, x_n$  and  $y_1, y_2, \dots, y_n$  each consisting of  $n$  distinct elements of  $X$ , there exists  $g \in G$  such that  $g * x_i = y_i$  for every  $i = 1, 2, \dots, n$ .

We note that if  $*$  is an action of  $G$  on  $X$ , then for any  $S \subseteq X$ ,  $*$  induces an action of  $G_S$  on  $X - S$ .

The next theorem is well known (see, for example, [1], Theorem 9.1).

**Theorem 1:** Let  $(G, X, *)$  be transitive. Then for  $n \geq 2$ ,  $(G, X, *)$  is  $n$ -fold transitive iff there exists an  $x \in X$  such that  $(G_x, X - \{x\}, *)$  is  $(n - 1)$ -fold transitive.

It is our purpose in this note to derive a corollary (Theorem 2) of this theorem which is sometimes more convenient to use. The essential idea is to replace the transitive condition on  $(G, X, *)$  by a restriction on the stability subgroups.

**Lemma 1:** If  $(G, X, *)$  is a transformation group, then  $(G, X, *)$  is 2-fold transitive iff there exists an  $x \in X$  such that  $G_x \neq G$  and  $(G_x, X - \{x\}, *)$  is transitive.

*Proof:* Clearly if  $(G, X, *)$  is 2-fold transitive then the given condition holds for any  $x \in X$ .

Now suppose  $x \in X$  such that  $G_x \neq G$  and  $(G_x, X - \{x\}, *)$  is transitive. Let  $y, z \in X$ . If  $y, z \in X - \{x\}$ , then there exists  $g \in G_x$  such that  $g * y = z$ . If  $y = z = x$ , then  $1 * y = z$ . If  $y = x$  and  $z \neq x$ , then since  $G_x \neq G$ , there exists  $h \in G$  such that  $h * x \neq x$ . So there is an  $r \in G_x$  such that  $r * (h * x) = z$  and so  $(rh) * x = z$ . If  $y \neq x$ ,  $z = x$  and  $h$  is as before, then there exists  $t \in G_x$  such that  $t * y = h * x = h * z$  so that  $(h^{-1}t) * y = z$ . Hence  $(G, X, *)$  is transitive so that by Theorem 1 it is 2-fold transitive.

**Lemma 2:** Let  $n \geq 2$  and  $|X| > 1$ . Then  $(G, X, *)$  is  $n$ -fold transitive iff there exists an  $x \in X$  such that  $G_x \neq G$  and  $(G_x, X - \{x\}, *)$  is  $(n - 1)$ -fold transitive.

*Proof:* Assume  $G_x \neq G$  and  $(G_x, X - \{x\}, *)$  is  $(n - 1)$ -fold transitive. Then by Lemma 1,  $(G, X, *)$  is transitive and hence by Theorem 1 it is  $n$ -fold transitive. If  $(G, X, *)$  is  $n$ -fold transitive, then the given condition holds for all  $x \in X$ .

**Theorem 2:** For  $|X| \geq n \geq 2$ ,  $(G, X, *)$  is  $n$ -fold transitive iff there exists  $S \subseteq X$  with  $S = \{t_1, t_2, \dots, t_{n-1}\}$  such that if  $S_k = \{t_1, t_2, \dots, t_k\}$  for each  $k = 1, 2, \dots, n - 1$ , then

- a)  $G_{t_1} \neq G$  and  $G_{S_k} \neq G_{S_{k+1}}$  for all  $k = 1, 2, \dots, n - 1$ ; and
- b)  $(G_S, X - S, *)$  is transitive.

*Proof:* Since any  $n$ -fold transformation group clearly satisfies (a) and (b), we need only show the other half.

The case  $n = 2$  is the content of Lemma 1.

Suppose the theorem holds for all integers greater than one and less than  $n$ . Let  $S \subseteq X$  be  $S = \{t_1, t_2, \dots, t_{n-1}\}$  such that Conditions (a) and (b) hold. Then  $S^* = \{t_2, \dots, t_{n-1}\}$  satisfies the conditions of the theorem for the transformation group  $(G_{t_1}, X - \{t_1\}, *)$  and hence this transformation group is  $(n - 1)$ -fold transitive. But then by Lemma 2,  $(G, X, *)$  is  $n$ -fold transitive.

We next consider an application of this result. Let  $k$  be a field and let  $G$  be the group  $GL(k, 2)$  of all nonsingular  $2 \times 2$  matrices over  $k$ . Let  $*$  be the action of  $G$  on  $k \cup \{\infty\}$  defined by

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} * z = \begin{cases} \frac{\alpha z + \beta}{\gamma z + \delta}, & \text{if } z \neq \infty, \gamma z + \delta \neq 0 \\ \infty, & \text{if } z \neq \infty, \gamma z + \delta = 0 \\ \alpha/\gamma, & \text{if } z = \infty, \gamma \neq 0 \\ \infty, & \text{if } z = \infty, \gamma = 0. \end{cases}$$

We will apply the previous result to show that  $(G, X, *)$  is 3-fold transitive. First we note the following special case of Theorem 2 obtained by letting  $n = 3$ .

**Theorem 3:** For  $|X| \geq 3$ ,  $(G, X, *)$  is 3-fold transitive iff there exist  $x, y \in X$  such that  $G_x \neq G$ ,  $G_{\{x,y\}} \neq G_x$  and  $(G_{\{x,y\}}, X - \{x, y\}, *)$  is transitive.

Note that  $G_{\{x,y\}} = G_x \cap G_y$ . It is easy to see that

$$G_\infty = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \mid a, b, c \in k, ac \neq 0 \right\}$$

and

$$G_0 = \left\{ \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \mid a, b, c \in k, ac \neq 0 \right\}.$$

So

$$G_{\{0,\infty\}} = G_0 \cap G_\infty = \left\{ \begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix} \mid a, c \in k, ac \neq 0 \right\}.$$

Hence  $G_0 \neq G$ ,  $G_{\{0,\infty\}} \neq G_0$ . It is also clear that  $(G_{\{0,\infty\}}, k - \{0\}, *)$  is transitive, for if  $x \neq 0$  and  $y \neq 0$ , then

$$\begin{bmatrix} y & 0 \\ 0 & x \end{bmatrix} * x = y.$$

Hence by Theorem 3,  $(G, X, *)$  is 3-fold transitive. We note that  $(G, X, *)$  is not 4-fold transitive, for then  $(G_{\{0,\infty\}}, k - \{0\}, *)$  would be 2-fold transitive.

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## On 1-Factorability and Edge-Colorability of Cartesian Products of Graphs

There is no characterization of 1-factorable graphs. Thus, it is natural that many of the results on this topic have been the determination of classes of 1-factorable graphs. The object of this paper is to present a sufficient condition for the 1-factorability of the cartesian product of two graphs. We begin with some notation and definitions.

The vertex set of a graph  $G$  will be denoted by  $V(G)$  and its edge set by  $E(G)$ . In this paper we consider only finite, undirected graphs without loops or multiple edges. Let  $G$  and  $H$  be two nonempty graphs for which  $V(G) = V(H)$  and  $E(G) \cap E(H) = \emptyset$ ; then the graph  $G'$  is the *sum* of  $G$  and  $H$ , written  $G' = G + H$ , if  $V(G') = V(G)$  and  $E(G') = E(G) \cup E(H)$ . A *1-factor* of a graph  $G$  is a spanning 1-regular subgraph of  $G$ . A graph is *1-factorable* if it can be expressed as a sum of edge-disjoint 1-factors. The *cartesian product* (or *product*) of the graph  $G$  with the graph  $H$ , denoted by  $G \times H$ , is defined by:  $V(G \times H) = V(G) \times V(H)$ ;  $E(G \times H) = \{[(u_1, v_1), (u_2, v_2)] \mid u_1 = u_2 \text{ and } v_1 v_2 \in E(H), \text{ or } v_1 = v_2 \text{ and } u_1 u_2 \in E(G)\}$ .

An assignment of  $n$  colors to the edges of a nonempty graph  $G$  so that adjacent edges are colored differently is an  *$n$ -edge-coloring* of  $G$ . The minimum  $n$  for which a graph  $G$  is  $n$ -edge-colorable is its *edge-chromatic number*  $\chi_1(G)$ . By a theorem of Vizing [2], the edge-chromatic number  $\chi_1(G)$  of a graph  $G$  is bounded by:  $\Delta(G) \leq \chi_1(G) \leq \Delta(G) + 1$ , where  $\Delta(G)$  is the maximum degree of  $G$ . If  $G$  is regular, then  $G$  is 1-factorable if and only if  $\chi_1(G) = \Delta(G)$ . Hence any theorem concerning the 1-factorability of regular graphs has as an immediate corollary a result concerning edge-colorability, which is useful since there is also no characterization of those graphs which are  $\Delta(G)$ -edge-colorable. For other notations and definitions, we follow [1].

If  $K_2$  denotes the complete graph on two vertices, then  $K_2 \times H$ , where  $H$  is any regular graph, is shown to be 1-factorable in the following lemma.