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## Non-Hamiltonian Square-minus-two

The square  $G^2$  of a graph  $G$  with vertex set  $V(G)$  is defined as the graph having  $V(G)$  for its vertex set, and two vertices of  $G^2$  are connected by an edge in  $G^2$  if and only if their distance in  $G$  is at most 2. Fleishner [2] proved Nash-Williams and Plummer's Conjecture that  $G^2$  is Hamiltonian for every 2-connected graph  $G$ ; Chartrand and Kapoor [1] proved that  $G^2 - v$  is Hamiltonian for every 2-connected graph  $G$  with  $\overline{V(G)} \geq 4$  and every  $v \in V(G)$ .

It has been conjectured [4] that  $G^2 - u - v$  is Hamiltonian for every 2-connected graph  $G$  with  $\overline{V(G)} \geq 5$  and for every  $u, v \in V(G)$ .

The purpose of this note is to show that this conjecture is false (see [3]), as follows:

**Theorem 1:** For every odd integer  $n$ ,  $n \geq 3$ , there exists a 2-connected graph  $G = G(n)$  with  $\overline{V(G)} = 3n + 2$ , such that  $G^2 - u - v$  is not Hamiltonian for some  $u, v \in V(G)$ .

The following is even a stronger result:

**Theorem 2:** For every integer  $n$ ,  $n \geq 2$ , there exists a 2-connected graph  $G = G(n)$  with  $\overline{V(G)} = 8n$  and  $V(G)$  contains  $2n$  vertices  $u_1, \dots, u_n, v_1, \dots, v_n$  such that  $G^2 - u_i - v_i$  is not Hamiltonian for every  $i$ ,  $1 \leq i \leq n$ .

We need the following simple

**Lemma:** If a graph  $G$  contains a simple  $u - v$  path  $u, ux, x, xy, y, yz, z, zv, v$  of length 4, such that  $x, y$  and  $z$  are 2-valent in  $G$ , then every simple path in  $G^2 - u - v$  that contains  $y$  as an inner vertex contains also the edges  $xy$  and  $yz$ .

*Proof:*  $xy$  and  $yz$  are the only edges of  $G^2 - u - v$  that contain  $y$  as one of their end points, hence they are contained in every simple path in  $G^2 - u - v$  that contains  $y$  as an inner vertex.

*Proof of Theorem 1:* For every odd  $n$ ,  $n \geq 3$ , let  $G(n)$  be the union of the  $n$   $u - v$  paths  $u, ux_i, x_i, x_i y_i, y_i, y_i z_i, z_i, z_i v, v$ , for all  $i = 1, \dots, n$ , where  $x_i, y_i$  and  $z_i$  are all different (see figure 1 with  $n = 5$ ); clearly  $G(n)$  is 2-connected, for all  $n \geq 3$ .

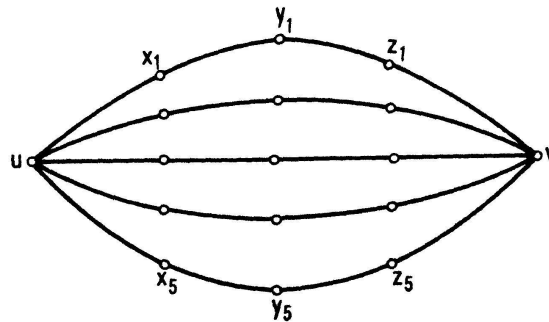


Figure 1

$G^2 - u - v$  consists of the two complete  $n$ -graphs  $K^*$  and  $K^{**}$ , on the vertices  $x_1, \dots, x_n$  and  $z_1, \dots, z_n$ , respectively, together with the vertices  $y_1, \dots, y_n$  and the edges  $x_i y_i, y_i z_i$  and  $x_i z_i$ , for all  $i = 1, \dots, n$ .

Suppose there exists a Hamiltonian cycle  $h$  in  $G^2 - u - v$ .  $h$  must contain all the vertices  $y_i$ , therefore it must contain by the Lemma all the edges  $x_i y_i$  and  $y_i z_i$ , for all  $i = 1, \dots, n$ ; since  $n > 1$ ,  $h$  does not contain any of the edges  $x_i z_i$ .  $h$  therefore runs from  $K^*$  to  $K^{**}$  and back an odd number of times, which is impossible; hence  $G^2 - u - v$  is non-Hamiltonian.

*Proof of Theorem 2:* For every integer  $n \geq 2$ , let the graph  $G = G(n)$  consist of the  $n$  cycles of length 8 of the vertices  $u_i, x_i, y_i, z_i, v_i, z'_i, y'_i, x'_i$  and  $u_i$  (in this cyclic order), for all  $1 \leq i \leq n$ , plus the edges  $v_i u_{i+1}$  for all  $1 \leq i \leq n$  (the last one of which being  $v_n u_1$ ).  $G$  is shown in Figure 2, with  $n = 4$ :

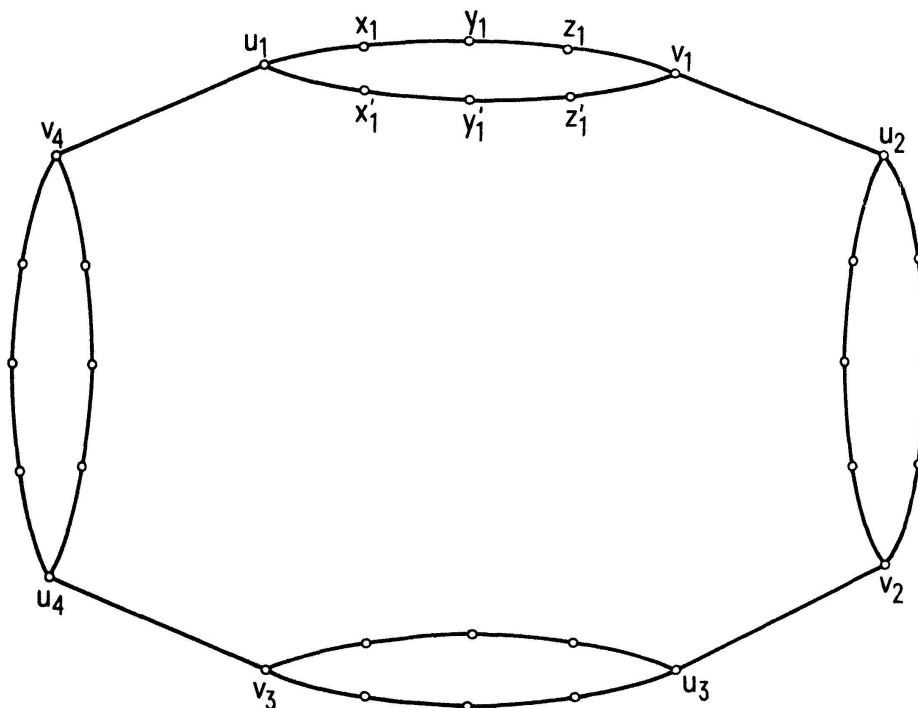


Figure 2

Clearly,  $G(n)$  is a 2-connected graph for all  $n \geq 2$ .

Suppose that for some  $i$ ,  $1 \leq i \leq n$ ,  $G^2 - u_i - v_i$  contains a simple cycle  $c$  that contains both  $y_i$  and  $y'_i$ ;  $c$  contains, by the Lemma, all the four edges  $x_i y_i$ ,  $y_i z_i$ ,  $x'_i y'_i$  and  $y'_i z'_i$ .  $c$  does not contain the edge  $x_i z_i$  ( $x'_i z'_i$ ), since otherwise  $c$  would have at least two connected components, one of which being a cycle of length 3. The vertex  $x_i$ , as any vertex of  $c$ , is of valence 2 in  $c$ , hence either the edge  $x_i v_{i-1}$  or else the edge  $x_i x'_i$  is in  $c$ ; similarly for the vertices  $x'_i, z_i$  and  $z'_i$ . It follows that  $c$  contains either the edge  $x_i x'_i$  or else the two edges  $x_i v_{i-1}$  and  $x'_i v_{i-1}$ ; similarly,  $c$  contains either the edge  $z_i z'_i$  or else the two edges  $z_i u_{i+1}$  and  $z'_i u_{i+1}$ . As a result,  $c$  is of length at most 8, while  $G^2 - u_i - v_i$  has  $8n - 2$  vertices,  $n \geq 2$ ; since  $8 < 8n - 2$ , for all  $n \geq 2$ , it follows that  $c$  does not contain all the vertices of  $G^2 - u_i - v_i$ , therefore  $G^2 - u_i - v_i$  is non-Hamiltonian. This completes the proof of Theorem 2.

*Remark:* It follows immediately from [1] that if  $G$  is a 2-connected graph, then  $G^2 - u - v$  has a Hamiltonian path for every  $u, v \in V(G)$ .

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## REFERENCES

- [1] G. CHARTRAND and S. F. KAPOOR, *The Square of Every 2-Connected Graph is 1-Hamiltonian*, to appear.
- [2] H. FLEISHNER, *The Square of Every Nonseparable Graph is Hamiltonian*, to appear.
- [3] J. ZAKS, announcement 1(B), Graph Theory Newsletter Vol. 1 (No. 4), 1972 edited by S. F. Kapoor, W.M.U. Kalamazoo, Michigan), p. 7.
- [4] (anonymous) Problem 1, Graph Theory Newsletter, Vol. 1. (No. 2), (1971), p. 3.

## Kleine Mitteilungen

## Proof of a Conjecture of H. Hadwiger

As part of a research problem [2], Hadwiger conjectured that every simple closed curve in  $E^3$  admits a nontrivial inscribed parallelogram. Schnirelman's method [4] [1] leads immediately to the following result:

**Theorem:** *Every simple closed  $C^2$  curve in  $E^3$  admits a nontrivial inscribed rhombus.*

Outline of proof: The statement for plane curves has been proved by Schnirelman [4] [1]. Every simple closed curve in  $E^3$  is homotopic to a plane Jordan curve. If the curve in  $E^3$  is not knotted, the homotopy is in fact an isotopy. If the curve is a knot, it may be deformed into a plane Jordan curve through a  $C^2$ -homotopy  $F(\alpha, t)$ ,  $0 \leq \alpha \leq 2\pi$ ,  $0 \leq t \leq 1$ , for which  $F(\alpha, t_0)$  is a simple closed curve except for finitely many values  $t_0$  for which  $F(\alpha, t_0)$ ,  $0 \leq \alpha \leq 2\pi$ , is a curve with one simple transversal selfintersection. Because of the compactness of the sets involved, a given smooth homotopy can be locally modified to satisfy the given conditions. The parametrization can be chosen so that the Jacobian matrix of  $F$  is nowhere singular. The theorem will be proved if we can show that it holds for all curves  $F(\alpha, t)$ ,  $t_0 \leq t < t_0 + \varepsilon$  if it holds for  $F(\alpha, t_0)$ .

By hypothesis, there exist four distinct parameter values  $\alpha_1, \alpha_2, \alpha_3, \alpha_4$  so that for  $F_i = F(\alpha_i, t_0)$  we have

$$|F_1 - F_2| = |F_2 - F_3| = |F_3 - F_4| = |F_4 - F_1| (\neq 0) \quad (1)$$

$$\det(F_1 - F_2, F_1 - F_3, F_1 - F_4) = 0$$

where  $\det$  denotes the determinant. The problem is to find four points  $F_i^*$  on  $F(\alpha, t)$ ,  $t_0 \leq t \leq t_0 + \varepsilon$ , that also satisfy conditions (1). We develop in a Taylor polynomial,

$$F_i^* = F_i + \frac{\delta F_i}{\delta \alpha_i} \Delta \alpha_i + \frac{\delta F_i}{\delta t} \Delta t + o(\Delta \alpha_i, \Delta t)$$

introduce the expression in (1) and develop as well. An appropriate form of the inverse function theorem says that under our differentiability assumptions the  $\Delta \alpha_i$  can be found if the linearized problem obtained by putting all  $o(\Delta \alpha_i, \Delta t) = 0$ , can be solved. From (1) one obtains a system of four nonhomogeneous linear equations (that can immediately be written down) for the four unknowns  $\Delta \alpha_i$  ( $i = 1, 2, 3, 4$ ). The matrix of the system has the form