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Autor:	Zaks, Joseph
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Non-Hamiltonian Square-minus-two

The square G^2 of a graph G with vertex set V(G) is defined as the graph having V(G) for its vertex set, and two vertices of G^2 are connected by an edge in G^2 if and only if their distance in G is at most 2. Fleishner [2] proved Nash-Williams and Plummer's Conjecture that G^2 is Hamiltonian for every 2-connected graph G; Chartrand and Kapoor [1] proved that $G^2 - v$ is Hamiltonian for every 2-connected graph G with $\overline{V(G)} \geq 4$ and every $v \in V(G)$.

It has been conjectured [4] that $G^2 - u - v$ is Hamiltonian for every 2-connected graph G with $\overline{V(G)} \ge 5$ and for every $u, v \in V(G)$.

The purpose of this note is to show that this conjecture is false (see [3]), as follows:

Theorem 1: For every odd integer $n, n \ge 3$, there exists a 2-connected graph G = G(n) with $\overline{V(G)} = 3n + 2$, such that $G^2 - u - v$ is not Hamiltonian for some $u, v \in V(G)$.

The following is even a stronger result:

Theorem 2: For every integer $n, n \ge 2$, there exists a 2-connected graph G = G(n) with $\overline{V(G)} = 8 n$ and V(G) contains 2 n vertices $u_1, \ldots, u_n, v_1, \ldots, v_n$ such that $G^2 - u_i - v_i$ is not Hamiltonian for every $i, 1 \le i \le n$.

We need the following simple

Lemma: If a graph G contains a simple u - v path u, ux, x, xy, y, yz, z, zv, v of length 4, such that x, y and z are 2-valent in G, then every simple path in $G^2 - u - v$ that contains y as an inner vertex contains also the edges x y and y z.

Proof: x y and y z are the only edges of $G^2 - u - v$ that contain y as one of their end points, hence they are contained in every simple path in $G^2 - u - v$ that contains y as an inner vertex.

Proof of Theorem 1: For every odd $n, n \ge 3$, let G(n) be the union of the n u - v paths $u, ux_i, x_i, x_iy_i, y_i, y_iz_i, z_i, z_iv, v$, for all i = 1, ..., n, where x_i, y_i and z_i are all different (see figure 1 with n = 5); clearly G(n) is 2-connected, for all $n \ge 3$.



 $G^2 - u - v$ consists of the two complete *n*-graphs K^* and K^{**} , on the vertices x_1, \ldots, x_n and z_1, \ldots, z_n , respectively, together with the vertices y_1, \ldots, y_n and the edges $x_i y_i, y_i z_i$ and $x_i z_i$, for all $i = 1, \ldots, n$.

Suppose there exists a Hamiltonian cycle h in $G^2 - u - v$. h must contain all the vertices y_i , therefore it must contain by the Lemma all the edges $x_i y_i$ and $y_i z_i$, for all i = 1, ..., n; since n > 1, h does not contain any of the egdes $x_i z_i$. h therefore runs from K^* to K^{**} and back an odd number of times, which is impossible; hence $G^2 - u - v$ is non-Hamiltonian.

Proof of Theorem 2: For every integer $n \ge 2$, let the graph G = G(n) consist of the *n* cycles of length 8 of the vertices u_i , x_i , y_i , z_i , v_i , z'_i , y'_i , x'_i and u_i (in this cyclic order), for all $1 \le i \le n$, plus the edges $v_i u_{i+1}$ for all $1 \le i \le n$ (the last one of which being $v_n u_1$). G is shown in Figure 2, with n = 4:



Figure 2

Clearly, G(n) is a 2-connected graph for all $n \ge 2$.

Suppose that for some $i, 1 \le i \le n, G^2 - u_i - v_i$ contains a simple cycle c that contains both y_i and y'_i ; c contains, by the Lemma, all the four edges $x_i y_i, y_i z_i, x'_i y'_i$ and $y'_i z'_i$. c does not contain the edge $x_i z_i (x'_i z'_i)$, since otherwise c would have at least two connected components, one of which being a cycle of length 3. The vertex x_i , as any vertex of c, is of valence 2 in c, hence either the edge $x_i v_{i-1}$ or else the edge $x_i x'_i$ is in c; similarly for the vertices x'_i, z_i and z'_i . It follows that c contains either the edge $x_i x'_i$ or else the two edges $x_i v_{i-1}$ and $x'_i v_{i-1}$; similarly, c contains either the edge $z_i z'_i$ or else the two edges $z_i u_{i+1}$ and $z'_i u_{i+1}$. As a result, c is of length at most 8, while $G^2 - u_i - v_i$ has 8n - 2 vertices, $n \ge 2$; since 8 < 8n - 2, for all $n \ge 2$, it follows that c does not contain all the vertices of $G^2 - u_i - v_i$, therefore $G^2 - u_i - v_i$ is non-Hamiltonian. This completes the proof of Theorem 2.

Remark: It follows immediately from [1] that if G is a 2-connected graph, then $G^2 - u - v$ has a Hamiltonian path for every $u, v \in V(G)$.

Joseph Zaks, Michigan State University, East Lansing, USA, and University of Haifa, Haifa, Israel.

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Kleine Mitteilungen

Proof of a Conjecture of H. Hadwiger

As part of a research problem [2], Hadwiger conjectured that every simple closed curve in E^3 admits a nontrivial inscribed parallelogram. Schnirelman's method [4] [1] leads immediately to the following result:

Theorem: Every simple closed C² curve in E³ admits a nontrivial inscribed rhombus.

Outline of proof: The statement for plane curves has been proved by Schnirelman [4] [1]. Every simple closed curve in E^3 is homotopic to a plane Jordan curve. If the curve in E^3 is not knotted, the homotopy is in fact an isotopy. If the curve is a knot, it may be deformed into a plane Jordan curve through a C²-homotopy $F(\alpha, t)$, $0 \le \alpha \le 2\pi$, $0 \le t \le 1$, for which $F(\alpha, t_0)$ is a simple closed curve except for finitely many values t_0 for which $F(\alpha, t_0)$, $0 \le \alpha \le 2\pi$, is a curve with one simple transversal selfintersection. Because of the compactness of the sets involved, a given smooth homotopy can be locally modified to satisfy the given conditions. The parametrization can be chosen so that the Jacobian matrix of F is nowhere singular. The theorem will be proved if we can show that it holds for all curves $F(\alpha, t)$, $t_0 \le t < t_0 + \varepsilon$ if it holds for $F(\alpha, t_0)$.

By hypothesis, there exist four distinct parameter values α_1 , α_2 , α_3 , α_4 so that for $F_i = F(\alpha_i, t_0)$ we have

$$|F_1 - F_2| = |F_2 - F_3| = |F_3 - F_4| = |F_4 - F_1| (\neq 0)$$

$$det (F_1 - F_2, F_1 - F_3, F_1 - F_4) = 0$$
(1)

where det denotes the determinant. The problem is to find four points F_i^* on $F(\alpha, t)$, $t_0 \leq t \leq t_0 + \varepsilon$, that also satisfy conditions (1). We develop in a Taylor polynomial,

$$F_i^* = F_i + \frac{\delta F_i}{\delta x_i} \Delta \alpha_i + \frac{\delta F_i}{\delta t} \Delta t + o(\Delta \alpha_i, \Delta t)$$

introduce the expression in (1) and develop as well. An appropriate form of the inverse function theorem says that under our differentiability assumptions the $\Delta \alpha_i$ can be found if the linearized problem obtained by putting all $o(\Delta \alpha_i, \Delta t) = 0$, can be solved. From (1) one obtains a system of four nonhomogeneous linear equations (that can immediately be written down) for the four unknowns $\Delta \alpha_i$ (i = 1, 2, 3, 4). The matrix of the system has the form