Zeitschrift: Elemente der Mathematik
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 29 (1974)
Heft: 2

Artikel: Non-hamiltonian square-minus-two
Autor: Zaks, Joseph
DOI: https://doi.org/10.5169/seals-29892

Non-Hamiltonian Square-minus-two

The square G^2 of a graph G with vertex set $V(G)$ is defined as the graph having $V(G)$ for its vertex set, and two vertices of G^2 are connected by an edge in G^2 if and only if their distance in G is at most 2. Fleishner [2] proved Nash-Williams and Plummer's Conjecture that G^2 is Hamiltonian for every 2-connected graph G; Chartrand and Kapoor [1] proved that $G^2 - v$ is Hamiltonian for every 2-connected graph G with $\overline{\overline{V(G)}} \geq 4$ and every $v \in V(G)$.

It has been conjectured [4] that $G^2 - u - v$ is Hamiltonian for every 2-connected graph G with $\overline{\overline{V(G)}} \geq 5$ and for every $u, v \in V(G)$.

The purpose of this note is to show that this conjecture is false (see [3]), as follows:

Theorem 1: For every odd integer n, $n \geq 3$, there exists a 2-connected graph $G = G(n)$ with $\overline{\overline{V(G)}} = 3n + 2$, such that $G^2 - u - v$ is not Hamiltonian for some $u, v \in V(G)$.

The following is even a stronger result:

Theorem 2: For every integer n, $n \geq 2$, there exists a 2-connected graph $G = G(n)$ with $\overline{\overline{V(G)}} = 8n$ and $V(G)$ contains $2n$ vertices $u_1, \ldots, u_n, v_1, \ldots, v_n$ such that $G^2 - u_i - v_i$ is not Hamiltonian for every i, $1 \leq i \leq n$.

We need the following simple

Lemma: If a graph G contains a simple $u - v$ path $u, ux, x, xy, y, yz, z, zv, v$ of length 4, such that x, y and z are 2-valent in G, then every simple path in $G^2 - u - v$ that contains y as an inner vertex contains also the edges xy and yz.

Proof: xy and yz are the only edges of $G^2 - u - v$ that contain y as one of their end points, hence they are contained in every simple path in $G^2 - u - v$ that contains y as an inner vertex.

Proof of Theorem 1: For every odd n, $n \geq 3$, let $G(n)$ be the union of the n $u - v$ paths $u, ux_i, x_i, x_iy_i, y_i, y_iz_i, z_i, z_iv, v$, for all $i = 1, \ldots, n$, where x_i, y_i and z_i are all different (see figure 1 with $n = 5$); clearly $G(n)$ is 2-connected, for all $n \geq 3$.

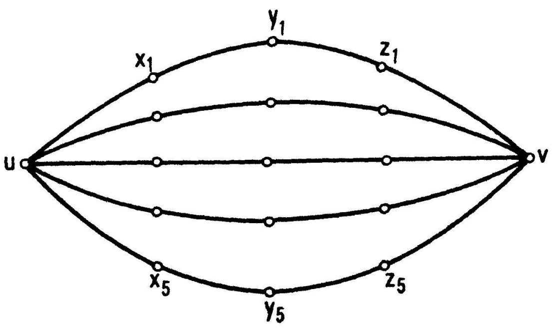

Figure 1

$G^2 - u - v$ consists of the two complete n-graphs K^* and K^{**}, on the vertices $x_1, \ldots, x_n$ and $z_1, \ldots, z_n$, respectively, together with the vertices $y_1, \ldots, y_n$ and the edges $x_i y_i$, $y_i z_i$ and $x_i z_i$, for all $i = 1, \ldots, n$.

Suppose there exists a Hamiltonian cycle h in $G^2 - u - v$. h must contain all the vertices y_i, therefore it must contain by the Lemma all the edges $x_i y_i$ and $y_i z_i$, for all $i = 1, \ldots, n$; since $n > 1$, h does not contain any of the egdes $x_i z_i$. h therefore runs from K^* to K^{**} and back an odd number of times, which is impossible; hence $G^2 - u - v$ is non-Hamiltonian.

Proof of Theorem 2: For every integer $n \geq 2$, let the graph $G = G(n)$ consist of the n cycles of length 8 of the vertices $u_i, x_i, y_i, z_i, v_i, z_i', y_i', x_i'$ and u_i (in this cyclic order), for all $1 \leq i \leq n$, plus the edges $v_i u_{i+1}$ for all $1 \leq i \leq n$ (the last one of which being $v_n u_1$). G is shown in Figure 2, with $n = 4$:

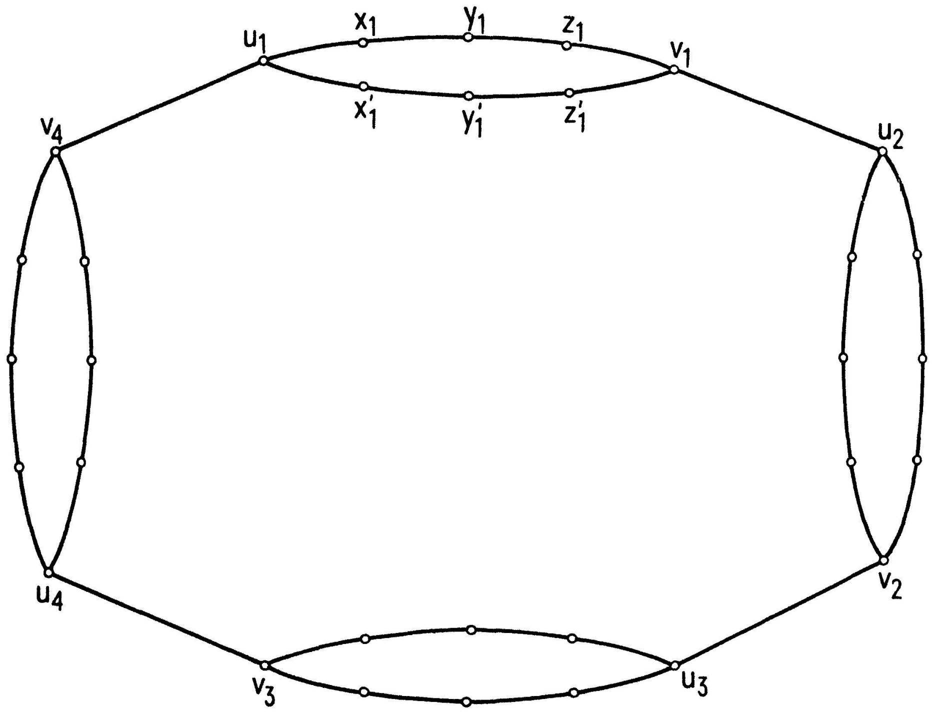

Figure 2

Clearly, $G(n)$ is a 2-connected graph for all $n \geq 2$.

Suppose that for some i, $1 \leq i \leq n$, $G^2 - u_i - v_i$ contains a simple cycle c that contains both y_i and y_i'; c contains, by the Lemma, all the four edges $x_i y_i$, $y_i z_i$, $x_i' y_i'$ and $y_i' z_i'$. c does not contain the edge $x_i z_i$ ($x_i' z_i'$), since otherwise c would have at least two connected components, one of which being a cycle of length 3. The vertex x_i, as any vertex of c, is of valence 2 in c, hence either the edge $x_i v_{i-1}$ or else the edge $x_i x_i'$ is in c; similarily for the vertices x_i', z_i and z_i'. It follows that c contains either the edge $x_i x_i'$ or else the two edges $x_i v_{i-1}$ and $x_i' v_{i-1}$; similarly, c contains either the edge $z_i z_i'$ or else the two edges $z_i u_{i+1}$ and $z_i' u_{i+1}$. As a result, c is of length at most 8, while $G^2 - u_i - v_i$ has $8n - 2$ vertices, $n \geq 2$; since $8 < 8n - 2$, for all $n \geq 2$, it follows that c does not contain all the vertices of $G^2 - u_i - v_i$, therefore $G^2 - u_i - v_i$ is non-Hamiltonian. This completes the proof of Theorem 2.

Remark: It follows immediately from [1] that if G is a 2-connected graph, then $G^2 - u - v$ has a Hamiltonian path for every $u, v \in V(G)$.

Joseph Zaks, Michigan State University, East Lansing, USA,
and University of Haifa, Haifa, Israel.

REFERENCES

[1] G. CHARTRAND and S. F. KAPOOR, *The Square of Every 2-Connected Graph is 1-Hamiltonian,* to appear.
[2] H. FLEISHNER, *The Square of Every Nonseparable Graph is Hamiltonian,* to appear.
[3] J. ZAKS, announcement 1(B), Graph Theory Newsletter Vol. 1 (No. 4), 1972 edited by S. F. Kapoor, W.M.U. Kalamazoo, Michigan), p. 7.
[4] (anonymous) Problem 1, Graph Theory Newsletter, Vol. 1. (No. 2), (1971), p. 3.

Kleine Mitteilungen

Proof of a Conjecture of H. Hadwiger

As part of a research problem [2], Hadwiger conjectured that every simple closed curve in E^3 admits a nontrivial inscribed parallelogram. Schnirelman's method [4] [1] leads immediately to the following result:

Theorem: *Every simple closed* C^2 *curve in* E^3 *admits a nontrivial inscribed rhombus.*

Outline of proof: The statement for plane curves has been proved by Schnirelman [4] [1]. Every simple closed curve in E^3 is homotopic to a plane Jordan curve. If the curve in E^3 is not knotted, the homotopy is in fact an isotopy. If the curve is a knot, it may be deformed into a plane Jordan curve through a C^2-homotopy $F(\alpha, t)$, $0 \leq \alpha \leq 2\pi$, $0 \leq t \leq 1$, for which $F(\alpha, t_0)$ is a simple closed curve except for finitely many values t_0 for which $F(\alpha, t_0)$, $0 \leq \alpha \leq 2\pi$, is a curve with one simple transversal selfintersection. Because of the compactness of the sets involved, a given smooth homotopy can be locally modified to satisfy the given conditions. The parametrization can be chosen so that the Jacobian matrix of F is nowhere singular. The theorem will be proved if we can show that it holds for all curves $F(\alpha, t)$, $t_0 \leq t < t_0 + \varepsilon$ if it holds for $F(\alpha, t_0)$.

By hypothesis, there exist four distinct parameter values $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ so that for $F_i = F(\alpha_i, t_0)$ we have

$$|F_1 - F_2| = |F_2 - F_3| = |F_3 - F_4| = |F_4 - F_1| \; (\neq 0) \qquad (1)$$
$$\det\,(F_1 - F_2,\ F_1 - F_3,\ F_1 - F_4) = 0$$

where det denotes the determinant. The problem is to find four points F_i^* on $F(\alpha, t)$, $t_0 \leq t \leq t_0 + \varepsilon$, that also satisfy conditions (1). We develop in a Taylor polynomial,

$$F_i^* = F_i + \frac{\delta F_i}{\delta x_i} \Delta\alpha_i + \frac{\delta F_i}{\delta t} \Delta t + o(\Delta\alpha_i, \Delta t)$$

introduce the expression in (1) and develop as well. An appropriate form of the inverse function theorem says that under our differentiability assumptions the $\Delta\alpha_i$ can be found if the linearized problem obtained by putting all $o(\Delta\alpha_i, \Delta t) = 0$, can be solved. From (1) one obtains a system of four nonhomogeneous linear equations (that can immediately be written down) for the four unknowns $\Delta\alpha_i$ $(i = 1, 2, 3, 4)$. The matrix of the system has the form