

Zeitschrift: Elemente der Mathematik
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 29 (1974)
Heft: 2

Artikel: Non-hamiltonian square-minus-two
Autor: Zaks, Joseph
DOI: <https://doi.org/10.5169/seals-29892>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 18.02.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

Non-Hamiltonian Square-minus-two

The square G^2 of a graph G with vertex set $V(G)$ is defined as the graph having $V(G)$ for its vertex set, and two vertices of G^2 are connected by an edge in G^2 if and only if their distance in G is at most 2. Fleishner [2] proved Nash-Williams and Plummer's Conjecture that G^2 is Hamiltonian for every 2-connected graph G ; Chartrand and Kapoor [1] proved that $G^2 - v$ is Hamiltonian for every 2-connected graph G with $\overline{V(G)} \geq 4$ and every $v \in V(G)$.

It has been conjectured [4] that $G^2 - u - v$ is Hamiltonian for every 2-connected graph G with $\overline{V(G)} \geq 5$ and for every $u, v \in V(G)$.

The purpose of this note is to show that this conjecture is false (see [3]), as follows:

Theorem 1: For every odd integer n , $n \geq 3$, there exists a 2-connected graph $G = G(n)$ with $\overline{V(G)} = 3n + 2$, such that $G^2 - u - v$ is not Hamiltonian for some $u, v \in V(G)$.

The following is even a stronger result:

Theorem 2: For every integer n , $n \geq 2$, there exists a 2-connected graph $G = G(n)$ with $\overline{V(G)} = 8n$ and $V(G)$ contains $2n$ vertices $u_1, \dots, u_n, v_1, \dots, v_n$ such that $G^2 - u_i - v_i$ is not Hamiltonian for every i , $1 \leq i \leq n$.

We need the following simple

Lemma: If a graph G contains a simple $u - v$ path $u, ux, x, xy, y, yz, z, zv, v$ of length 4, such that x, y and z are 2-valent in G , then every simple path in $G^2 - u - v$ that contains y as an inner vertex contains also the edges xy and yz .

Proof: xy and yz are the only edges of $G^2 - u - v$ that contain y as one of their end points, hence they are contained in every simple path in $G^2 - u - v$ that contains y as an inner vertex.

Proof of Theorem 1: For every odd n , $n \geq 3$, let $G(n)$ be the union of the n $u - v$ paths $u, ux_i, x_i, x_iy_i, y_i, y_iz_i, z_i, z_iv, v$, for all $i = 1, \dots, n$, where x_i, y_i and z_i are all different (see figure 1 with $n = 5$); clearly $G(n)$ is 2-connected, for all $n \geq 3$.

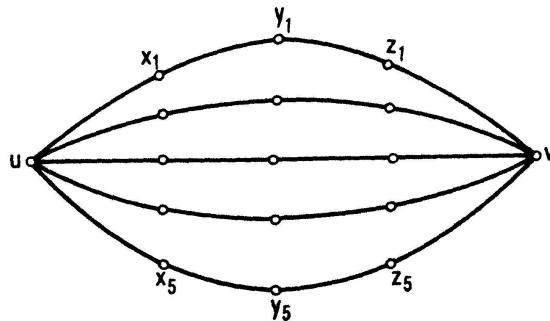


Figure 1

$G^2 - u - v$ consists of the two complete n -graphs K^* and K^{**} , on the vertices x_1, \dots, x_n and z_1, \dots, z_n , respectively, together with the vertices y_1, \dots, y_n and the edges $x_i y_i, y_i z_i$ and $x_i z_i$, for all $i = 1, \dots, n$.

Suppose there exists a Hamiltonian cycle h in $G^2 - u - v$. h must contain all the vertices y_i , therefore it must contain by the Lemma all the edges $x_i y_i$ and $y_i z_i$, for all $i = 1, \dots, n$; since $n > 1$, h does not contain any of the edges $x_i z_i$. h therefore runs from K^* to K^{**} and back an odd number of times, which is impossible; hence $G^2 - u - v$ is non-Hamiltonian.

Proof of Theorem 2: For every integer $n \geq 2$, let the graph $G = G(n)$ consist of the n cycles of length 8 of the vertices $u_i, x_i, y_i, z_i, v_i, z'_i, y'_i, x'_i$ and u_i (in this cyclic order), for all $1 \leq i \leq n$, plus the edges $v_i u_{i+1}$ for all $1 \leq i \leq n$ (the last one of which being $v_n u_1$). G is shown in Figure 2, with $n = 4$:

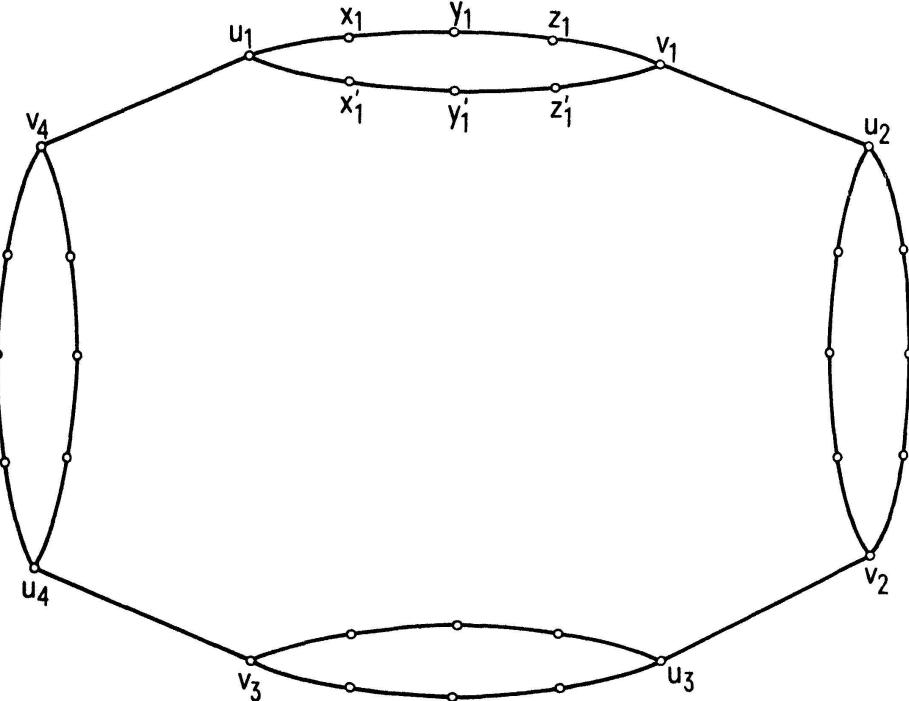


Figure 2

Clearly, $G(n)$ is a 2-connected graph for all $n \geq 2$.

Suppose that for some i , $1 \leq i \leq n$, $G^2 - u_i - v_i$ contains a simple cycle c that contains both y_i and y'_i ; c contains, by the Lemma, all the four edges $x_i y_i, y_i z_i, x'_i y'_i$ and $y'_i z'_i$. c does not contain the edge $x_i z_i$ ($x'_i z'_i$), since otherwise c would have at least two connected components, one of which being a cycle of length 3. The vertex x_i , as any vertex of c , is of valence 2 in c , hence either the edge $x_i v_{i-1}$ or else the edge $x_i x'_i$ is in c ; similarly for the vertices x'_i, z_i and z'_i . It follows that c contains either the edge $x_i x'_i$ or else the two edges $x_i v_{i-1}$ and $x'_i v_{i-1}$; similarly, c contains either the edge $z_i z'_i$ or else the two edges $z_i u_{i+1}$ and $z'_i u_{i+1}$. As a result, c is of length at most 8, while $G^2 - u_i - v_i$ has $8n - 2$ vertices, $n \geq 2$; since $8 < 8n - 2$, for all $n \geq 2$, it follows that c does not contain all the vertices of $G^2 - u_i - v_i$, therefore $G^2 - u_i - v_i$ is non-Hamiltonian. This completes the proof of Theorem 2.

Remark: It follows immediately from [1] that if G is a 2-connected graph, then $G^2 - u - v$ has a Hamiltonian path for every $u, v \in V(G)$.

Joseph Zaks, Michigan State University, East Lansing, USA,
and University of Haifa, Haifa, Israel.

REFERENCES

- [1] G. CHARTRAND and S. F. KAPOOR, *The Square of Every 2-Connected Graph is 1-Hamiltonian*, to appear.
- [2] H. FLEISHNER, *The Square of Every Nonseparable Graph is Hamiltonian*, to appear.
- [3] J. ZAKS, announcement 1(B), Graph Theory Newsletter Vol. 1 (No. 4), 1972 edited by S. F. Kapoor, W.M.U. Kalamazoo, Michigan), p. 7.
- [4] (anonymous) Problem 1, Graph Theory Newsletter, Vol. 1. (No. 2), (1971), p. 3.

Kleine Mitteilungen

Proof of a Conjecture of H. Hadwiger

As part of a research problem [2], Hadwiger conjectured that every simple closed curve in E^3 admits a nontrivial inscribed parallelogram. Schnirelman's method [4] [1] leads immediately to the following result:

Theorem: *Every simple closed C^2 curve in E^3 admits a nontrivial inscribed rhombus.*

Outline of proof: The statement for plane curves has been proved by Schnirelman [4] [1]. Every simple closed curve in E^3 is homotopic to a plane Jordan curve. If the curve in E^3 is not knotted, the homotopy is in fact an isotopy. If the curve is a knot, it may be deformed into a plane Jordan curve through a C^2 -homotopy $F(\alpha, t)$, $0 \leq \alpha \leq 2\pi$, $0 \leq t \leq 1$, for which $F(\alpha, t_0)$ is a simple closed curve except for finitely many values t_0 for which $F(\alpha, t_0)$, $0 \leq \alpha \leq 2\pi$, is a curve with one simple transversal selfintersection. Because of the compactness of the sets involved, a given smooth homotopy can be locally modified to satisfy the given conditions. The parametrization can be chosen so that the Jacobian matrix of F is nowhere singular. The theorem will be proved if we can show that it holds for all curves $F(\alpha, t)$, $t_0 \leq t < t_0 + \varepsilon$ if it holds for $F(\alpha, t_0)$.

By hypothesis, there exist four distinct parameter values $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ so that for $F_i = F(\alpha_i, t_0)$ we have

$$|F_1 - F_2| = |F_2 - F_3| = |F_3 - F_4| = |F_4 - F_1| (\neq 0) \quad (1)$$

$$\det (F_1 - F_2, F_2 - F_3, F_3 - F_4) = 0$$

where \det denotes the determinant. The problem is to find four points F_i^* on $F(\alpha, t)$, $t_0 \leq t \leq t_0 + \varepsilon$, that also satisfy conditions (1). We develop in a Taylor polynomial,

$$F_i^* = F_i + \frac{\delta F_i}{\delta x_i} \Delta \alpha_i + \frac{\delta F_i}{\delta t} \Delta t + o(\Delta \alpha_i, \Delta t)$$

introduce the expression in (1) and develop as well. An appropriate form of the inverse function theorem says that under our differentiability assumptions the $\Delta \alpha_i$ can be found if the linearized problem obtained by putting all $o(\Delta \alpha_i, \Delta t) = 0$, can be solved. From (1) one obtains a system of four nonhomogeneous linear equations (that can immediately be written down) for the four unknowns $\Delta \alpha_i$ ($i = 1, 2, 3, 4$). The matrix of the system has the form