Rationals not expressible as a sum of three unit fractions

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Rationals Not Expressible as a Sum of Three Unit Fractions

I. Introduction

Let k be a positive integer and define

$$\delta_k = \left\{ n : \frac{k}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z};\ x, y, z \text{ positive integers} \right\} \tag{1.1}$$

and

$$\lambda_k = \sup_{n \notin \delta_k} n\,. \tag{1.2}$$

It has been conjectured by Schinzel [5] that $\lambda_k < \infty$ for all k; some specific cases of this conjecture are due to Erdös [3] – $k = 4$; Sierpiński [5] – $k = 5$; and Aigner [1] – $k = 6, 7$.

Since the set of fractions expressible as a sum of three unit fractions is nowhere dense ([1, 7]) we must have $\lim\sup \frac{k}{\lambda_k} = 0$, that is λ_k must approach ∞ faster than k does. However, for small k, λ_k also seems small. For example if the conjectures are correct $\lambda_4 = 1$, $\lambda_5 = 1$, $\lambda_6 = 1$ and $\lambda_7 = 2$, where the magnitude of $7/2$ alone implies that $2 \notin \delta_7$. The first case where $\lambda_k > k$ is $k = 8$. Aigner [1] noted that $17 \notin \delta_8$ so $\lambda_8 \geq 17$. We will see later that actually $\lambda_8 \geq 241$.

In this paper we will look for examples of $n \notin \delta_k$ for $k \geq 6$.

II. General Methods

One of the principal tools we will use in our search for $n \notin \delta_k$ is the following lemma.

Lemma 2.1 $a/b = 1/x + 1/y$ if and only if there exist positive divisors d_1 and d_2 of b such that $a \mid d_1 + d_2$.

This is a modified form of a result of Aigner [1] Satz 6. For a complete proof of some generalizations of Lemma 2.1 see [7].

If $k/n = 1/x + 1/y + 1/z$, $0 < x \leq y \leq z$, then $x \leq \frac{3n}{k}$. Hence, in checking if $n \in \delta_k$ it suffices to apply Lemma 2.1 to $k/n - 1/x$ for each x such that $\frac{n}{k} \leq x \leq \frac{3n}{k}$.

Also, since $n \in \delta_k$ implies $n\,m \in \delta_k$ for all $m \geq 1$, we can limit ourselves to the case where n is a prime.

The following result shows that we may restrict the values of x we must check even further.

Theorem 2.2 Let $k \geq 4$, p a prime such that $(k, p) = 1$, and let $x \leq y \leq z$ be positive integers such that

$$\frac{k}{p} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}; \tag{2.1}$$

then $x \leq (2p+2)/k$. Moreover, if $x > 2p/k$ then either $k \mid 2p+1$ and $x = (2p+1)/k$ or $k \mid p+1$ and $x = (2p+2)/k$.

Proof. Now by Lemma 2.1

$$\frac{k}{p} - \frac{1}{x} = \frac{k\,x - p}{p\,x} = \frac{a}{b} = \frac{1}{y} + \frac{1}{z}; \tag{2.2}$$

if and only if there exist $d_1, d_2 \mid px$ such that $d_1 + d_2 = s(k\,x - p)$ for some integer s; in which case

$$\frac{a}{b} = \frac{1}{s\left(\dfrac{b}{d_1}\right)} + \frac{1}{s\left(\dfrac{b}{d_2}\right)}. \tag{2.3}$$

Hence, $x \leq s\,b/d_1 \leq s\,b/d_2$ since we may assume $d_1 \geq d_2$.

Now, assume that $x > 2p/k$. This implies that $k\,x - p > p$. We know that $x \leq 3p/k \leq 3p/4$ so $p \nmid x$. Hence $(p, k\,x) = 1$ and $(p, k\,x - p) = 1$.

If $d \leq e$ are any two positive divisors of x, then $kx - p \nmid d + e$. This may be seen as follows:

Assume $k\,x - p \mid d + e$. Then $d + e \leq 2\,x \leq \frac{3}{2} p \leq \frac{3}{2}(k\,x - p)$, so $k\,x - p = d + e$. Also, if $e < x, d + e \leq x < k\,x - p$; so $e = x$. Hence, $d = (k-1)\,x - p$ which implies $d \mid p$, and so $d = 1$ since $(p, x) = 1$. Therefore

$$x = \frac{p+1}{k-1} \leq \frac{2p}{k}, \text{ a contradiction}.$$

Since we have that $d_1, d_2 \mid p\,x$ and $k\,x - p \mid d_1 + d_2$, by the above, we cannot have $d_1, d_2 \mid x$. Similarly we cannot have $d_1 = p\,d_3$ and $d_2 = p\,d_4$ where $d_3, d_4 \mid x$, since this would mean that $k\,x - p \mid p\,(d_3 + d_4)$ which implies $k\,x - p \mid d_3 + d_4$ again contradicting what we have shown above. Therefore, and since $d_1 \geqslant d_2$, we must have $d_1 = p\,d$ where $d \mid x$ and $d_2 \mid x$.

Now $p\,d + d_2 = s\,(k\,x - p) > s\,p$ and $d_2 \leq x < p$ which implies $d \geq s$. Hence,

$$\frac{s\,b}{d_1} = \frac{s\,p\,x}{d_1} = \frac{s\,p\,x}{p\,d} = \frac{s\,x}{d} \leq x; \tag{2.4}$$

but $x \leq s\,b/d_1$ and so $x = s\,x/d = s\,b/d_1 = y$. Thus from (2.2), (2.3) and (2.4) we see that

$$\frac{k}{p} - \frac{2}{x} = \frac{k\,x - 2p}{p\,x} = \frac{1}{z}, \tag{2.5}$$

and so $k\,x - 2p \mid p\,x$.

Clearly $(p,\, k\,x - 2p) = 1$, so $k\,x - 2p \mid x$ which implies $k\,x - 2p \mid 2p$ which implies $k\,x - 2p = 1$ or 2. If $k\,x - 2p = 1$ then $k \mid 2p + 1$ and $x = (2p + 1)/k$. If $kx - 2p = 2$, then (2.5) holds if and only if x is even, which implies $x = (2p + 2)/k$ and $k \mid p + 1$.

This completes the proof of our theorem.

This theorem says that in looking for a solution of (2.1) we need only check $x \leqslant (2p + 2)/k$. The theorem is best possible as the cases $k \mid 2p + 1$ and $k \mid p + 1$ show; since if $k \mid 2\,p + 1$ then

$$\frac{k}{p} = \frac{1}{(2p + 1)/k} + \frac{1}{(2p + 1)/k} + \frac{1}{p\,(2p + 1)/k},$$

and if $k \mid p + 1$ then

$$\frac{k}{p} = \frac{1}{2\,(p + 1)/k} + \frac{1}{2\,(p + 1)/k} + \frac{1}{p\,(p + 1)/k}.$$

Although these examples show that x may be as large as $(2p + 2)/k$ in a solution of (2.1), in all of the examples so far encountered, there have been other solutions of (2.1) with a smaller value of x. For example in the case $k \mid p + 1$ mentioned above, we may take $x = (p + 1)/k$ since

$$\frac{k}{p} = \frac{1}{(p + 1)/k} + \frac{1}{2p\,(p + 1)/k} + \frac{1}{2p\,(p + 1)/k}.$$

If x_0 denotes the smallest value of x such that (2.1) is solvable, how close to $(2p + 2)/k$ can x_0 be? If $k = 12$, $p = 727$ then $x_0 = 108$;

$$\frac{12}{727} = \frac{1}{108} + \frac{1}{138} + \frac{1}{1805868}.$$

Since $108 > (1.78)\,\frac{727}{12}$, we see that the bound $2p/k$ cannot be improved too much. Probably even better examples can be found.

III. Numerical Results

In this section we will give some examples of primes p for which (2.1) has no solution. Using Lemma 2.1 it can be shown that for a fixed k, (2.1) is solvable if p is an element of certain residue classes. For small values of k we can restrict our search for counterexamples to a very few residue classes. For example, for $k = 5$ Stewart [6] has shown that if (2.1) is unsolvable then $p \equiv 1 \pmod{278460}$. Using this he has shown that (2.1) is solvable for all $p < 1{,}057{,}438{,}801$. Yamamoto [8] has shown that for $k = 4$, (2.1) is solvable for all $p < 10^7$. This extends work done by Aigner [1], Bernstein [2], Palamà [4]. As k becomes larger this procedure becomes less effective since it eliminates a smaller fraction of the integers from consideration, and it multiplies the number of cases to be considered. For example consider the cases $k = 5$ and $k = 10$. If $k = 5$ we need consider only one residue class modulo 278460. But if $k = 10$, although we can easily eliminate the residue classes 2, 4, 5, 6, 7, 8, 9, 10 (mod 10), we are left with the possibilities $p \equiv 1, 3, 7 \pmod{10}$. It is not hard to show that actually we may restrict ourselves to the residue classes 1 (mod 10), 3 (mod 140), 43 (mod 140) and 7 (mod 60). Furthermore we can replace the residue class 1 (mod 10) by the residue classes 1, 11, 31, 41, 61 (mod 90) which would reduce the number of numbers considered slightly, but we would have five separate cases instead of only one. It is probably possibly to find a procedure somewhat more efficient for $k = 10$ than this; but as we shall see in Theorem 3.2, since there are several primes p for which (2.1) is unsolvable with $k = 10$ and these p fall in three different residue classes modulo 10, it is impossible to do nearly as well as we can in the case $k = 5$.

The procedure used in this paper is to remove as many residue classes as seems practical by the above method, and then use a computer to search for counter-examples in the remaining residue classes. The method used in the computer search is essentially that given near the beginning of section 2.

As an example we take $k = 6$, and assume (2.1) is not solvable. It is easily seen that $p = 6n + 1$. We now eliminate various residue classes for n as follows.

$$\frac{6}{6n+1} - \frac{1}{n+1} = \frac{5}{(6n+1)(n+1)}. \tag{3.1}$$

Now apply Lemma 2.1.

n	d_1	d_2	
$\equiv 4 \pmod 5$			$5 \mid n+1$
$\equiv 3 \pmod 5$	$n+1$	1	
$\equiv 2 \pmod 5$	$(6n+1)(n+1)$	1	
$\equiv 1 \pmod 5$	$(6n+1)(n+1)$	1	
$\equiv 5 \pmod{10}$	$(n+1)/2$	2	

Hence $n \equiv 0 \pmod{10}$ and $p = 60m + 1$. Then

$$\frac{6}{60m+1} - \frac{1}{10m+2} = \frac{11}{2(60m+1)(5m+1)}. \tag{3.2}$$

Apply Lemma 2.1 again.

m	d_1	d_2	
$\equiv 10 \pmod{11}$	$2(60m+1)(5m+1)$	1	
$\equiv 8 \pmod{11}$	$(60m+1)(5m+1)$	2	
$\equiv 7 \pmod{11}$	$(60m+1)(5m+1)$	2	
$\equiv 6 \pmod{11}$	$5m+1$	2	
$\equiv 5 \pmod{11}$	$2(60m+1)(5m+1)$	1	
$\equiv 4 \pmod{11}$	$5m+1$	1	
$\equiv 3 \pmod{11}$	$2(5m+1)$	1	
$\equiv 2 \pmod{11}$			$11 \mid 5m+1$

Thus $m \equiv 0$, 1 or 9 (mod 11) and so $p \equiv 1$, 61 or 541 (mod 660).

Using the methods described above we have the following results.

Theorem 3.1 The equations

$$\frac{6}{p} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \tag{3.3}$$

and

$$\frac{7}{p} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \tag{3.4}$$

are solvable for all $3 \leq p \leq 100{,}000$.

Theorem 3.2. The following table lists values of $p > k/3$ for which

$$\frac{k}{p} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \tag{3.5}$$

is not solvable.

k	p
8	11, 17, 131, 241
9	5, 11, 19
10	7, 11, 43, 61, 67, 181
11	37
12	5, 13, 29, 37, 73, 97, 193, 433, 577, 1129, 1657, 1873, 2521, 2593, 3433, 10369, 12049, 12241
13	5, 7, 53, 61, 67, 79, 211, 281
14	5, 17, 19, 29, 59, 257, 353
15	17, 19, 23, 31, 47, 53, 61, 79, 113, 137, 151, 197, 233, 271, 541, 1103, 1171, 1367, 4201
16	7, 11, 13, 17, 23, 37, 73, 97, 113, 131, 167, 193, 241, 257, 421, 577, 593, 641, 769, 1201, 1489, 2113, 2521, 2689, 3169, 3361, 4801, 4993
17	7, 13, 19, 23, 41, 53, 71, 73, 157, 281, 421, 1123, 2081
18	7, 11, 13, 19, 23, 29, 31, 37, 41, 47, 59, 61, 73, 109, 113, 131, 137, 149, 181, 193, 223, 239, 281, 379, 389, 397, 443, 457, 541, 599, 613, 661, 761, 811, 821, 911, 1009, 1297, 1381, 2269, 2819

For $8 \leq k \leq 11$ the table is complete for $p \leq 25000$, for $13 \leq k \leq 18$ the table is complete for $p \leq 5000$ and for $k = 12$ it is complete for $p \leq 100000$. The case $k = 12$ was carried further than the others since it is the first case for which (3.5) is unsolvable for some relatively large primes. It is quite likely that more counterexamples for $13 \leq k \leq 18$ can be found, if we check for $p > 5000$. The total computer time used in verifying theorems 3.1 and 3.2 was approximately 20 minutes using an IBM 360/67.

These examples show that λ_k gets relatively large for even some small values of k ($\lambda_{12} \geq 12241$), something that was not evident for $k \leq 7$ studied previously. However, in the case $k = 12$ which was carried out further than the others, there were no more counterexamples for $12241 < p \leq 100000$. Thus these examples appear to give some evidence both for and against the general conjecture that $\lambda_k < \infty$ for all k.

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Eine bemerkenswerte Abbildung der Punkte des Raumes auf die Kreise einer Ebene

Einleitung

Die im folgenden untersuchte Abbildung hat gewisse Ähnlichkeiten mit der *Zyklographie* [1, 2]. Bei ihr werden die Punkte des dreidimensionalen Raumes R^3 auf die Kreise einer im R^3 enthaltenen, waagrecht gedachten Ebene abgebildet, wobei statt der in der Zyklographie für den Abbildungsvorgang verwendeten Drehkegel mit lotrechter Achse Drehparaboloide mit lotrechter Achse und festem Parameter (= 1/2) treten.

Ein wesentlicher Unterschied gegenüber der Zyklographie liegt darin, dass die Abbildung ohne Orientierung der Kreise auskommt und trotzdem umkehrbar eindeutig ist. Durch zyklographische Deutung der Bildkreise wird im Raum eine zweieindeutige Punkttransformation induziert, die einen klaren Einblick in das Wesen der neuen Abbildung gewährt.

Einer Geraden des R^3 entspricht die Menge der eine Parabel der Bildebene doppelt berührenden Kreise, während die Scharen der einen Mittelpunktskegelschnitt doppelt berührenden Kreise von Parabeln mit lotrechter Achse herrühren. Diese Tatsachen lassen die konstruktive Lösung verschiedener damit zusammenhängender Aufgaben zu.