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If we now let $(a', b', c') = (a^2, b^2, c^2)$ and restrict ABC to be an acute triangle, then a', b', c' are sides of a general triangle of area Δ'. By virtue of the known inequality, $4\,\Delta^2 \geq \sqrt{3}\,\Delta'$, of Finsler and Hadwiger [1, p. 91] together with (9), gives

$$\frac{p}{q+r}\, b'\, c' + \frac{q}{r+p}\, c'\, a' + \frac{r}{p+q}\, a'\, b' \geq 2\sqrt{3}\,\Delta' \tag{10}$$

or equivalently

$$\frac{p \csc A'}{q+r} + \frac{q \csc B'}{r+p} + \frac{r \csc C'}{p+q} \geq \sqrt{3}\,. \tag{10'}$$

The last two forms generalize the known special case corresponding to $p = q = r$ [2, p. 31, 43].

Other related extensions will be given in a subsequent paper. Also, for other examples of non-negative quadratic forms and their associated triangle inequalities, see [5], [6] and the references therein.

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A Note on Discontinuous Functions

Let $\mathfrak{J}$ denote the class of real-valued functions defined and everywhere discontinuous on an interval $[a, b]$. F. Fricker [1] considered questions concerning the set $\mathcal{H}(f) = \{x\colon \lim_{y\to x} f(y) \text{ exists}\}$ for $f \in \mathfrak{J}$. He asked whether it is possible for $\mathcal{H}(f)$ to be dense in $[a, b]$. A negative answer to this question was obtained by R. Jeltsch [2]. The purpose of this note is to characterize those sets H for which there exists $f \in \mathfrak{J}$ such that $H = \mathcal{H}(f)$.

We begin with three lemmas.

Lemma 1. For any real-valued function f defined on $[a, b]$ the set $\mathcal{H}(f) = \{x\colon \lim_{y\to x} f(y) \text{ exists}\}$ is of type G_δ.

Proof. For each $x \in [a, b]$ and $\delta > 0$ let

$$\omega_\delta(x) = \sup\{|\,f(y) - f(z)\,|\colon 0 < |\,y - x\,| < \delta,\ 0 < |\,z - x\,| < \delta\}$$

and let $\omega(x) = \lim_{\delta\to 0}\omega_\delta(x)$. Thus $\omega(x)$ is the deleted oscillation of f at x and $\lim_{y\to x} f(y)$ exists if and only if $\omega(x) = 0$. Let $H_n = \{x: \omega(x) < 1/n\}$. It is easy to verify that H_n is open for each n and that $\mathcal{H}(f) = \bigcap_{n=1}^{\infty} H_n$. Thus $\mathcal{H}(f)$ is of type G_δ.

Lemma 2. For any real-valued function f defined on $[a, b]$, the set $\mathcal{D}(f) = \{x \in \mathcal{H}(f): \lim_{y\to x} f(y) \neq f(x)\}$ is denumerable.

Proof. For each positive integer n and each rational number r, let

$$A_{nr} = \{x \in \mathcal{H}(f) : f(x) < r < f(y) \text{ for all } y \text{ satisfying } 0 < |y - x| < 1/n\}$$

and

$$B_{nr} = \{x \in \mathcal{H}(f) : f(x) > r > f(y) \text{ for all } y \text{ satisfying } 0 < |y - x| < 1/n\}.$$

It is clear that for each n and r, the sets A_{nr} and B_{nr} are finite subsets of $[a, b]$. Thus the union of all these sets is denumerable. Since $\mathcal{D}(f)$ is contained in this union, $\mathcal{D}(f)$ is also denumerable.

Lemma 3. Let H be a denumerable set of type G_δ. Then there exists a descending sequence of $\{G_n\}_{n=1}^{\infty}$ open sets such that $H = \bigcap_{n=1}^{\infty} G_n$ and $G_n \sim G_{n+1}$ is dense-in-itself for each n.

Proof. Since H is of type G_δ, there exists a decreasing sequence $\{H_n\}_{n=1}^{\infty}$ of open sets such that $H = \bigcap_{n=1}^{\infty} H_n$, and since H is denumerable we may choose H_n such that for each n, $H_n - H_{n+1} \neq \varnothing$. Let $G_1 = H_1$. Let C consist of the isolated points of $H_1 - H_2$. If $C = \varnothing$, choose $G_2 = H_2$. If $C \neq \varnothing$, then C is denumerable and there exists a denumerable family of disjoint intervals contained in H_1 and covering C, each of which contains exactly one point of C. Let $x \in C$ and let B be such an interval. Then there exists a component interval I of H_2 having x as an endpoint. Since H is a denumerable set of type G_δ, H is nowhere dense. It follows that these exists a monotonic sequence of disjoint nondegenerate closed intervals $\{I_n\}_{n=1}^{\infty}$ such that $I_n \to x$ and for each n, $I_n \subset I \cap B \sim H$. Let $I(x) = \bigcup_{n=1}^{\infty} I_n$ and $G_2 = H_2 - \bigcup\{I(x) : x \in C\}$. Then G_2 is open, $H \subseteqq G_2$, and $G_1 - G_2$ is nonvoid and dense-in-itself. Carrying out the above construction inductively, we arrive at the desired sequence $\{G_n\}_{n=1}^{\infty}$.

Theorem. Let $H \subset [a, b]$. A necessary and sufficient condition for there to exist an everywhere discontinuous function f such that $H = \{x: \lim_{y\to x} f(y) \text{ exists}\}$ is that H be a denumerable set of type G_δ.

Proof. The necessity of the condition follows immediately from Lemmas 1 and 2.

We turn now to the sufficiency of the condition. By Lemma 3, there exists a decreasing sequence $\{G_n\}_{n=1}^{\infty}$ of open sets such that $H = \bigcap_{n=1}^{\infty} G_n$ and $G_n - G_{n+1}$ is nonvoid and dense-in-itself for each n. Let $h_1, h_2, \ldots$ be an enumeration of H. For each n, let A_n and B_n be nonvoid, dense subsets of $G_n - G_{n+1}$ such that $A_n \cap B_n = \varnothing$ and $A_n \cup B_n = G_n - G_{n+1}$. Define a function f by

$$f(x) = \begin{cases} \dfrac{1}{k} & \text{if} \quad x = h_k \quad \text{for some} \quad k \\[2ex] \dfrac{1}{n} & \text{if} \quad x \in A_n \quad \text{for some} \quad n\,. \\[2ex] -\dfrac{1}{n} & \text{if} \quad x \in B_n \quad \text{for some} \quad n \end{cases}$$

We show f is everywhere discontinuous and $\lim\limits_{y \to x} f(y)$ exists if and only if $x \in H$. First, suppose $x \in H$. If $x_n \to x$, $x_n \in H$, then $x_n = h_{k_n}$ so $f(x_n) = 1/k_n$ and $\lim\limits_{n \to \infty} f(x_n) = 0$. If $x_n \to x$, $x_n \notin H$, then for each n there exists a natural number q_n such that $x_n \in G_{q_n} - G_{q_n+1}$ so $f(x_n) = 1/q_n$. It is easy to verify that $\lim\limits_{n \to \infty} q_n = \infty$ so $\lim\limits_{n \to \infty} f(x_n) = 0$. It follows that $\lim\limits_{y \to x} f(y) = 0$ for all $x \in H$. Since $f(x) = 1/k$ for some k, f is discontinuous at x.

Now suppose $x \notin H$. There exists a natural number n such that $x \in G_n - G_{n+1}$. But the sets A_n and B_n are each dense in $G_n - G_{n+1}$ so that, by the definition of f, the numbers $1/n$ and $-1/n$ are both in the cluster set of f at x. It follows that $\lim\limits_{y \to x} f(y)$ does not exist.

This completes the proof of the theorem.

Remark 1: The foregoing proof can be easily modified to apply to nowhere continuous functions on a complete separable metric space which is dense in itself.

Remark 2: Since a denumerable set of type G_δ is nowhere dense (in fact, nowhere dense-in-itself), we see that the question posed by F. Fricker has a negative answer.

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Kleine Mitteilungen

There is no Odd Super Perfect Number of the Form $p^{2\alpha}$

In [4] the author defined super perfect numbers as positive integers n such that $\sigma(\sigma(n)) = 2n$, where $\sigma(n)$ denotes the sum of all the positive divisors of n. It has been shown in [4] that an even integer n is super perfect if and only if $n = 2^r$, where $2^{r+1} - 1$ is a prime and posed the existence of odd super perfect numbers as a problem. This is still an open problem. In [2] H. J. Kanold has shown that if n is an odd super perfect number, then n must be a square. In [1] P. Bundschuh posed the problem,

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