Zeitschrift: Elemente der Mathematik
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 28 (1973)
Heft: 6

Artikel: A note on discontinuous functions
Autor: Bruckner, A.M.
DOI: https://doi.org/10.5169/seals-29464

Nutzungsbedingungen
Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Mehr erfahren

Conditions d'utilisation
L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. En savoir plus

Terms of use
The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. Find out more

Download PDF: 08.01.2026

ETH-Bibliothek Zürich, E-Periodica, https://www.e-periodica.ch

If we now let $(a', b', c') = (a^2, b^2, c^2)$ and restrict ABC to be an acute triangle, then a', b', c' are sides of a general triangle of area Δ'. By virtue of the known inequality, $4\,\Delta^2 \geq \sqrt{3}\,\Delta'$, of Finsler and Hadwiger [1, p. 91] together with (9), gives

$$\frac{p}{q+r}\, b'\, c' + \frac{q}{r+p}\, c'\, a' + \frac{r}{p+q}\, a'\, b' \geq 2\sqrt{3}\,\Delta' \tag{10}$$

or equivalently

$$\frac{p \csc A'}{q+r} + \frac{q \csc B'}{r+p} + \frac{r \csc C'}{p+q} \geq \sqrt{3}\,. \tag{10$'$}$$

The last two forms generalize the known special case corresponding to $p = q = r$ [2, p. 31, 43].

Other related extensions will be given in a subsequent paper. Also, for other examples of non-negative quadratic forms and their associated triangle inequalities, see [5], [6] and the references therein.

Murray S. Klamkin, Ford Motor Company, Dearborn/USA

REFERENCES

[1] Amer. Math. Monthly, March 1971.
[2] O. BOTTEMA, et al., *Geometric Inequalities* (Noordhoff, Groningen 1969).
[3] M. S. KLAMKIN, *Duality in Triangle Inequalities*, Ford Motor Company Preprint, July 1971. (Also, see Notices of A.M.S., August 1971, p. 782).
[4] G. H. HARDY, J. E. LITTLEWOOD and G. POLYÁ, *Inequalities* (Cambridge University Press, London 1952), p. 57–60.
[5] M. S. KLAMKIN, *Non-Negative Quadratic Forms and Triangle Inequalities*, Ford Motor Company Preprint, June 1971. (Also, see Notices of A.M.S., October 1971, p. 966).
[6] M. S. KLAMKIN, *Asymmetric Triangle Inequalities*, Univ. Beograd Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 367, *1971*, p. 33–44.

A Note on Discontinuous Functions

Let $\mathfrak{J}$ denote the class of real-valued functions defined and everywhere discontinuous on an interval $[a, b]$. F. Fricker [1] considered questions concerning the set $\mathfrak{H}(f) = \{x\colon \lim_{y\to x} f(y) \text{ exists}\}$ for $f \in \mathfrak{J}$. He asked whether it is possible for $\mathfrak{H}(f)$ to be dense in $[a, b]$. A negative answer to this question was obtained by R. Jeltsch [2]. The purpose of this note is to characterize those sets H for which there exists $f \in \mathfrak{J}$ such that $H = \mathfrak{H}(f)$.

We begin with three lemmas.

Lemma 1. For any real-valued function f defined on $[a, b]$ the set $\mathfrak{H}(f) = \{x\colon \lim_{y\to x} f(y) \text{ exists}\}$ is of type G_δ.

Proof. For each $x \in [a, b]$ and $\delta > 0$ let

$$\omega_\delta(x) = \sup\{|\, f(y) - f(z)\,|\colon\ 0 < |\, y - x\,| < \delta,\ 0 < |\, z - x\,| < \delta\}$$

and let $\omega(x) = \lim_{\delta \to 0} \omega_\delta(x)$. Thus $\omega(x)$ is the deleted oscillation of f at x and $\lim_{y \to x} f(y)$ exists if and only if $\omega(x) = 0$ .Let $H_n = \{x: \omega(x) < 1/n\}$. It is easy to verify that H_n is open for each n and that $\mathcal{H}(f) = \bigcap_{n=1}^{\infty} H_n$. Thus $\mathcal{H}(f)$ is of type G_δ.

Lemma 2. For any real-valued function f defined on $[a, b]$, the set $\mathcal{D}(f) = \{x \in \mathcal{H}(f): \lim_{y \to x} f(y) \neq f(x)\}$ is denumerable.

Proof. For each positive integer n and each rational number r, let

$$A_{nr} = \{x \in \mathcal{H}(f) : f(x) < r < f(y) \text{ for all } y \text{ satisfying } 0 < | y - x | < 1/n\}$$

and

$$B_{nr} = \{x \in \mathcal{H}(f) : f(x) > r > f(y) \text{ for all } y \text{ satisfying } 0 < | y - x | < 1/n\}.$$

It is clear that for each n and r, the sets A_{nr} and B_{nr} are finite subsets of $[a, b]$. Thus the union of all these sets is denumerable. Since $\mathcal{D}(f)$ is contained in this union, $\mathcal{D}(f)$ is also denumerable.

Lemma 3. Let H be a denumerable set of type G_δ. Then there exists a descending sequence of $\{G_n\}_{n=1}^{\infty}$ open sets such that $H = \bigcap_{n=1}^{\infty} G_n$ and $G_n \sim G_{n+1}$ is dense-in-itself for each n.

Proof. Since H is of type G_δ, there exists a decreasing sequence $\{H_n\}_{n=1}^{\infty}$ of open sets such that $H = \bigcap_{n=1}^{\infty} H_n$, and since H is denumerable we may choose H_n such that for each n, $H_n - H_{n+1} \neq \varnothing$. Let $G_1 = H_1$. Let C consist of the isolated points of $H_1 - H_2$. If $C = \varnothing$, choose $G_2 = H_2$. If $C \neq \varnothing$, then C is denumerable and there exists a denumerable family of disjoint intervals contained in H_1 and covering C, each of which contains exactly one point of C. Let $x \in C$ and let B be such an interval. Then there exists a component interval I of H_2 having x as an endpoint. Since H is a denumerable set of type G_δ, H is nowhere dense. It follows that these exists a monotonic sequence of disjoint nondegenerate closed intervals $\{I_n\}_{n=1}^{\infty}$ such that $I_n \to x$ and for each n, $I_n \subset I \cap B \sim H$. Let $I(x) = \bigcup_{n=1}^{\infty} I_n$ and $G_2 = H_2 - \bigcup\{I(x) : x \in C\}$. Then G_2 is open, $H \subseteqq G_2$, and $G_1 - G_2$ is nonvoid and dense-in-itself. Carrying out the above construction inductively, we arrive at the desired sequence $\{G_n\}_{n=1}^{\infty}$.

Theorem. Let $H \subset [a, b]$. A necessary and sufficient condition for there to exist an everywhere discontinuous function f such that $H = \{x: \lim_{y \to x} f(y) \text{ exists}\}$ is that H be a denumerable set of type G_δ.

Proof. The necessity of the condition follows immediately from Lemmas 1 and 2.

We turn now to the sufficiency of the condition. By Lemma 3, there exists a decreasing sequence $\{G_n\}_{n=1}^{\infty}$ of open sets such that $H = \bigcap_{n=1}^{\infty} G_n$ and $G_n - G_{n+1}$ is nonvoid and dense-in-itself for each n. Let $h_1, h_2, \ldots$ be an enumeration of H. For each n, let A_n and B_n be nonvoid, dense subsets of $G_n - G_{n+1}$ such that $A_n \cap B_n = \varnothing$ and $A_n \cup B_n = G_n - G_{n+1}$. Define a function f by

$$f(x) = \begin{cases} \frac{1}{k} & \text{if} \quad x = h_k \quad \text{for some} \quad k \\ \frac{1}{n} & \text{if} \quad x \in A_n \quad \text{for some} \quad n\,. \\ -\frac{1}{n} & \text{if} \quad x \in B_n \quad \text{for some} \quad n \end{cases}$$

We show f is everywhere discontinuous and $\lim_{y \to x} f(y)$ exists if and only if $x \in H$. First, suppose $x \in H$. If $x_n \to x$, $x_n \in H$, then $x_n = h_{k_n}$ so $f(x_n) = 1/k_n$ and $\lim_{n \to \infty} f(x_n) = 0$. If $x_n \to x$, $x_n \notin H$, then for each n there exists a natural number q_n such that $x_n \in G_{q_n} - G_{q_n+1}$ so $f(x_n) = 1/q_n$. It is easy to verify that $\lim_{n \to \infty} q_n = \infty$ so $\lim_{n \to \infty} f(x_n) = 0$. It follows that $\lim_{y \to x} f(y) = 0$ for all $x \in H$. Since $f(x) = 1/k$ for some k, f is discontinuous at x.

Now suppose $x \notin H$. There exists a natural number n such that $x \in G_n - G_{n+1}$. But the sets A_n and B_n are each dense in $G_n - G_{n+1}$ so that, by the definition of f, the numbers $1/n$ and $-1/n$ are both in the cluster set of f at x. It follows that $\lim_{y \to x} f(y)$ does not exist.

This completes the proof of the theorem.

Remark 1: The foregoing proof can be easily modified to apply to nowhere continuous functions on a complete separable metric space which is dense in itself.

Remark 2: Since a denumerable set of type G_δ is nowhere dense (in fact, nowhere dense-in-itself), we see that the question posed by F. Fricker has a negative answer.

A. M. Bruckner[1]) and Jack Ceder, University of California, Santa Barbara

REFERENCES

[1] F. FRICKER, *Über hebbare Unstetigkeiten*, El. Math. *25*, 107–113 (1970).
[2] R. JELTSCH, *Über hebbare Unstetigkeiten*, El. Math. *27*, 129–132 (1972).

Kleine Mitteilungen

There is no Odd Super Perfect Number of the Form $p^{2\alpha}$

In [4] the author defined super perfect numbers as positive integers n such that $\sigma(\sigma(n)) = 2n$, where $\sigma(n)$ denotes the sum of all the positive divisors of n. It has been shown in [4] that an even integer n is super perfect if and only if $n = 2^r$, where $2^{r+1} - 1$ is a prime and posed the existence of odd super perfect numbers as a problem. This is still an open problem. In [2] H. J. Kanold has shown that if n is an odd super perfect number, then n must be a square. In [1] P. Bundschuh posed the problem,

[1]) This author was supported in part by NSF Grant GP-18968.