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und daher

$$\begin{aligned}\varphi(U \cup W) f((U + \lambda t) \cup W) &= \varphi(U) (f(U) + \lambda t) + \varphi(W) f(W) \\ &= \varphi(U \cup W) f(U \cup W) + \lambda \varphi(U) t \\ &\notin H^+\end{aligned}$$

für alle grossen  $\lambda$ . Wegen

$$f((U + \lambda t) \cup W) \in \text{konv}((U + \lambda t) \cup W) \subset H^+$$

muss also  $\varphi(U \cup W) < 0$  sein. Andererseits gilt für hinreichend grosse  $\lambda$

$$\varphi(U \cup W) f(U \cup (W + \lambda t)) = \varphi(U \cup W) f(U \cup W) + \lambda t \in H^+$$

und

$$f(U \cup (W + \lambda t)) \in \text{konv}(U \cup (W + \lambda t)) \subset H^+,$$

was  $\varphi(U \cup W) > 0$  nach sich zieht. Die Annahme war also falsch, das heisst  $\varphi$  ist definit.

Rolf Schneider, TU Berlin

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## Two Non-Negative Quadratic Forms

### I. Introduction

In problem E 2348 [1], L. Carlitz has given the inequality

$$\sum R_1 (r_2 + r_3) \geq \sum (r_1 + r_2) (r_1 + r_3) \quad (1)$$

where  $R_1, R_2, R_3$  and  $r_1, r_2, r_3$  denote the distances from an interior point of a triangle  $ABC$  to the vertices  $A, B, C$  and the sides  $a, b, c$ , respectively. Coupling (1) with the known lower bounds  $R_1 \geq (r_2 c + r_3 b)/a$ , etc. [2, p. 107], suggests the stronger inequality

$$\left. \begin{aligned} F_1 \equiv & \left\{ \frac{b}{c} + \frac{c}{b} - 1 \right\} r_1^2 + \left\{ \frac{c}{a} + \frac{a}{c} - 1 \right\} r_2^2 + \left\{ \frac{a}{b} + \frac{b}{a} - 1 \right\} r_3^2 \\ & - \left\{ 3 - \frac{b+c}{a} \right\} r_2 r_3 - \left\{ 3 - \frac{c+a}{b} \right\} r_3 r_1 - \left\{ 3 - \frac{a+b}{c} \right\} r_1 r_2 \geq 0. \end{aligned} \right\} \quad (2)$$

We shall show that  $F_1 \geq 0$  is indeed valid for all triangles  $ABC$  and *any* real values of  $r_1, r_2, r_3$ . Then by using the lower bounds  $b/c + c/b \geq 2$ , etc., inequality (2) can be partially strengthened to

$$F_2 \equiv r_1^2 + r_2^2 + r_3^2 - \left\{ 3 - \frac{b+c}{a} \right\} r_2 r_3 - \left\{ 3 - \frac{c+a}{b} \right\} r_3 r_1 - \left\{ 3 - \frac{a+b}{c} \right\} r_1 r_2 \geq 0. \quad (3)$$

However, here  $r_1, r_2, r_3$  are to be arbitrary non-negative numbers.

### II. $F_1 \geq 0$

A standard way of showing  $F_1$  is non-negative is to show that its associated matrix

$$M = \begin{vmatrix} \frac{b^2 - bc + c^2}{bc} & \frac{a + b - 3c}{2c} & \frac{a + c - 3b}{2b} \\ \frac{b + a - 3c}{2c} & \frac{c^2 - ca + a^2}{ca} & \frac{b + c - 3a}{2a} \\ \frac{c + a - 3b}{2b} & \frac{c + b - 3a}{2a} & \frac{a^2 - ab + b^2}{ab} \end{vmatrix}$$

is positive semidefinite. As is well known,  $F_1$  is a non-negative form *iff* all the principal minors of  $M$  are  $\geq 0$ . The first two leading ones,  $M_1$  and  $M_2$  are easy to establish. For

$$bc M_1 = (b - c)^2 + bc > 0,$$

and after some manipulation

$$4abc^2 M_2 = 4c^2 \{ \sum a^2 - \sum ab \} + ab \{ 2 \sum ab - \sum a^2 \}.$$

That  $M_2$  is non-negative, follows from two elementary triangle inequalities [2, p. 11]. The non-negativity of the remaining 1st and 2nd order principal minors follows by symmetry.

To simplify the valuation of  $M_3 = \det M$ , we make the duality transformation [3],

$$a = y + z, \quad b = z + x, \quad c = x + y$$

where  $x, y, z$  are arbitrary non-negative numbers, not all zero. After some simple algebra, we obtain

$$(x + y)^2 (y + z)^2 (z + x)^2 M_3 = \begin{vmatrix} P - 2yz & (z - x - y)(z + x) & (y - z - x)(y + x) \\ (z - x - y)(x + y) & P - 2zx & (x - y - z)(x + y) \\ (y - z - x)(y + z) & (x - y - z)(x + z) & P - 2xy \end{vmatrix}$$

where  $P = \sum x^2 + \sum xy$ . We now add row 2 and row 3 to row 1, giving a row of constant terms,  $\sum x^2 - \sum xy$ , which can be factored out. Next, we subtract column 2 from

column 3 and then column 1 from column 2, leading to a  $2 \times 2$  determinant. Then adding the rows together, we finally obtain

$$(x + y)^2 (y + z)^2 (z + x)^2 M_3 = \left\{ \sum xy \right\} \left\{ \sum x^2 - \sum xy \right\} \left\{ \sum x^2 + 3 \sum xy \right\} \geq 0$$

with equality *iff*  $x = y = z$  or two of  $x, y, z$  are zero. Consequently, (1) and (2) are valid with equality *iff*  $ABC$  is equilateral (we are excluding degenerate triangles).

### III. $F_2 \geq 0$

By just considering the case  $r_3 = 0, r_1 r_2 < 0$  and  $(a + b)/c$  large, it follows that inequality (3) is not valid for all real  $r_1, r_2, r_3$ . Consequently, we restrict  $r_1, r_2, r_3$  to non-negative values and replace them by  $x^2, y^2, z^2$ , respectively. Also, we replace  $a, b, c$  by  $q + r, r + p, p + q$ , respectively, where  $p, q, r$  are arbitrary non-negative numbers. Inequality (3) now takes the form

$$F'_2 \equiv x^4 + y^4 + z^4 - 2uy^2z^2 - 2vz^2x^2 - 2wx^2y^2 \geq 0 \tag{3}'$$

where

$$u = 1 - \frac{p}{q + r}, \quad v = 1 - \frac{q}{r + p}, \quad w = 1 - \frac{r}{p + q}.$$

Since  $F'_2$  is a biquadratic in  $x, y, z$ , it follows by a theorem of Hilbert [4] that if  $F'_2 \geq 0$ , then it can be expressed as the sum of squares of three real polynomials and conversely. Consequently, our proof of (3)' is based on exhibiting such a representation.

Without loss of generality, we can assume that  $p \geq q \geq r$ . Whence,  $u < 1, 0 \leq v, w \leq 1$ .  $F'_2$  can now be expressed in the form

$$F'_2 = \{x^2 - vz^2 - wy^2\}^2 + \{y^2(1 - w^2)^{1/2} - z^2(1 - v^2)^{1/2}\}^2 + 2Gy^2z^2$$

where

$$G = (1 - v^2)^{1/2}(1 - w^2)^{1/2} - u - vw.$$

In order to show that  $G$  is non-negative, we consider, separately, three intervals for  $u$ , i.e.,  $(C_1) u \leq u_0, (C_2) 0 \leq u \leq 1$  and  $(C_3) u_0 < u < 0$ . The first case  $(C_1)$  is the easiest since  $u_0$  is the negative root of a certain cubic such that  $u + vw \leq 0$ . For  $(C_2)$ ,  $p, q, r$  are possible sides of a triangle. Thus, we can make the duality transformation  $p = \beta + \gamma, q = \gamma + \alpha, r = \alpha + \beta$  where  $\alpha, \beta, \gamma \geq 0$ . On squaring out  $G = 0$ , we get

$$1 \geq u^2 + v^2 + w^2 + 2uvw. \tag{4}$$

After a considerable amount of simple algebra, (4) can be expressed in the form

$$\left. \begin{aligned} & \frac{7}{3} T_1^2 T_2 (T_1^2 - 3 T_2) + \frac{4}{9} T_1^3 (T_1 T_2 - 9 T_3) + \frac{10}{9} T_1 T_2 (T_1^3 - 27 T_3) \\ & + \frac{1}{9} (T_1^3 \cdot T_1 T_2 - 27 T_3 \cdot 9 T_3) \geq 0 \end{aligned} \right\} \tag{5}$$

where

$$T_1 = \alpha + \beta + \gamma, \quad T_2 = \beta\gamma + \gamma\alpha + \alpha\beta, \quad T_3 = \alpha\beta\gamma.$$

Since it is well known that

$$T_1^2 \geq 3 T_2, \quad T_1^3 \geq 27 T_3, \quad T_1 T_2 \geq 9 T_3,$$

inequality (5) is valid and with equality *iff*  $\alpha = \beta = \gamma$ .

For the last case ( $C_3$ ), we resort to a geometric proof. First, we note that  $u, v, w$  satisfy the identity

$$\begin{vmatrix} 1 & u-1 & u-1 \\ v-1 & 1 & v-1 \\ w-1 & w-1 & 1 \end{vmatrix} = 0$$

or equivalently

$$H \equiv \left\{ v - \frac{4-3u}{3-2u} \right\} \left\{ w - \frac{4-3u}{3-2u} \right\} - \left\{ \frac{2-u}{3-2u} \right\}^2 = 0. \quad (6)$$

For a fixed  $u$ , permissible values of  $v, w$  will then lie on the lower branch of the hyperbola  $H = 0$  which is contained in the unit square  $0 \leq v, w \leq 1$ . If we also wish to have  $u + vw \leq 0$ , then the critical value of  $u$  (denoted by  $u_0$ ) is determined by requiring the lower branch of  $H = 0$  to be tangent to the positive branch of  $vw = -u$  (see figure).  $u_0$  is then the negative root of  $\sqrt{-u} = 2(1-u)/(3-2u)$  or  $4u^3 - 8u^2 + u + 4 = 0$ . Here,  $u_0 \approx -.57$ .

In order to show that (4) is valid for each fixed  $u$  in  $(u_0, 0)$ , it suffices to show that the part of the lower branch of  $H = 0$  lying above the positive branch of  $vw = -u$  and within the square  $0 \leq v, w \leq 1$ , also lies inside the ellipse (see figure)

$$E \equiv v^2 + w^2 + 2uvw - (1-u^2) = 0.$$

The semi-axes of  $E$  lie on the lines  $v \pm w = 0$  and their lengths are  $\sqrt{1-u}$  and  $\sqrt{1+u}$ , respectively.

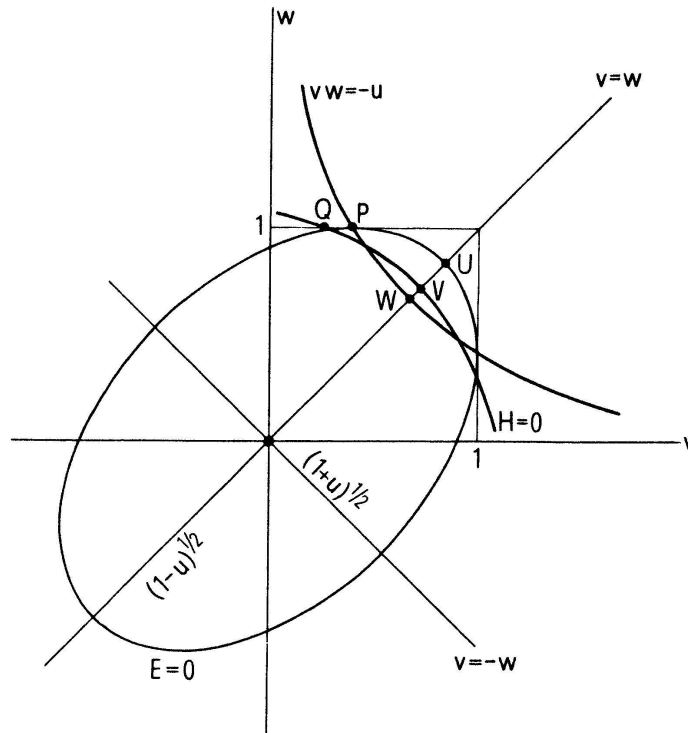
The ellipse is inscribed in the square  $-1 \leq v, w \leq 1$ . The positive branch of the hyperbola  $vw = -u$  passes through two points of tangency. The three curves are symmetric with respect to the line  $v = w$ . The coordinates of the indicated points are given by

$$P = (-u, 1), \quad Q = \left( \frac{u}{u-1}, 1 \right),$$

$$U_v = \sqrt{(1-u)/2}, \quad V_v = \frac{2(1-u)}{3-2u}, \quad W_v = \sqrt{-u/2}$$

( $U_v$  denotes the abscissa of  $U$ , etc.).

Since  $V$  lies between  $W$  and  $U$  and  $Q$  is to the left of  $P$ , our inequality is established.



In addition to having equality in (3)' for the case,  $a = b = c, x = y = z$ , we also have equality for a degenerate triangle corresponding to  $P$  and  $Q$  coinciding at  $(0,1)$ . For this case,  $r = 0, x = y, a = q, b = p, c = p + q$ .

#### IV. Triangle inequalities

Numerous triangle inequalities can be obtained from the forms  $F_1$  and  $F_2$  by letting  $r_1, r_2, r_3$  be particular functions of the sides, e.g.,  $r_1 = a, a^2, ab, b$  etc. ( $r_2$  and  $r_3$  are then chosen by a cyclic interchange of  $a, b, c$ ). Most of these inequalities are not particularly elegant, e.g.,

$$abc \sum a^2 + \sum (a + b) a^2 b^2 \geq 3abc \sum ab, \tag{7}$$

$$2a^2 b^2 c^2 \sum ab + \sum b^4 c^4 \geq 3a^2 b^2 c^2 \sum c^2. \tag{8}$$

However, we can rewrite (3)' into the more appealing form (geometrically)

$$\frac{p}{q+r} b^2 c^2 + \frac{q}{r+p} c^2 a^2 + \frac{r}{p+q} a^2 b^2 \geq 8\Delta^2 \tag{9}$$

where  $\Delta$  denotes the area of triangle  $ABC$ . It is to be noted that if  $|a|, |b|, |c|$  did not form a triangle, the r.h.s. of (9) would be negative giving a trivial inequality. In terms of angles, (9) is given by

$$\frac{p \csc^2 A}{q+r} + \frac{q \csc^2 B}{r+p} + \frac{r \csc^2 C}{p+q} \geq 2. \tag{9}'$$

There is equality in (9) and (9)' iff  $A = B = C = \pi/3, p = q = r$ . If we also allow degenerate triangles, there is also equality iff  $A = B = \pi/2, p = q, r = 0$  (assuming  $p \geq q \geq r$ ). Inequalities (9) and (9)' generalize the known special case corresponding to  $p = q = r$  [1, pp. 31, 45].

If we now let  $(a', b', c') = (a^2, b^2, c^2)$  and restrict  $ABC$  to be an acute triangle, then  $a', b', c'$  are sides of a general triangle of area  $\Delta'$ . By virtue of the known inequality,  $4 \Delta^2 \geq \sqrt{3} \Delta'$ , of Finsler and Hadwiger [1, p. 91] together with (9), gives

$$\frac{p}{q+r} b' c' + \frac{q}{r+p} c' a' + \frac{r}{p+q} a' b' \geq 2 \sqrt{3} \Delta' \quad (10)$$

or equivalently

$$\frac{p \csc A'}{q+r} + \frac{q \csc B'}{r+p} + \frac{r \csc C'}{p+q} \geq \sqrt{3} . \quad (10)'$$

The last two forms generalize the known special case corresponding to  $p = q = r$  [2, p. 31, 43].

Other related extensions will be given in a subsequent paper. Also, for other examples of non-negative quadratic forms and their associated triangle inequalities, see [5], [6] and the references therein.

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## A Note on Discontinuous Functions

Let  $\mathcal{J}$  denote the class of real-valued functions defined and everywhere discontinuous on an interval  $[a, b]$ . F. Fricker [1] considered questions concerning the set  $\mathcal{H}(f) = \{x: \lim_{y \rightarrow x} f(y) \text{ exists}\}$  for  $f \in \mathcal{J}$ . He asked whether it is possible for  $\mathcal{H}(f)$  to be dense in  $[a, b]$ . A negative answer to this question was obtained by R. Jeltsch [2]. The purpose of this note is to characterize those sets  $H$  for which there exists  $f \in \mathcal{J}$  such that  $H = \mathcal{H}(f)$ .

We begin with three lemmas.

**Lemma 1.** For any real-valued function  $f$  defined on  $[a, b]$  the set  $\mathcal{H}(f) = \{x: \lim_{y \rightarrow x} f(y) \text{ exists}\}$  is of type  $G_\delta$ .

*Proof.* For each  $x \in [a, b]$  and  $\delta > 0$  let

$$\omega_\delta(x) = \sup \{|f(y) - f(z)| : 0 < |y - x| < \delta, 0 < |z - x| < \delta\}$$