Hypo-eulerian and hypo-traversable graphs

Proof. Let T_0 be a simplex of minimal volume containing K. By the theorem of Day [2], the centroids of the facets of T_0 touch K. Let t be the simplex whose vertices are those centroids, and let T be the simplex parallel to t and circumscribed about K. Then $t = (n^{-n})\, T_0$ and $T \geq T_0$, so

$$K^n \geq t^{n-1}\, T \geq (n^{-n(n-1)}\, T_0^{n-1})\, (T_0)\,, \tag{11}$$

so $T_0 \leq (n^{n-1})\, K$, as we wanted to prove.

G. D. Chakerian, University of California, Davis

REFERENCES

[1] G. D. CHAKERIAN and L. H. LANGE, *Geometric Extremum Problems*, Math. Mag. *44*, 57–69 (1971).
[2] M. M. DAY, *Polygons Circumscribed About Closed Convex Curves*, Trans. Amer. Math. Soc. *62*, 315–319 (1947).
[3] C. H. DOWKER, *On Minimum Circumscribed Polygons*, Bull. Amer. Math. Soc. *50*, 120–122 (1944).
[4] L. FEJES TÓTH, *Lagerungen in der Ebene, auf der Kugel und im Raum* (Berlin 1953).
[5] L. A. LYUSTERNIK, *Convex Figures and Polyhedra*, Moscow 1956 (Russian; English translation, New York 1963,).

Hypo-Eulerian and Hypo-Traversable Graphs

Introduction

If a graph G does not possess a given property P, and for each vertex v of G the graph $G - v$ enjoys property P, then G is said to be a *hypo-P* graph. Recently, studies have been made where P stands for the graph being *hamiltonian*, *planar*, and *outerplanar* (e.g., see [3]). Here we obtain a characterization of *hypo-eulerian* and *hypo-randomly-eulerian* graphs, and investigate in this respect some of the other concepts arising out of Euler's solution of the classical Königsberg Seven Bridges Problem.

Preliminaries

Following the terminology of [2], a *graph* will be finite, undirected, without loops or multiple edges. A *walk* of a graph G is an alternating sequence $v_0, e_1, v_1, e_2, v_2, \ldots, v_{n-1}, e_n, v_n$ of vertices and edges of G, beginning and ending with vertices and where the edge $e_i = v_{i-1} v_i$ for $i = 1, 2, \ldots, n$. This is a $v_0 - v_n$ walk, and is usually denoted $v_0\, v_1\, v_2 \ldots v_n$; it is *closed* if $v_0 = v_n$ and *open* otherwise. A walk is a *trail* if all its edges are distinct; it is a *path* if all its vertices are distinct. A closed trail is a circuit and a circuit on distinct vertices is a *cycle*. A cycle on p vertices is denoted C_p, and C_3 is called a *triangle*.

If for every two distinct vertices u and v of a graph G there exists a $u - v$ path, then G is *connected*. A *component* of G is a maximal connected subgraph of G. A vertex

v is a *cutpoint* of G if $G - v$ has more components than G. An eulerian circuit of a graph G is a circuit which contains all the vertices and edges of G, and an *eulerian trail* of G is an open trail which contains all the vertices and edges of G; in either case G has to be connected. We will assume that an eulerian circuit or an eulerian trail has at least one edge in it.

The number of edges incident with a vertex v is the *degree* of v which is written as *deg v*. Let $\delta(G) = \min_v \deg v$ and $\Delta(G) = \max_v \deg v$. A graph G is *regular of degree r* (or *r-regular*) if $\delta(G) = \Delta(G) = r$. A *cubic* graph is 3-regular. We use $p(G)$ and $q(G)$ (often simply p and q) for the number of vertices and edges of a graph G. The *trivial* graph has $p = 1$ and the *complete graph* K_p on p vertices has $q = p\,(p-1)/2$. The *complete bipartite graph* $K(m, n)$ has its vertex set partitioned into nonempty sets V_1 and V_2 containing m and n elements respectively such that uv is an edge of $K(m, n)$ if and only if $u \in V_i$ and $v \in V_j$, $i \neq j$.

An edge $x = u\,v$ of a graph H is said to be *subdivided* if it is replaced by a new vertex w together with the edges $u\,w$ and $w\,v$. A graph G is *homeomorphic from* a graph H if G can be obtained from H by a finite sequence of such subdivisions. Two graphs G_1 and G_2 are *homeomorphic* if there exists a graph G such that G_1 and G_2 are both homeomorphic from G.

Let $\theta(G)$ ($\xi(G)$) consist of the vertices of G having their degrees odd (even). Let the number of elements in $\theta(G)$ be called the *euler number* of G, and let this be written as $\in(G)$. Then $\in(G)$ is a nonnegative even integer.

Hypo-eulerian Graphs

A graph G on $p \geq 3$ vertices is defined to be *eulerian* if it possesses an eulerian circuit. The next result is well known.

Theorem (Euler). Let G be a connected graph. Then G is eulerian if and only if $\in(G) = 0$.

By definition, a graph G is *hypo-eulerian* if G is not eulerian, but the graph $G - v$ is eulerian for each vertex v of G.

Theorem 1. Let G be a connected nontrivial graph. Then G is hypo-eulerian if and only if $G = K_{2n}$, $n \geq 2$.

Proof. Clearly, $\in(K_{2n}) = 2\,n > 0$ and $\in(K_{2n} - v) = \in(K_{2n-1}) = 0$ imply the sufficiency part. So let G be a nontrivial connected hypo-eulerian graph. As $G - v$ is eulerian, $p(G) \geq 4$.

First we show that every vertex of G must be odd. Assume that $\xi(G) \neq \phi$, and let $u \in \xi(G)$. Now u must be adjacent with only odd vertices otherwise $\in(G - u) > 0$. On the other hand if $v \in \theta(G)$, then for the same reason v must also be adjacent with only odd vertices. This contradicts $\xi(G) \neq \phi$. Hence $p(G) = \in(G) = 2\,n$ for some $n \geq 2$.

Secondly, we assert that G is complete. For if not, there exist two nonadjacent odd vertices u and v in G. Now the vertex v has odd degree in $G - u$ and contradicts $\in(G - u) = 0$. This completes the proof.

If G is an eulerian graph with $p \geq 3$ and v is any vertex of G, then $G - v$ necessarily contains odd vertices and must be noneulerian. This we mention next.

Theorem 2. Let G be a connected nontrivial graph. Then G is hypo-noneulerian if and only if G is eulerian.

Ore [4] called an eulerian graph G *randomly eulerian from a vertex* v if every trail of G beginning at v can be extended to an eulerian circuit of G; a graph G is *randomly eulerian* if it is randomly eulerian from each of its vertices. Ore characterized graphs which are randomly eulerian from a vertex v as those graphs in which v belongs to every cycle of G. This leads to the result that G is randomly eulerian if and only if G is a cycle.

Theorem 3. A graph G is hypo-randomly-eulerian if and only if $G = K_4$.

Proof. Since a cycle is obtained by deleting any vertex of K_4, this graph certainly has the desired property. Conversely, let G be a hypo-randomly-eulerian graph. Observe that in view of Theorem 2, G and $G - v$ cannot be both eulerian for any vertex v. Hence G is necessarily hypo-eulerian, and by Theorem 1, $G = K_{2n}$ for some $n \geq 2$. Moreover, since $G - v$ must be a cycle for each vertex v of K_{2n}, we conclude that $G = K_4$.

Chartrand and White [1] proved that if G is an eulerian graph which is randomly eulerian from k vertices, then $k = 0, 1, 2$ or $p(G)$, and following this we will denote a graph which is randomly eulerian from k vertices as an $RE(k)$ graph. A study of *hypo-*$RE(k)$ graphs is now in order. Let G be a graph which is not $RE(k)$, but let $G - v$ be randomly eulerian from k vertices. Then, as stated earlier, G must be a hypo-eulerian graph with the additional property that for all v, $G - v$ is an $RE(k)$ graph. So by Theorem 1, $G = K_{2n}$ and $G - v = K_{2n-1}$, $n \geq 2$. When $n \geq 3$, for every vertex u of $G - v$ we can find a cycle, namely a triangle, which avoids u, and so $G - v$ is an $RE(o)$ graph. The case $n = 2$ yields that $G - v$ is an $RE(p)$ graph. Also, $G - v$ is not an $RE(k)$ graph for $k = 1$ and $k = 2$. These remarks lead to the next result where we note that the hypo-$RE(p)$ graphs have already been described in Theorem 3.

Theorem 4.

(a) A graph G on $p \geq 4$ vertices is hypo-$RE(o)$ if and only if $G = K_{2n}$, $n \geq 3$.
(b) No graph is hypo-$RE(1)$ or hypo-$RE(2)$.
(c) A graph G on $p \geq 4$ vertices is hypo-$RE(p)$ if and only if $G = K_4$.

We conclude this section by stating a result analogous to Theorem 2.

Theorem 5. A graph G is hypo-non$RE(k)$ if and only if G is an $RE(k)$ graph.

Hypo-traversable Graphs

A graph G on $p \geq 2$ vertices is said to be *traversable* if G has an eulerian trail, i.e., G has an open trail which contains all the vertices and edges of G (and in view of the next result, this trail begins at one of the odd vertices and ends at the other).

Theorem (*Euler*). Let G be a connected graph. Then G is traversable if and only if $\in (G) = 2$.

Let G be a hypo-traversable graph. Then $\in (G) \neq 2$, and $\in (G - v) = 2$ for each vertex v of G. It is clear that G is a block, and $\delta(G) \geq 2$. Also, $\in (G)$ is even and $0 \leq \in (G) \leq p$. From the first possible value we readily get the following.

Theorem 6. Let G be any connected graph which has euler number 0. Then G is hypo-traversable if and only if G is a cycle.

Proof. The sufficiency is immediate, and for the necessity we note that $\in(G) = 0$ implies that $V(G) = \xi(G)$. Now $\in(G - v) = 2$ for any vertex v of G gives $\deg v = 2$. By connectedness, G has to be a cycle.

Now let $\in(G) = 2m$, $m \geq 2$, and let G be hypo-traversable. Let $u \in \xi(G)$ and $v \in \theta(G)$. Then it can be seen that $\deg u = 2m - 2, 2m$ or $2m + 2$ and $\deg v = 2m - 3$, $2m - 1$ or $2m + 1$, otherwise $\in(G - w) \neq 2$ for some vertex w of G. This fact is useful in considering individual cases. Should $m = 2$, the possible values of $\deg v$ will be 3 or 5 since $\delta(G) \geq 2$. It can be verified that for $p \leq 5$, cycles are the only hypo-traversable graphs. Figure 1 shows all graphs on 6 vertices which are hypo-traversable.

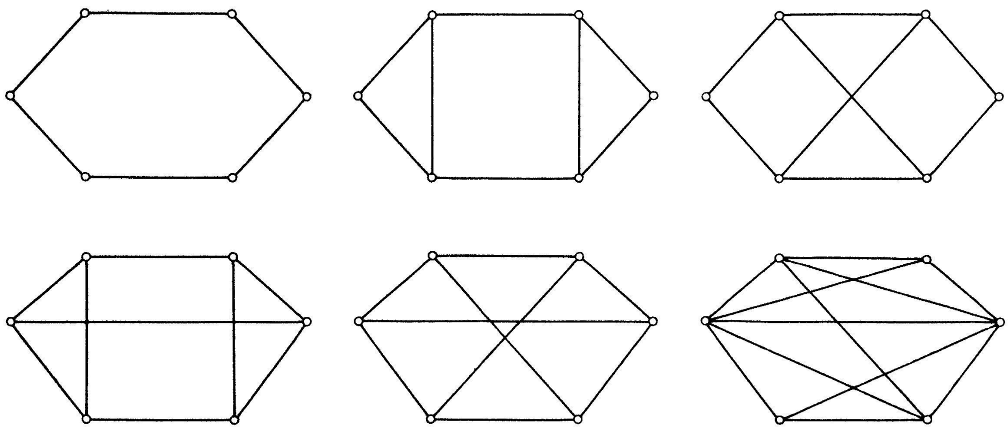

Figure 1
Hypo-traversable graphs on 6 vertices.

The preceding theorem dealt with the case when the graph had all vertices even. The next result treats graphs possessing no even vertices.

Theorem 7. Let G be any connected graph having euler number $\in(G) = p(G) \geq 6$. Then G is hypo-traversable if and only if G is regular of degree $p - 3$.

Proof. Here $\xi(G) = \phi$ and $p = 2m = \in(G)$. By the above remarks, every vertex of G is odd and has possible degrees $2m - 3$ or $2m - 1$. But if any vertex is adjacent with all the other $p - 1$ vertices, its deletion gives an eulerian graph. The necessity now follows.

Conversely, let G be a connected $(p - 3)$-regular graph and $\in(G) = p(G) \geq 6$. Then $\in(G - v) = 2$ for all v, and the proof is complete.

Theorem 8. Let G be a connected graph having euler number $\in(G) = p(G) - 1$, and let $p(G) \geq 5$. Then G is hypo-traversable if and only if the even vertex u of G has degree $p - 3$, the vertices a and b that are nonadjacent with u have degree $p - 4$, and every other vertex has degree $p - 2$.

Proof. Let $\xi(G) = \{u\}$, and assume that G is hypo-traversable. Since every vertex adjacent with u becomes even in the traversable graph $G - u$, we need $\deg u = p - 3$. Let a and b be the vertices nonadjacent with u, and let $v \in \theta(G) - \{a, b\}$. Now the traversable graph $G - w$ contains exactly 2 odd vertices, for each $w \in V(G)$.

Hence deg $v = p - 2$ and deg $a =$ deg $b = p - 4$. For the sufficiency we note that $\in(G) \geq 4$, and by hypothesis, $\in(G - w) = 2$ for each vertex w of G.

It is possible that a complete classification of hypo-traversable graphs may get involved with discussing individual cases, and this suggests scope for further research.

Let G be a hypo-nontraversable graph, i.e., $\in(G) = 2$ and $\in(G - v) \neq 2$ for each vertex v. Moreover, since it is meaningful to require that $G - v$ be connected, we further assume that G has no cutpoints and $p \geq 4$ (so that $\delta(G) \geq 2$). Designate the two odd vertices of G as a and b. If $a\,b$ is not an edge in G, then $\in(G - a)$ and $\in(G - b)$ are 4 or more. On the other hand, if a and b are adjacent, we must have deg $a \geq 5$ and deg $b \geq 5$. Now let $v \in \xi(G)$. This imposes the following restrictions: If deg $v = 2$, then v is adjacent with either both or neither of a and b; if deg $v = 4$, then v is not simultaneously joined to both a and b. These present a set of necessary conditions for G to have the desired property, and it can be verified that they are also sufficient.

Theorem 9. Let G be a block with $p \geq 4$. Then G is hypo-nontraversable if and only if $\theta(G) = \{a, b\}$ and

(i) $a\,b \varepsilon E(G) \Rightarrow$ deg $a \geq 5$ and deg $b \geq 5$,

(ii) deg $v = 2 \Rightarrow v$ is joined to both or neither of a, b, and

(iii) deg $v = 4 \Rightarrow v$ is not joined to both a and b.

In [1] a traversable graph G is called *randomly traversable from a vertex v* if every trail in G with initial vertex v can be extended to an eulerian trail of G. Clearly, a traversable graph can be randomly traversable from $k = 0$, 1 or 2 vertices, and we may, as before, denote this class of graphs as $RT(k)$, where $RT(2)$ will refer to the class of *randomly traversable graphs*. It was also proved in [1] that if a and b are the two odd vertices of a traversable graph G, then G is randomly traversable from a if and only if every cycle of G contains b. Moreover, a graph G is in $RT(2)$ if and only if the two odd vertices of G lie on every cycle of G. This suggests the problem of studying *hypo-RT(k)* and *hypo-nonRT(k)* graphs.

We conclude by presenting a complete classification of $RT(2)$ graphs.

Theorem 10. Let G be a traversable graph with $\theta(G) = \{a, b\}$. Then G is randomly traversable if and only if G is homeomorphic from K_2, $K(2, 2m - 1)$ or $K(2, 2m) + a\,b$, where $m \geq 1$.

Proof. It is obvious that the graphs described are randomly traversable. To prove the converse, first we note that if deg $a = 1$, then any $b - a$ path must be G itself, otherwise there exists a cycle which avoids a or b. Thus, deg $b = 1$, and the graph G is homeomorphic from K_2. So we assume that each of a and b has degree at least 3.

Let v be any vertex of G other than a or b. Since G is connected, there exist $v - a$ and $v - b$ paths. Clearly these paths have v as their only common vertex otherwise some cycle of G avoids a or b. Moreover, the union of these paths gives an $a - b$ path which contains v. With every vertex $v \in V(G) - \theta(G)$ we can associate an $a - b$ path $P(v)$ such that $P(v)$ contains v. Let us consider the collection of all $a - b$ paths, where, for obvious reasons, any two paths are disjoint, i.e., the only vertices common to them are a and b. So $P(v)$ is unique, and the union of all these paths must be G itself. We therefore conclude that every vertex other than a and b has degree 2, and deg a = deg b is odd. Also, if a and b are adjacent, then $G - ab$ is homeomorphic from $K(2, 2m)$; and if a, b are nonadjacent, then G is homeomorphic from $K(2, 2m - 1)$, where $m \geq 1$.

S.F. Kapoor[1]), Western Michigan University, USA

REFERENCES

[1] G. CHARTRAND and A. T. WHITE, *Randomly Traversable Graphs*, El. Math. *25*, 101–107 (1970).
[2] F. HARARY, *Graph Theory* (Addison-Wesley, Reading 1969).
[3] J. MITCHEM, *Hypo-Properties in Graphs, The Many Facets of Graph Theory* (G. Chartrand and S. F. Kapoor editors), Lecture Notes in Mathematics No. 110 (Springer-Verlag, Berlin 1969), p. 223–230.
[4] O. ORE, *A Problem Regarding the Tracing of Graphs*, El. Math. *6*, 49–53 (1951).

[1]) Research partially supported by National Science Foundation grant GP 9435.

Kleine Mitteilungen

New Quadratic Forms with High Density of Primes

Let p_{min} be the smallest prime contained in a quadratic form of the shape $f(x) = A x^2 + A x - C$ and let n_{icp} be the number of initial consecutive primes of $f(x)$, then, by means of a CDC 6400 computer, all $f(x) = A x^2 + A x - C$ were investigated for $A < 10$, $C < 2.10^5$, and $p_{min} > 47$. In Table 1, the number below C is the number of all primes of $f(x)$ for $x < 100$, and p_{min} is the number in parentheses.

For each form $x^2 + x - C$ we have also a form $9y^2 + 9y - (C - 2)$, because the substitution $x = 3y + 1$ transforms $x^2 + x - C$ into $9y^2 + 9y - (C - 2)$; hence, each third term of $x^2 + x - C$ (starting with the second) belongs to $9y^2 + 9y - (C - 2)$. Similarly, for each form $2x^2 - C$ we have also a form $8z^2 + 8z - (C - 2)$, because the substitution $x = 2z + 1$ transforms $2x^2 - C$ into $8z^2 + 8z - (C - 2)$; hence, each second term of $2x^2 - C$ (starting with the second) belongs to $8z^2 + 8z - (C - 2)$. For the forms $2x^2 - 119131$ and $2x^2 - 186871$, related to the forms with $A = 8$ in Table 1, we have 64 and 61 primes, respectively, for $x < 100$.

Table 1 gives the impression that there might be no forms with $A = 4$. This is not so. In a test run with $A < 10$, $10^8 - 5000 < C < 10^8$, and $p_{min} > 47$, the forms $x^2 + x - 99995659$, $9x^2 + 9x - 99995657$, and $4x^2 + 4x - 99996937$ were discovered, all with $p_{min} = 53$.

The form $x^2 + x - 53509$ with $p_{min} = 61$ is due to N.G.W.H. Beeger [1] in 1938, the forms $x^2 + x - 90073$ with $p_{min} = 53$ and $x^2 + x - 169933$ with $p_{min} = 59$ are due to the author [2] in 1967.

Two hundred years ago, Euler published his famous quadratic form $x^2 + x + 41$ with $p_{min} = 41$ and $n_{icp} = 40$. This form was believed to have the highest density of primes of all quadratic forms $A x^2 + B x \pm C$ discovered till now. Many forms were found with $p_{min} > 41$ and the second differences greater than 2; but the corresponding