

Zeitschrift: Elemente der Mathematik
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 28 (1973)
Heft: 5

Artikel: Hypo-eulerian and hypo-traversable graphs
Autor: Kapoor, S.F.
DOI: <https://doi.org/10.5169/seals-29459>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 09.12.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

Proof. Let T_0 be a simplex of minimal volume containing K . By the theorem of Day [2], the centroids of the facets of T_0 touch K . Let t be the simplex whose vertices are those centroids, and let T be the simplex parallel to t and circumscribed about K . Then $t = (n^{-n}) T_0$ and $T \geq T_0$, so

$$K^n \geq t^{n-1} T \geq (n^{-n(n-1)} T_0^{n-1}) (T_0), \quad (11)$$

so $T_0 \leq (n^{n-1}) K$, as we wanted to prove.

G. D. Chakerian, University of California, Davis

REFERENCES

- [1] G. D. CHAKERIAN and L. H. LANGE, *Geometric Extremum Problems*, Math. Mag. 44, 57–69 (1971).
- [2] M. M. DAY, *Polygons Circumscribed About Closed Convex Curves*, Trans. Amer. Math. Soc. 62, 315–319 (1947).
- [3] C. H. DOWKER, *On Minimum Circumscribed Polygons*, Bull. Amer. Math. Soc. 50, 120–122 (1944).
- [4] L. FEJES TÓTH, *Lagerungen in der Ebene, auf der Kugel und im Raum* (Berlin 1953).
- [5] L. A. LYUSTERNIK, *Convex Figures and Polyhedra*, Moscow 1956 (Russian; English translation, New York 1963).

Hypo-Eulerian and Hypo-Traversable Graphs

Introduction

If a graph G does not possess a given property P , and for each vertex v of G the graph $G - v$ enjoys property P , then G is said to be a *hypo- P* graph. Recently, studies have been made where P stands for the graph being *hamiltonian*, *planar*, and *outerplanar* (e.g., see [3]). Here we obtain a characterization of *hypo-eulerian* and *hypo-randomly-eulerian* graphs, and investigate in this respect some of the other concepts arising out of Euler's solution of the classical Königsberg Seven Bridges Problem.

Preliminaries

Following the terminology of [2], a *graph* will be finite, undirected, without loops or multiple edges. A *walk* of a graph G is an alternating sequence $v_0, e_1, v_1, e_2, v_2, \dots, v_{n-1}, e_n, v_n$ of vertices and edges of G , beginning and ending with vertices and where the edge $e_i = v_{i-1} v_i$ for $i = 1, 2, \dots, n$. This is a $v_0 - v_n$ walk, and is usually denoted $v_0 v_1 v_2 \dots v_n$; it is *closed* if $v_0 = v_n$ and *open* otherwise. A walk is a *trail* if all its edges are distinct; it is a *path* if all its vertices are distinct. A closed trail is a circuit and a circuit on distinct vertices is a *cycle*. A cycle on p vertices is denoted C_p , and C_3 is called a *triangle*.

If for every two distinct vertices u and v of a graph G there exists a $u - v$ path, then G is *connected*. A *component* of G is a maximal connected subgraph of G . A vertex

v is a *cutpoint* of G if $G - v$ has more components than G . An eulerian circuit of a graph G is a circuit which contains all the vertices and edges of G , and an *eulerian trail* of G is an open trail which contains all the vertices and edges of G ; in either case G has to be connected. We will assume that an eulerian circuit or an eulerian trail has at least one edge in it.

The number of edges incident with a vertex v is the *degree* of v which is written as $\deg v$. Let $\delta(G) = \min_v \deg v$ and $\Delta(G) = \max_v \deg v$. A graph G is *regular of degree r* (or *r -regular*) if $\delta(G) = \Delta(G) = r$. A *cubic* graph is 3-regular. We use $p(G)$ and $q(G)$ (often simply p and q) for the number of vertices and edges of a graph G . The *trivial* graph has $p = 1$ and the *complete graph* K_p on p vertices has $q = p(p-1)/2$. The *complete bipartite graph* $K(m, n)$ has its vertex set partitioned into nonempty sets V_1 and V_2 containing m and n elements respectively such that uv is an edge of $K(m, n)$ if and only if $u \in V_i$ and $v \in V_j$, $i \neq j$.

An edge $x = uv$ of a graph H is said to be *subdivided* if it is replaced by a new vertex w together with the edges uw and wv . A graph G is *homeomorphic from* a graph H if G can be obtained from H by a finite sequence of such subdivisions. Two graphs G_1 and G_2 are *homeomorphic* if there exists a graph G such that G_1 and G_2 are both homeomorphic from G .

Let $\theta(G)$ ($\xi(G)$) consist of the vertices of G having their degrees odd (even). Let the number of elements in $\theta(G)$ be called the *euler number* of G , and let this be written as $\epsilon(G)$. Then $\epsilon(G)$ is a nonnegative even integer.

Hypo-eulerian Graphs

A graph G on $p \geq 3$ vertices is defined to be *eulerian* if it possesses an eulerian circuit. The next result is well known.

Theorem (Euler). Let G be a connected graph. Then G is eulerian if and only if $\epsilon(G) = 0$.

By definition, a graph G is *hypo-eulerian* if G is not eulerian, but the graph $G - v$ is eulerian for each vertex v of G .

Theorem 1. Let G be a connected nontrivial graph. Then G is hypo-eulerian if and only if $G = K_{2n}$, $n \geq 2$.

Proof. Clearly, $\epsilon(K_{2n}) = 2n > 0$ and $\epsilon(K_{2n} - v) = \epsilon(K_{2n-1}) = 0$ imply the sufficiency part. So let G be a nontrivial connected hypo-eulerian graph. As $G - v$ is eulerian, $p(G) \geq 4$.

First we show that every vertex of G must be odd. Assume that $\xi(G) \neq \emptyset$, and let $u \in \xi(G)$. Now u must be adjacent with only odd vertices otherwise $\epsilon(G - u) > 0$. On the other hand if $v \in \theta(G)$, then for the same reason v must also be adjacent with only odd vertices. This contradicts $\xi(G) \neq \emptyset$. Hence $p(G) = \epsilon(G) = 2n$ for some $n \geq 2$.

Secondly, we assert that G is complete. For if not, there exist two nonadjacent odd vertices u and v in G . Now the vertex v has odd degree in $G - u$ and contradicts $\epsilon(G - u) = 0$. This completes the proof.

If G is an eulerian graph with $p \geq 3$ and v is any vertex of G , then $G - v$ necessarily contains odd vertices and must be noneulerian. This we mention next.

Theorem 2. Let G be a connected nontrivial graph. Then G is hypo-noneulerian if and only if G is eulerian.

Ore [4] called an eulerian graph G *randomly eulerian from a vertex v* if every trail of G beginning at v can be extended to an eulerian circuit of G ; a graph G is *randomly eulerian* if it is randomly eulerian from each of its vertices. Ore characterized graphs which are randomly eulerian from a vertex v as those graphs in which v belongs to every cycle of G . This leads to the result that G is randomly eulerian if and only if G is a cycle.

Theorem 3. A graph G is hypo-randomly-eulerian if and only if $G = K_4$.

Proof. Since a cycle is obtained by deleting any vertex of K_4 , this graph certainly has the desired property. Conversely, let G be a hypo-randomly-eulerian graph. Observe that in view of Theorem 2, G and $G - v$ cannot be both eulerian for any vertex v . Hence G is necessarily hypo-eulerian, and by Theorem 1, $G = K_{2n}$ for some $n \geq 2$. Moreover, since $G - v$ must be a cycle for each vertex v of K_{2n} , we conclude that $G = K_4$.

Chartrand and White [1] proved that if G is an eulerian graph which is randomly eulerian from k vertices, then $k = 0, 1, 2$ or $p(G)$, and following this we will denote a graph which is randomly eulerian from k vertices as an $RE(k)$ graph. A study of *hypo-RE(k)* graphs is now in order. Let G be a graph which is not $RE(k)$, but let $G - v$ be randomly eulerian from k vertices. Then, as stated earlier, G must be a hypo-eulerian graph with the additional property that for all v , $G - v$ is an $RE(k)$ graph. So by Theorem 1, $G = K_{2n}$ and $G - v = K_{2n-1}$, $n \geq 2$. When $n \geq 3$, for every vertex u of $G - v$ we can find a cycle, namely a triangle, which avoids u , and so $G - v$ is an $RE(o)$ graph. The case $n = 2$ yields that $G - v$ is an $RE(p)$ graph. Also, $G - v$ is not an $RE(k)$ graph for $k = 1$ and $k = 2$. These remarks lead to the next result where we note that the *hypo-RE(p)* graphs have already been described in Theorem 3.

Theorem 4.

- (a) A graph G on $p \geq 4$ vertices is *hypo-RE(o)* if and only if $G = K_{2n}$, $n \geq 3$.
- (b) No graph is *hypo-RE(1)* or *hypo-RE(2)*.
- (c) A graph G on $p \geq 4$ vertices is *hypo-RE(p)* if and only if $G = K_4$.

We conclude this section by stating a result analogous to Theorem 2.

Theorem 5. A graph G is *hypo-nonRE(k)* if and only if G is an $RE(k)$ graph.

Hypo-traversable Graphs

A graph G on $p \geq 2$ vertices is said to be *traversable* if G has an eulerian trail, i. e., G has an open trail which contains all the vertices and edges of G (and in view of the next result, this trail begins at one of the odd vertices and ends at the other).

Theorem (Euler). Let G be a connected graph. Then G is traversable if and only if $\epsilon(G) = 2$.

Let G be a hypo-traversable graph. Then $\epsilon(G) \neq 2$, and $\epsilon(G - v) = 2$ for each vertex v of G . It is clear that G is a block, and $\delta(G) \geq 2$. Also, $\epsilon(G)$ is even and $0 \leq \epsilon(G) \leq p$. From the first possible value we readily get the following.

Theorem 6. Let G be any connected graph which has euler number 0. Then G is hypo-traversable if and only if G is a cycle.

Proof. The sufficiency is immediate, and for the necessity we note that $\epsilon(G) = 0$ implies that $V(G) = \xi(G)$. Now $\epsilon(G - v) = 2$ for any vertex v of G gives $\deg v = 2$. By connectedness, G has to be a cycle.

Now let $\epsilon(G) = 2m$, $m \geq 2$, and let G be hypo-traversable. Let $u \in \xi(G)$ and $v \in \theta(G)$. Then it can be seen that $\deg u = 2m - 2, 2m$ or $2m + 2$ and $\deg v = 2m - 3, 2m - 1$ or $2m + 1$, otherwise $\epsilon(G - w) \neq 2$ for some vertex w of G . This fact is useful in considering individual cases. Should $m = 2$, the possible values of $\deg v$ will be 3 or 5 since $\delta(G) \geq 2$. It can be verified that for $p \leq 5$, cycles are the only hypo-traversable graphs. Figure 1 shows all graphs on 6 vertices which are hypo-traversable.

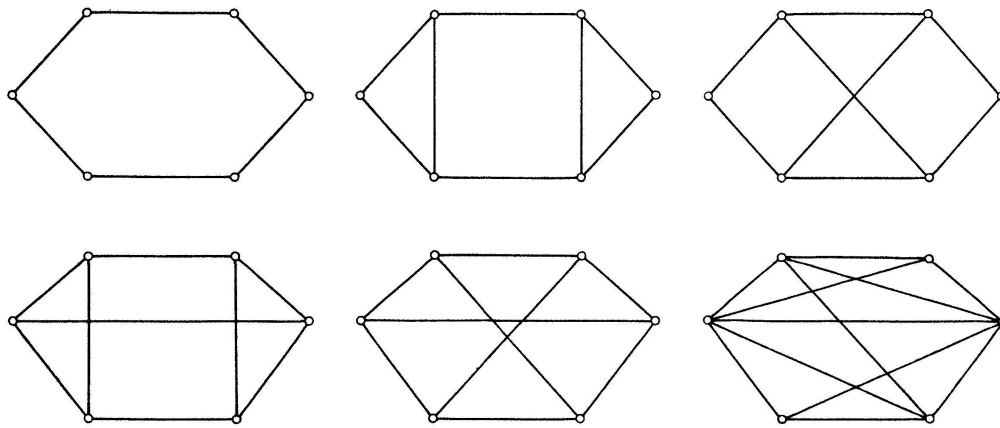


Figure 1

Hypo-traversable graphs on 6 vertices.

The preceding theorem dealt with the case when the graph had all vertices even. The next result treats graphs possessing no even vertices.

Theorem 7. Let G be any connected graph having euler number $\epsilon(G) = p(G) \geq 6$. Then G is hypo-traversable if and only if G is regular of degree $p - 3$.

Proof. Here $\xi(G) = \emptyset$ and $p = 2m = \epsilon(G)$. By the above remarks, every vertex of G is odd and has possible degrees $2m - 3$ or $2m - 1$. But if any vertex is adjacent with all the other $p - 1$ vertices, its deletion gives an eulerian graph. The necessity now follows.

Conversely, let G be a connected $(p - 3)$ -regular graph and $\epsilon(G) = p(G) \geq 6$. Then $\epsilon(G - v) = 2$ for all v , and the proof is complete.

Theorem 8. Let G be a connected graph having euler number $\epsilon(G) = p(G) - 1$, and let $p(G) \geq 5$. Then G is hypo-traversable if and only if the even vertex u of G has degree $p - 3$, the vertices a and b that are nonadjacent with u have degree $p - 4$, and every other vertex has degree $p - 2$.

Proof. Let $\xi(G) = \{u\}$, and assume that G is hypo-traversable. Since every vertex adjacent with u becomes even in the traversable graph $G - u$, we need $\deg u = p - 3$. Let a and b be the vertices nonadjacent with u , and let $v \in \theta(G) - \{a, b\}$. Now the traversable graph $G - w$ contains exactly 2 odd vertices, for each $w \in V(G)$.

Hence $\deg v = p - 2$ and $\deg a = \deg b = p - 4$. For the sufficiency we note that $\epsilon(G) \geq 4$, and by hypothesis, $\epsilon(G - w) = 2$ for each vertex w of G .

It is possible that a complete classification of hypo-traversable graphs may get involved with discussing individual cases, and this suggests scope for further research.

Let G be a hypo-nontraversable graph, i.e., $\epsilon(G) = 2$ and $\epsilon(G - v) \neq 2$ for each vertex v . Moreover, since it is meaningful to require that $G - v$ be connected, we further assume that G has no cutpoints and $p \geq 4$ (so that $\delta(G) \geq 2$). Designate the two odd vertices of G as a and b . If ab is not an edge in G , then $\epsilon(G - a)$ and $\epsilon(G - b)$ are 4 or more. On the other hand, if a and b are adjacent, we must have $\deg a \geq 5$ and $\deg b \geq 5$. Now let $v \in \xi(G)$. This imposes the following restrictions: If $\deg v = 2$, then v is adjacent with either both or neither of a and b ; if $\deg v = 4$, then v is not simultaneously joined to both a and b . These present a set of necessary conditions for G to have the desired property, and it can be verified that they are also sufficient.

Theorem 9. Let G be a block with $p \geq 4$. Then G is hypo-nontraversable if and only if $\theta(G) = \{a, b\}$ and

- (i) $ab \in E(G) \Rightarrow \deg a \geq 5$ and $\deg b \geq 5$,
- (ii) $\deg v = 2 \Rightarrow v$ is joined to both or neither of a, b , and
- (iii) $\deg v = 4 \Rightarrow v$ is not joined to both a and b .

In [1] a traversable graph G is called *randomly traversable from a vertex v* if every trail in G with initial vertex v can be extended to an eulerian trail of G . Clearly, a traversable graph can be randomly traversable from $k = 0, 1$ or 2 vertices, and we may, as before, denote this class of graphs as $RT(k)$, where $RT(2)$ will refer to the class of *randomly traversable graphs*. It was also proved in [1] that if a and b are the two odd vertices of a traversable graph G , then G is randomly traversable from a if and only if every cycle of G contains b . Moreover, a graph G is in $RT(2)$ if and only if the two odd vertices of G lie on every cycle of G . This suggests the problem of studying *hypo-RT(k)* and *hypo-nonRT(k)* graphs.

We conclude by presenting a complete classification of $RT(2)$ graphs.

Theorem 10. Let G be a traversable graph with $\theta(G) = \{a, b\}$. Then G is randomly traversable if and only if G is homeomorphic from K_2 , $K(2, 2m - 1)$ or $K(2, 2m) + ab$, where $m \geq 1$.

Proof. It is obvious that the graphs described are randomly traversable. To prove the converse, first we note that if $\deg a = 1$, then any $b - a$ path must be G itself, otherwise there exists a cycle which avoids a or b . Thus, $\deg b = 1$, and the graph G is homeomorphic from K_2 . So we assume that each of a and b has degree at least 3.

Let v be any vertex of G other than a or b . Since G is connected, there exist $v - a$ and $v - b$ paths. Clearly these paths have v as their only common vertex otherwise some cycle of G avoids a or b . Moreover, the union of these paths gives an $a - b$ path which contains v . With every vertex $v \in V(G) - \theta(G)$ we can associate an $a - b$ path $P(v)$ such that $P(v)$ contains v . Let us consider the collection of all $a - b$ paths, where, for obvious reasons, any two paths are disjoint, i.e., the only vertices common to them are a and b . So $P(v)$ is unique, and the union of all these

paths must be G itself. We therefore conclude that every vertex other than a and b has degree 2, and $\deg a = \deg b$ is odd. Also, if a and b are adjacent, then $G - ab$ is homeomorphic from $K(2, 2m)$; and if a, b are nonadjacent, then G is homeomorphic from $K(2, 2m - 1)$, where $m \geq 1$.

S.F. Kapoor¹⁾, Western Michigan University, USA

REFERENCES

- [1] G. CHARTRAND and A. T. WHITE, *Randomly Traversable Graphs*, *El. Math.* 25, 101–107 (1970).
- [2] F. HARARY, *Graph Theory* (Addison-Wesley, Reading 1969).
- [3] J. MITCHEM, *Hypo-Properties in Graphs*, *The Many Facets of Graph Theory* (G. Chartrand and S. F. Kapoor editors), *Lecture Notes in Mathematics* No. 110 (Springer-Verlag, Berlin 1969), p. 223–230.
- [4] O. ORE, *A Problem Regarding the Tracing of Graphs*, *El. Math.* 6, 49–53 (1951).

¹⁾ Research partially supported by National Science Foundation grant GP 9435.

Kleine Mitteilungen

New Quadratic Forms with High Density of Primes

Let p_{min} be the smallest prime contained in a quadratic form of the shape $f(x) = Ax^2 + Ax - C$ and let n_{icp} be the number of initial consecutive primes of $f(x)$, then, by means of a CDC 6400 computer, all $f(x) = Ax^2 + Ax - C$ were investigated for $A < 10$, $C < 2 \cdot 10^5$, and $p_{min} > 47$. In Table 1, the number below C is the number of all primes of $f(x)$ for $x < 100$, and p_{min} is the number in parentheses.

For each form $x^2 + x - C$ we have also a form $9y^2 + 9y - (C - 2)$, because the substitution $x = 3y + 1$ transforms $x^2 + x - C$ into $9y^2 + 9y - (C - 2)$; hence, each third term of $x^2 + x - C$ (starting with the second) belongs to $9y^2 + 9y - (C - 2)$. Similarly, for each form $2x^2 - C$ we have also a form $8z^2 + 8z - (C - 2)$, because the substitution $x = 2z + 1$ transforms $2x^2 - C$ into $8z^2 + 8z - (C - 2)$; hence, each second term of $2x^2 - C$ (starting with the second) belongs to $8z^2 + 8z - (C - 2)$. For the forms $2x^2 - 119131$ and $2x^2 - 186871$, related to the forms with $A = 8$ in Table 1, we have 64 and 61 primes, respectively, for $x < 100$.

Table 1 gives the impression that there might be no forms with $A = 4$. This is not so. In a test run with $A < 10$, $10^8 - 5000 < C < 10^8$, and $p_{min} > 47$, the forms $x^2 + x - 99995659$, $9x^2 + 9x - 99995657$, and $4x^2 + 4x - 99996937$ were discovered, all with $p_{min} = 53$.

The form $x^2 + x - 53509$ with $p_{min} = 61$ is due to N.G.W.H. Beeger [1] in 1938, the forms $x^2 + x - 90073$ with $p_{min} = 53$ and $x^2 + x - 169933$ with $p_{min} = 59$ are due to the author [2] in 1967.

Two hundred years ago, Euler published his famous quadratic form $x^2 + x + 41$ with $p_{min} = 41$ and $n_{icp} = 40$. This form was believed to have the highest density of primes of all quadratic forms $Ax^2 + Bx \pm C$ discovered till now. Many forms were found with $p_{min} > 41$ and the second differences greater than 2; but the corresponding