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## Minimum Area of Circumscribed Polygons

### 1. Introduction

In [1] some estimates on minimal areas of polygons circumscribed about a plane convex set were considered. In what follows we shall prove a theorem that leads to very concise proofs of those estimates and some other results concerning circumscribed polygons.

We shall deal mainly with plane convex bodies. If  $K$  is a plane convex body, the area of  $K$  will usually be denoted by the same symbol  $K$  in order to simplify notation. We shall say that two convex  $n$ -gons are *parallel* if corresponding sides are parallel. Then we can state the main theorem as follows.

*Theorem 1.* Suppose  $K$  is a plane convex body,  $p$  is a polygon inscribed in  $K$ , and  $P$  is a polygon parallel to  $p$  and circumscribed about  $K$ . Then

$$K^2 \geq pP. \quad (1)$$

### 2. Proof of the main theorem

The proof of Theorem 1 depends on Minkowski's concept of the *mixed area*,  $A(K, L)$ , of two plane convex bodies  $K$  and  $L$ . In case  $p$  and  $P$  are parallel  $n$ -gons,  $A(p, P)$  is easily described as follows. Let  $O$  be a point fixed interior to  $P$ . If  $l_i$  is the length of a side of  $p$ , let  $d_i$  be the distance from  $O$  to the corresponding parallel side of  $P$ . Then

$$A(p, P) = \frac{1}{2} \sum d_i l_i, \quad (2)$$

summed over all sides of  $p$ . In [5] one can find a treatment of the properties of mixed areas and a proof of the following fundamental inequality of Minkowski:

$$A(K, L)^2 \geq KL. \quad (3)$$

Now consider a plane convex body  $K$ , with inscribed  $n$ -gon  $p$  and parallel circumscribed  $n$ -gon  $P$ . Each side of  $P$  contains at least one point of  $K$ . If we choose one such point on each side of  $P$ , then these points, taken together with the vertices of  $p$ , are the vertices of a convex  $2n$ -gon  $Q$  inscribed in  $K$ . Fix a point  $O$  inside  $p$ . If  $l_i$  is the length of a side of  $p$ , let  $d_i$  be the distance from  $O$  to the corresponding parallel side of  $P$ . Upon making a sketch of the situation, the reader will readily see that the area of  $Q$  is given by

$$Q = \frac{1}{2} \sum d_i l_i = A(p, P). \quad (4)$$

Using the fact that  $Q \subset K$ , and Minkowski's inequality, we then have,

$$K^2 \geq Q^2 = A(p, P)^2 \geq pP, \quad (5)$$

which proves Theorem 1.

### 3. Applications of the main theorem

We now derive a number of corollaries of Theorem 1, with all proofs following basically the same pattern.

*Corollary 1.* Any plane convex body  $K$  is contained in a triangle  $T_0$  of area not more than twice that of  $K$ .

*Proof.* Let  $T_0$  be a triangle of minimal area containing  $K$ . Then the midpoints of the sides of  $T_0$  touch  $K$  (see [1] for a proof). Let  $t$  be the triangle inscribed in  $K$  formed by joining these midpoints, and let  $T$  be the triangle parallel to  $t$  and circumscribed about  $K$ . We have that  $t = \frac{1}{4} T_0$  and  $T \geq T_0$ . Hence

$$K^2 \geq tT \geq \left(\frac{1}{4} T_0\right) (T_0) = \frac{1}{4} T_0^2, \quad (6)$$

so  $T_0 \leq 2K$ , as we wanted to prove.

*Corollary 2.* Any plane convex body  $K$  is contained in a quadrilateral  $Q_0$  of area not more than  $\sqrt{2}$  times that of  $K$ .

*Proof.* Let  $Q_0$  be a quadrilateral of minimal area containing  $K$ . Again (see [1]) the midpoints of the sides of  $Q_0$  touch  $K$ . Let  $q$  be the quadrilateral inscribed in  $K$  formed by joining the midpoints of the sides of  $Q_0$ . Let  $Q$  be the quadrilateral parallel to  $q$  circumscribed about  $K$ . We have  $Q \geq Q_0$ , and it is easy to see  $q$  is a parallelogram with  $q = \frac{1}{2} Q_0$ . Hence

$$K^2 \geq qQ \geq \left(\frac{1}{2} Q_0\right) (Q_0) = \frac{1}{2} Q_0^2, \quad (7)$$

so  $Q_0 \leq (\sqrt{2}) K$ , as required.

The result given in Corollary 1 is in a sense the best possible, since a parallelogram  $K$  is not contained in any triangle of area less than twice the area of  $K$ . On the other hand, it is not known if the estimate for minimal circumscribed quadrilaterals in Corollary 2 is best possible, and good estimates for minimal circumscribed  $n$ -gons,  $n > 4$ , are apparently not known. However, the next corollary of Theorem 1 shows how to obtain an inequality by utilizing the maximum inscribed  $n$ -gon.

*Corollary 3.* Any plane convex body  $K$  is contained in an  $n$ -gon  $P$  of area not more than  $\frac{2\pi}{n} \csc \frac{2\pi}{n}$  times that of  $K$ .

*Proof.* Let  $p$  be an  $n$ -gon of maximal area inscribed in  $K$ , and let  $P$  be the circumscribed  $n$ -gon parallel to  $p$ . By a theorem of Sas (see [4]), we have  $p \geq \left( \frac{n}{2\pi} \sin \frac{2\pi}{n} \right) K$ . Hence

$$K^2 \geq pP \geq \left( \frac{n}{2\pi} \sin \frac{2\pi}{n} \right) KP, \quad (8)$$

from which the result follows.

Suppose  $K$  is a centrally symmetric plane convex body. By a *lattice packing* of  $K$  we mean a distribution of translates of  $K$ , no pair having interior points in common, with their centers forming a plane lattice. The density of such a packing measures the fraction of the plane covered by these translates of  $K$ . The following result, proved in [4] in a different manner, follows readily from Theorem 1.

*Corollary 4.* Any centrally symmetric plane convex body  $K$  can be lattice packed with density at least  $\frac{\sqrt{3}}{2}$ .

*Proof.* By a theorem of Dowker [3], there is a centrally symmetric hexagon  $H_0$  of minimum area circumscribed about  $K$ . A theorem of Day [2] implies that the mid-points of the sides of  $H_0$  touch  $K$ . Let  $h$  be the hexagon formed by joining the mid-points of the sides of  $H_0$ . Then it is not a difficult exercise to verify that  $h$  is the affine image of a regular hexagon, with  $h = \frac{3}{4} H_0$ . Let  $H$  be the centrally symmetric hexagon parallel to  $h$  and circumscribed about  $K$ . Then  $H \geq H_0$ , and

$$K^2 \geq hH \geq \left( \frac{3}{4} H_0 \right) (H_0) = \frac{3}{4} H_0^2, \quad (9)$$

so  $K \geq \left( \frac{\sqrt{3}}{2} \right) H_0$ . Since  $H_0$  tiles the plane in a lattice manner, the required result follows.

#### 4. Generalization to higher dimensions

Using mixed volumes in place of mixed areas, the following higher dimensional analogue of Theorem 1 is easily proved.

*Theorem 2.* Let  $K$  be a convex body in Euclidean  $n$ -space. Let  $p$  be a convex polytope contained in  $K$  and let  $P$  be a polytope circumscribed about  $K$  and parallel to  $p$  (that is, the facets of  $P$  parallel to corresponding facets of  $p$ ). Then

$$K^n \geq p^{n-1} P, \quad (10)$$

where we are now using the same notational convention for volumes that we used before for areas.

*Corollary 5.* Any convex body  $K$  in Euclidean  $n$ -space is contained in a simplex  $T_0$  of volume not more than  $n^{n-1}$  times that of  $K$ .

*Proof.* Let  $T_0$  be a simplex of minimal volume containing  $K$ . By the theorem of Day [2], the centroids of the facets of  $T_0$  touch  $K$ . Let  $t$  be the simplex whose vertices are those centroids, and let  $T$  be the simplex parallel to  $t$  and circumscribed about  $K$ . Then  $t = (n^{-n}) T_0$  and  $T \geq T_0$ , so

$$K^n \geq t^{n-1} T \geq (n^{-n(n-1)} T_0^{n-1}) (T_0), \quad (11)$$

so  $T_0 \leq (n^{n-1}) K$ , as we wanted to prove.

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# Hypo-Eulerian and Hypo-Traversable Graphs

## Introduction

If a graph  $G$  does not possess a given property  $P$ , and for each vertex  $v$  of  $G$  the graph  $G - v$  enjoys property  $P$ , then  $G$  is said to be a *hypo- $P$*  graph. Recently, studies have been made where  $P$  stands for the graph being *hamiltonian*, *planar*, and *outerplanar* (e.g., see [3]). Here we obtain a characterization of *hypo-eulerian* and *hypo-randomly-eulerian* graphs, and investigate in this respect some of the other concepts arising out of Euler's solution of the classical Königsberg Seven Bridges Problem.

## Preliminaries

Following the terminology of [2], a *graph* will be finite, undirected, without loops or multiple edges. A *walk* of a graph  $G$  is an alternating sequence  $v_0, e_1, v_1, e_2, v_2, \dots, v_{n-1}, e_n, v_n$  of vertices and edges of  $G$ , beginning and ending with vertices and where the edge  $e_i = v_{i-1} v_i$  for  $i = 1, 2, \dots, n$ . This is a  $v_0 - v_n$  walk, and is usually denoted  $v_0 v_1 v_2 \dots v_n$ ; it is *closed* if  $v_0 = v_n$  and *open* otherwise. A walk is a *trail* if all its edges are distinct; it is a *path* if all its vertices are distinct. A closed trail is a circuit and a circuit on distinct vertices is a *cycle*. A cycle on  $p$  vertices is denoted  $C_p$ , and  $C_3$  is called a *triangle*.

If for every two distinct vertices  $u$  and  $v$  of a graph  $G$  there exists a  $u - v$  path, then  $G$  is *connected*. A *component* of  $G$  is a maximal connected subgraph of  $G$ . A vertex