Kleine Mitteilungen

On Moser's Problem of Accommodating Closed Curves in Triangles

In problem 11 of his well-known collection [2] of problems in combinatorial geometry, Leo Moser asks, 'What is the largest number $f = f(a, b, c)$ such that every closed curve of length f can be accommodated in the triangle (if it exists) of sides a, b and c? How is it for arcs?'

A closed curve Γ can be 'accommodated' in a triangular region (ABC) in different ways. We might demand that (ABC) contain a *translate* of Γ, or a *displacement* of Γ (i.e., the image of Γ under a direct isometry), or a *congruent copy* of Γ. In each of the first two interpretations, Moser's problem is the dual of a problem solved in [3]. The solution of the problem in all three interpretations is given below.

In theorem 2 of [3] we showed using Fagnano's problem that a triangular region (ABC) with angles α, β and γ and perimeter p contains a translate of every closed curve of length L if and only if

$$p \geq \frac{L}{2} \frac{\sin\alpha + \sin\beta + \sin\gamma}{\operatorname{Sin}\alpha \operatorname{Sin}\beta \operatorname{Sin}\gamma}, \tag{1}$$

where $\operatorname{Sin}\theta = \sin\theta$ when θ is acute, and $\operatorname{Sin}\theta = 1$ otherwise. It follows since (1) is sharp that the largest number L such that the triangular region (ABC) contains a translate of all closed curves of length L is precisely

$$L = \frac{2p \operatorname{Sin}\alpha \operatorname{Sin}\beta \operatorname{Sin}\gamma}{\sin\alpha + \sin\beta + \sin\gamma}.$$

Rewriting this formula in terms of the sides a, b and c (see formula (1) of [3]), we find

Theorem 1. If a, b and c are the sides of a triangle, then the largest number $f_T = f_T(a, b, c)$ so that the triangular region (ABC) with sides a, b and c contains a translate of every closed curve of length f_T is

$$f_T = \begin{cases} \dfrac{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}{2abc} & \text{if } (ABC) \text{ is acute,} \\[2ex] \dfrac{[(a+b+c)(-a+b+c)(a-b+c)(a+b-c)]^{1/2}}{\max\{a, b, c\}} & \text{if } (ABC) \text{ is not acute.} \end{cases}$$

In geometric terms, f_T is the perimeter of the orthic triangle of (ABC) when (ABC) is acute, and f_T is twice the longest altitude of (ABC) when (ABC) is not acute.

In theorem 5 of [3] we showed using an inequality due to Eggleston that a triangular region (ABC) with angles α, β and γ and perimeter p contains a displacement of every closed curve of length L if and only if

$$p \geq \frac{L}{2\pi} \frac{(\sin\alpha + \sin\beta + \sin\gamma)^2}{\sin\alpha \sin\beta \sin\gamma}. \tag{2}$$

It follows since (2) is sharp that the largest number L such that the triangular region (ABC) contains a displacement of all closed curves of length L is precisely

$$L = \frac{2\pi p \sin\alpha \sin\beta \sin\gamma}{(\sin\alpha + \sin\beta + \sin\gamma)^2} .$$

Rewriting this formula in terms of the sides a, b and c, we find

Theorem 2. If a, b and c are the sides of a triangle, then the largest number $f_\delta = f_\delta(a, b, c)$ so that the triangular region (ABC) with sides a, b and c contains a displacement of every closed curve of length f_δ is

$$f_\delta = \pi \left[\frac{(-a+b+c)\,(a-b+c)\,(a+b-c)}{a+b+c}\right]^{1/2} . \tag{3}$$

In geometric terms, f_δ is precisely the circumference of the inscribed circle of (ABC).

If $f_\mu = f_\mu(a, b, c)$ is the largest number so that the triangular region (ABC) with sides a, b and c contains a congruent copy of every closed curve of length f_μ, we see that $f_\mu \geqq f_\delta$ because every direct isometry is an isometry. On the other hand, $f_\mu \leqq f_\delta$ because (ABC), whose incircle has circumference f_δ, must contain a circle with circumference f_μ. Thus we find

Theorem 3. If a, b and c are the sides of a triangle, then the largest number $f_\mu = f_\mu(a, b, c)$ so that the triangular region (ABC) with sides a, b and c contains a congruent copy of every closed curve of length f_μ is given by (3).

Only fragmentary results are known on the corresponding problem for arcs. A partial result for the equilateral triangle appears in [1].

John E. Wetzel, University of Illinois, Urbana, USA

REFERENCES

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[2] Leo Moser, *Poorly Formulated Unsolved Problems of Combinatorial Geometry* (Mimeographed).
[3] J. E. Wetzel, *Triangular Covers for Closed Curves of Constant Length*, El. Math. *25*, 78–81 (1970).

A Note on the Elliptic Integral $K(k)$

We wish to point out the following particularly simple proof (cf. [1, 2]) of the well-known limit relation

$$\boldsymbol{K}(k) - \log \frac{4}{\sqrt{1-k^2}} \to 0 , \quad k \to 1-0$$

satisfied by the complete elliptic integral of the first kind

$$\boldsymbol{K}(k) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)\,(1-k^2 t^2)}} , \quad 0 < k < 1 .$$

Write

$$\boldsymbol{K}(k) = \int_0^{\varkappa} + \int_{\varkappa}^{1} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}},$$

choosing $\varkappa = \varkappa(k)$ as the fixed point of the continuous mapping $t \mapsto \tau$ of $[0, 1]$ onto $[0, 1]$ defined by

$$(1-k^2t^2)(1-k^2\tau^2) = 1-k^2 .$$

Clearly τ decreases from 1 to 0 as t increases from 0 to 1, so that $\varkappa$ is a well-defined point of $(0, 1)$; furthermore

$$(1-k^2\tau^2)\, t\, dt + (1-k^2t^2)\, \tau\, d\tau = 0 ,$$

$$t\sqrt{1-k^2\tau^2} = \sqrt{1-\tau^2}, \quad \tau\sqrt{1-k^2t^2} = \sqrt{1-t^2},$$

$$\frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} + \frac{d\tau}{\sqrt{(1-\tau^2)(1-k^2\tau^2)}} = 0 .$$

Hence

$$\boldsymbol{K}(k) = 2\int_0^{\varkappa} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} .$$

This at once yields

$$\boldsymbol{K}(k) > 2\int_0^{\varkappa} \frac{k\, dt}{1-k^2t^2} = 2\log\frac{1+k\varkappa}{\sqrt{1-k^2\varkappa^2}},$$

$$\boldsymbol{K}(k) < 2\int_0^{\varkappa} \frac{dt}{1-t^2} = 2\log\frac{1+\varkappa}{\sqrt{1-\varkappa^2}} .$$

Put $k' = \sqrt{1-k^2}$. Then $k\varkappa = \sqrt{1-k'}$, $1/\varkappa = \sqrt{1+k'}$ and we have

$$2\log\left(\frac{1+\sqrt{1-k'}}{\sqrt{k'}}\right) < \boldsymbol{K}(k) < 2\log\left(\frac{1+\sqrt{1+k'}}{\sqrt{k'}}\right)$$

from which the conclusion is immediate.

T. S. Nanjundiah, University of Mysore, India

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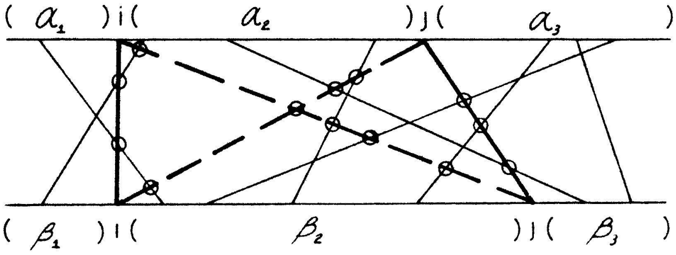
[1] M. Abraham and R. Becker, *The Classical Theory of Electricity and Magnetism*, 2nd Ed. (Blackie, London and Glasgow 1950), p. 173–174.

[2] E. Cesàro, *Elementares Lehrbuch der algebraischen Analysis und der Infinitesimalrechnung* (B. G. Teubner, Leipzig und Berlin 1904), p. 772–773.

The Parity of Permutations

From time to time various articles have appeared in which it was shown that a permutation cannot be expressed both as a product of an even number and as a product of an odd number of transpositions. These proofs, and the usual textbook proof, are algebraic in nature; in contrast we offer the following simple geometric proof.

Let $b = \{b_1, \dots, b_n\}$ be a permutation on $\{1, \dots, n\}$. Let α and β be two distinct horizontal lines in the plane with n distinct points labelled $1, \dots, n$ in the order $1, \dots, n$ from left to right on α and n distinct points labelled $1, \dots, n$ in the order $b_1, \dots, b_n$ on β so that no three of the segments joining some i on α to some j on β are concurrent. Let B be the number of intersection points of the set of segments γ_i joining i on α to i on β. It suffices to show that a transposition (i, j), $i < j$ changes the



parity of B for then it follows that for every sequence of transpositions $t_1, \dots, t_k$ determining a permutation p, the parity of k is equal to the parity of the associated set P of intersection points.

We may assume i is to the left of j on β also. Let α_1, α_2 and α_3 be the set of numbered points on α to the left of i, between i and j and to the right of j respectively. Define β_1, β_2 and β_3 in the same manner for β and let $|\alpha_r \beta_s|$ be the number of γ segments joining points of α_r to points of β_s. If i and j are now transposed on β then, excluding the intersection of γ_i and γ_j, there is a net increase of $|\alpha_2 \beta_2| + |\alpha_3 \beta_2| - |\alpha_1 \beta_2|$ intersection points on γ_i and $|\alpha_1 \beta_2| + |\alpha_2 \beta_2| - |\alpha_3 \beta_2|$ on γ_j. This gives a total increase of $2\,|\alpha_2 \beta_2| + 1$ intersection points which results in a change in the parity of B.

R. C. Entringer, University of New Mexico, USA

A Theorem which is Equivalent to the Axiom of Choice

The purpose of this note is to establish the following simple theorem which we have been unable to find in the literature.

Theorem. Every relation R is a union of functions with the same domain as R.

Proof. Let R be a relation with domain X. We may assume that X is nonempty. Let $F = \{f \mid f$ is a function with domain X, and $f \subset R\}$. The set F is nonempty.

Indeed, if $x \in X$, let $R(x) = \{y \mid (x, y) \in R\}$, and set $B = \{R(x) \mid x \in X\}$. By the axiom of choice there is a function $f: X \to \cup B$ such that $f(x) \in R(x)$ for each $x \in X$; clearly, $f \in F$. Let U denote the union of the functions in F. Then $U \subset R$. Assume that there is an element (x, y) in R which is not in U. Let $f \in F$. Then $(x, y) \notin f$. Since f is a function with domain X there is a unique $z \in X$ such that $(x, z) \in f \subset R$. Define a new function g from f by replacing the element (x, z) in f by the original element (x, y). Since $f \subset R$ and $(x, y) \in R$ we have $g \subset R$; that is, $g \in F$. It follows that $(x, y) \in g \subset U$, contrary to the way that (x, y) was chosen. The proof is now complete.

It is easy to see that the axiom of choice is a consequence of the theorem. In fact, if $E = \{X_\alpha \mid \alpha \in A\}$ is a nonempty family of nonempty sets, let R be the relation from A into $\cup E$ given by $R = \{(\alpha, x_\alpha) \mid \alpha \in A, x_\alpha \in X_\alpha\}$. By the theorem there is a function $f \subset R$ such that $f(\alpha) \in X_\alpha$ for each $\alpha \in A$, i.e., f is a choice function. Hence, the theorem and the axiom of choice are equivalent.

R. S. Doran, Texas Christian University, USA

Aufgaben

Aufgabe 642. Am ebenen Dreieck mit Seiten abc, Inradius r, Umradius R und Flächeninhalt rs beweise man die Verschärfung

$$(b-c)^2 + (c-a)^2 + (a-b)^2 + k\,[r s \sqrt{3} + (4 - 2\sqrt{3})\, r\, (R - 2r)] \leqslant \begin{cases} a^2 + b^2 + c^2 & \text{für} \quad k = 4 \\ (a+b+c)^2/2 & \text{für} \quad k = 6 \end{cases}$$

zweier Ungleichungen von H. Hadwiger, JBer. DMV *49*, 2. Abt. S. 35–39.

I. Paasche, München

Solution: Since

$$3[(b-c)^2 + (c-a)^2 + (a-b)^2 - a^2 - b^2 - c^2]$$

$$= 2\left[(b-c)^2 + (c-a)^2 + (a-b)^2 - \frac{1}{2}(a+b+c)^2\right]$$

the two stated inequalities are equivalent. It accordingly suffices to prove the first.

For $k = 4$ we have to prove

$$a^2 + b^2 + c^2 + 4\, rs \sqrt{3} + 4(4 - 2\sqrt{3})\, r\, (R - 2r) \leqslant 2(bc + ca + ab)\,, \qquad (*)$$

or what is the same thing,

$$(a+b+c)^2 + 4\, rs \sqrt{3} + 4\,(4 - 2\sqrt{3})\, r\, (R - 2r) \leqslant 4\,(bc + ca + ab)\,.$$

Since

$$bc + ca + ab = s^2 + 4\, Rr + r^2\,,$$