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Some Trigonometric Inequalities

1. Introduction

Let α_1, α_2 and α_3 be the angles of some triangle with perimeter $2s$, inradius r and circumradius R . The identities

$$\prod_{i=1}^3 \sin \frac{\alpha_i}{2} = \frac{r}{4R}, \quad \prod_{i=1}^3 \tan \frac{\alpha_i}{2} = \frac{r}{s}, \quad \prod_{i=1}^3 \cos \frac{\alpha_i}{2} = \frac{s}{4R},$$

together with some familiar inequalities ([2], §5), show that

$$\begin{aligned} \prod \sin \frac{\alpha_i}{2} &\leq \left(\frac{\sqrt{3}}{2}\right)^3 \prod \tan \frac{\alpha_i}{2} \leq \left(\frac{\sqrt{3}}{3}\right)^3 \prod \cos \frac{\alpha_i}{2} \leq \frac{1}{8} \\ &\leq \left(\frac{\sqrt{3}}{4}\right)^3 \prod \sec \frac{\alpha_i}{2} \leq \left(\frac{\sqrt{3}}{6}\right)^3 \prod \cot \frac{\alpha_i}{2} \leq \left(\frac{1}{4}\right)^3 \prod \operatorname{cosec} \frac{\alpha_i}{2}; \end{aligned}$$

in each case, equality holds if and only if $\alpha_1 = \alpha_2 = \alpha_3 = \pi/3$.

Another way of writing this chain of inequalities is

$$\left. \begin{aligned} M_0 \left(\sin \frac{\alpha}{2} \right) &\leq \frac{\sqrt{3}}{2} M_0 \left(\tan \frac{\alpha}{2} \right) \leq \frac{\sqrt{3}}{3} M_0 \left(\cos \frac{\alpha}{2} \right) \leq \frac{1}{2} \\ &\leq \frac{\sqrt{3}}{4} M_0 \left(\sec \frac{\alpha}{2} \right) \leq \frac{\sqrt{3}}{6} M_0 \left(\cot \frac{\alpha}{2} \right) \leq \frac{1}{4} M_0 \left(\operatorname{cosec} \frac{\alpha}{2} \right), \end{aligned} \right\} \quad (1)$$

where $M_r(x)$ denotes the mean of order r of the positive numbers $(x) = (x_1, x_2, \dots, x_n)$, defined by

$$M_r(x) = \begin{cases} \min(x) & \text{for } r = -\infty, \quad \max(x) & \text{for } r = +\infty, \\ \left(\prod_{i=1}^n x_i \right)^{1/n} & \text{for } r = 0, \\ \left(\frac{1}{n} \sum_{i=1}^n x_i^r \right)^{1/r} & \text{otherwise.} \end{cases}$$

$M_r(x)$ is a continuous function of r for $-\infty < r < +\infty$, and a strictly increasing function of r on the same interval unless all the x_i are equal ([3], pp. 12, 15 and 26).

In this note, we propose to show how several of the inequalities in (1) can be extended to other values of r . Our main tool will be an inequality for convex functions due to Hardy, Littlewood and Pólya ([3], p. 45 and 89), rediscovered by Karamata [5]. Other applications of this inequality to elementary geometry may be found in [1] and [7].

2. Preliminaries

We state the inequality to which we have just alluded as

Theorem A (Hardy, Littlewood, Pólya). *Let $(x) = (x_1, \dots, x_n)$ and $(y) = (y_1, \dots, y_n)$ be real. A necessary and sufficient condition for the inequality*

$$\phi(x_1) + \dots + \phi(x_n) \leq \phi(y_1) + \dots + \phi(y_n)$$

to hold for every real function ϕ continuous and convex in some interval containing all the numbers (x) and (y) is that

$$x_1 \leq x_2 \leq \dots \leq x_n, \quad y_1 \leq y_2 \leq \dots \leq y_n,$$

$$x_\nu + \dots + x_n \leq y_\nu + \dots + y_n, \quad \nu = 2, 3, \dots, n,$$

$$x_1 + \dots + x_n = y_1 + \dots + y_n.$$

If $\phi''(t) > 0$ for $y_1 \leq t \leq y_2$, equality holds if and only if $(x) = (y)$.

We require only the case $n = 3$ of this inequality. In order to apply it, we need the following lemma.

Lemma 1. *If $0 < \alpha_1 \leq \alpha_2 \leq \alpha_3 < \pi$ and $\alpha_1 + \alpha_2 + \alpha_3 = \pi$, then*

$$\sin \frac{\alpha_1}{2} \leq \frac{\sqrt{3}}{2} \sec \frac{\alpha_1}{2}, \tag{2}$$

$$2 \cos \frac{\alpha_1}{2} \leq \cot \frac{\alpha_1}{2} \quad \text{and} \quad 2 \cos \frac{\alpha_3}{2} \geq \cot \frac{\alpha_3}{2}, \tag{3}$$

$$\frac{\sqrt{3}}{3} \cos \frac{\alpha_1}{2} \leq \sin \frac{\alpha_3}{2} \leq \frac{\sqrt{3}}{2} \tan \frac{\alpha_3}{2}, \tag{4}$$

$$\frac{\sqrt{3}}{3} \cos \frac{\alpha_3}{2} \geq \sin \frac{\alpha_1}{2} \geq \frac{\sqrt{3}}{2} \tan \frac{\alpha_1}{2}, \tag{5}$$

with equality if and only if $\alpha_3 = \pi/3$.

Proof. Clearly, (2), (3) and the right-hand inequalities of (4) and (5) follow at once from $\alpha_1 \leq \pi/3 \leq \alpha_3$. For the left-hand side of (4), we first note that since $\alpha_3 \geq (\alpha_2 + \alpha_3)/2 = (\pi - \alpha_1)/2$, we have $\sin \alpha_3 > \cos \alpha_1/2$ if also $\alpha_3 \leq (\pi + \alpha_1)/2$. Then, as $\sin \alpha_3 = 2 \cos \alpha_3/2 \sin \alpha_3/2 \leq \sqrt{3} \sin \alpha_3/2$, we have the desired inequality. And if $\alpha_3 > (\pi + \alpha_1)/2$, then $\pi/2 > \alpha_3/2 > \pi/4$, so that $\sin \alpha_3/2 > \sqrt{2}/2 > \sqrt{3}/3 > \sqrt{3}/3 \cos \alpha_1/2$.

The left-hand side of (5) is established similarly. The case of equality is obvious. In § 8, we require the case $n = 3$ of the following result ([8], §4).

Theorem B. *If $(x) = (x_1, \dots, x_n)$ and $(y) = (y_1, \dots, y_n)$ are positive, and if the function $F(r) = M_r(x) - M_r(y)$ has more than $n - 1$ real zeros, then $F(r) \equiv 0$.*

In all that follows, we assume $0 < \alpha_1 \leq \alpha_2 \leq \alpha_3 < \pi$ and $\alpha_1 + \alpha_2 + \alpha_3 = \pi$. Any identities not proved in the paper can be deduced from those in [4], §150-154.

3. Sines and cosines

We shall extend the inequality $\sqrt{3} M_0(\sin \alpha/2) \leq M_0(\cos \alpha/2)$ by proving the existence of a number s , $0 < s < 2$, such that

$$\left. \begin{aligned} \sqrt{3} M_r \left(\sin \frac{\alpha}{2} \right) &\leq M_r \left(\cos \frac{\alpha}{2} \right) && \text{for } r < s, \\ \sqrt{3} M_r \left(\sin \frac{\alpha}{2} \right) &\geq M_r \left(\cos \frac{\alpha}{2} \right) && \text{for } r > s; \end{aligned} \right\} \quad (6)$$

the inequalities are strict unless $\alpha_1 = \alpha_2 = \alpha_3 = \pi/3$, when equality holds for all r .

Indeed, we have

$$M_2 \left(\cos \frac{\alpha}{2} \right) \leq \frac{\sqrt{3}}{2} \leq \sqrt{3} M_2 \left(\sin \frac{\alpha}{2} \right), \quad (7)$$

since $\sum \cos^2 \alpha_i/2 = (r + 4R)/2R$ and $R \geq 2r$. Now $M_r(\cos \alpha/2) - \sqrt{3} M_r(\sin \alpha/2)$ is a continuous function of r ; since it is positive for $r = 0$ and negative for $r = 2$, there exists a number s , $0 < s < 2$, for which

$$M_s \left(\cos \frac{\alpha}{2} \right) = \sqrt{3} M_s \left(\sin \frac{\alpha}{2} \right). \quad (8)$$

We can write (8) as

$$\cos^s \frac{\alpha_1}{2} + \cos^s \frac{\alpha_2}{2} + \cos^s \frac{\alpha_3}{2} = (\sqrt{3})^s \left(\sin^s \frac{\alpha_1}{2} + \sin^s \frac{\alpha_2}{2} + \sin^s \frac{\alpha_3}{2} \right). \quad (9)$$

Since $0 < \alpha_1 \leq \alpha_2 \leq \alpha_3 < \pi$ and $s > 0$, we have

$$\cos^s \frac{\alpha_1}{2} \geq \cos^s \frac{\alpha_2}{2} \geq \cos^s \frac{\alpha_3}{2}, \quad \sin^s \frac{\alpha_3}{2} \geq \sin^s \frac{\alpha_2}{2} \geq \sin^s \frac{\alpha_1}{2}. \quad (10)$$

Further, by (4),

$$\cos^s \frac{\alpha_1}{2} \leq (\sqrt{3})^s \sin^s \frac{\alpha_3}{2}, \quad (11)$$

while (5) and (9) together give

$$\cos^s \frac{\alpha_1}{2} + \cos^s \frac{\alpha_2}{2} \leq (\sqrt{3})^s \left(\sin^s \frac{\alpha_2}{2} + \sin^s \frac{\alpha_3}{2} \right). \quad (12)$$

Now (9), (10), (11) and (12) are precisely the conditions of the Hardy-Littlewood-Pólya inequality (in the case $n = 3$). We may now affirm that

$$\sum_{i=1}^3 \phi\left(\cos^s \frac{\alpha_i}{2}\right) \leq \sum_{i=1}^3 \phi\left(3^{s/2} \sin^s \frac{\alpha_i}{2}\right) \quad (13)$$

for all real functions ϕ continuous and convex on an interval containing $(\cos^s \alpha/2)$ and $(3^{s/2} \sin^s \alpha/2)$.

In particular, taking successively for ϕ the functions defined by

$$\phi(x) = x^{r/s} \quad \text{if } r < 0 \text{ or } r > s ,$$

$$\phi(x) = -x^{r/s} \quad \text{if } 0 < r < s ,$$

$$\phi(x) = -\log x ,$$

which are convex for $x > 0$, we obtain (6). It follows from the case of equality in Theorem A that for any r , equality holds in (6) if and only if the triangle is equilateral. In other terms, $M_r(\cos \alpha/2) - \sqrt{3} M_r(\sin \alpha/2)$, as function of r , is either identically zero, or has exactly one zero, situated in the interval $0 < r < 2$.

The exact value of s in (8) will depend on (α) ; for instance, $s > 1$ for $(\alpha) = (10^\circ, 10^\circ, 160^\circ)$, but $s < 1$ when $(\alpha) = (30^\circ, 70^\circ, 80^\circ)$.

4. Cosines and tangents

Since

$$(x_1 + x_2 + x_3)^2 \geq 3(x_1 x_2 + x_2 x_3 + x_3 x_1) \quad (14)$$

for any real numbers x_1, x_2, x_3 , and since

$$\tan \frac{\alpha_1}{2} \tan \frac{\alpha_2}{2} + \tan \frac{\alpha_2}{2} \tan \frac{\alpha_3}{2} + \tan \frac{\alpha_3}{2} \tan \frac{\alpha_1}{2} = 1 , \quad (15)$$

we have $\sqrt{3} M_1(\tan \alpha/2) \geq 1$. This, together with the left-hand side of (7) and the fact that $M_r(x)$ is a non-decreasing function of r , shows that

$$2 M_1\left(\cos \frac{\alpha}{2}\right) \leq \sqrt{3} \leq 3 M_1\left(\tan \frac{\alpha}{2}\right) .$$

Reasoning as in §3, we deduce the existence of a number $t, 0 < t < 1$, with the property that

$$\left. \begin{aligned} 3 M_r\left(\tan \frac{\alpha}{2}\right) &\leq 2 M_r\left(\cos \frac{\alpha}{2}\right) && \text{for } r < t , \\ 3 M_r\left(\tan \frac{\alpha}{2}\right) &\geq 2 M_r\left(\cos \frac{\alpha}{2}\right) && \text{for } r > t ; \end{aligned} \right\} \quad (16)$$

the case of equality is the same as for (6).

5. Cosines and cotangents

We have

$$\tan \frac{\alpha_{i-1}}{2} + \tan \frac{\alpha_{i+1}}{2} = \frac{\cos \frac{\alpha_i}{2}}{\cos \frac{\alpha_{i-1}}{2} \cos \frac{\alpha_{i+1}}{2}}, \quad i = 1, 2, 3,$$

where the indices are taken mod 3. By addition, we get

$$2 \sum_{i=1}^3 \tan \frac{\alpha_i}{2} = \sum_{i=1}^3 \left(\frac{\cos \frac{\alpha_{i-1}}{2} \cos \frac{\alpha_{i+1}}{2}}{\cos \frac{\alpha_i}{2}} \right)^{-1}.$$

But

$$x_1^2 (x_2 - x_3)^2 + x_2^2 (x_3 - x_1)^2 + x_3^2 (x_1 - x_2)^2 \geq 0,$$

whence

$$\left(\frac{x_1 x_2}{x_3} \right) + \left(\frac{x_2 x_3}{x_1} \right) + \left(\frac{x_3 x_1}{x_2} \right) \geq x_1 + x_2 + x_3$$

when $x_1 x_2 x_3 > 0$, with equality if and only if $x_1 = x_2 = x_3$.

Consequently,

$$2 \sum_{i=1}^3 \tan \frac{\alpha_i}{2} \geq \sum_{i=1}^3 \left(\cos \frac{\alpha_i}{2} \right)^{-1},$$

or

$$M_{-1} \left(\cot \frac{\alpha}{2} \right) \leq 2 M_{-1} \left(\cos \frac{\alpha}{2} \right). \tag{17}$$

As before we conclude from Theorem A, using (1), (3) and (17) that unless the triangle is equilateral, the function $M_r (\cot \alpha/2) - 2 M_r (\cos \alpha/2)$ has a single zero, situated in the interval $-1 < r < 0$. In other words, there exists a u , $-1 < u < 0$, such that

$$\left. \begin{aligned} M_r \left(\cot \frac{\alpha}{2} \right) &\leq 2 M_r \left(\cos \frac{\alpha}{2} \right) && \text{for } r < u \\ M_r \left(\cot \frac{\alpha}{2} \right) &\geq 2 M_r \left(\cos \frac{\alpha}{2} \right) && \text{for } r > u, \end{aligned} \right\} \tag{18}$$

with equality as in (6).

6. Sines and tangents

From (1) we have

$$2 M_0 \left(\sin \frac{\alpha}{2} \right) \leq \sqrt{3} M_0 \left(\tan \frac{\alpha}{2} \right),$$

and we shall show presently that

$$2 M_{-1} \left(\sin \frac{\alpha}{2} \right) \geq \sqrt{3} M_{-1} \left(\tan \frac{\alpha}{2} \right). \quad (19)$$

Hence, using (4) and (5), we can prove the existence of a number v , $-1 < v < 0$, with the following property:

$$\left. \begin{aligned} \sqrt{3} M_r \left(\tan \frac{\alpha}{2} \right) &\leq 2 M_r \left(\sin \frac{\alpha}{2} \right) && \text{for } r < v, \\ \sqrt{3} M_r \left(\tan \frac{\alpha}{2} \right) &\geq 2 M_r \left(\sin \frac{\alpha}{2} \right) && \text{for } r > v; \end{aligned} \right\} \quad (20)$$

equality holds under the same condition as in (6).

To prove (19), we first observe that this inequality is equivalent to

$$\sqrt{3} \sum_{i=1}^3 \left(\sin \frac{\alpha_i}{2} \right)^{-1} \leq 2 \sum_{i=1}^3 \cot \frac{\alpha_i}{2}. \quad (21)$$

But since

$$\cot \frac{\alpha_{i-1}}{2} + \cot \frac{\alpha_{i+1}}{2} = \frac{\cos \frac{\alpha_i}{2}}{\sin \frac{\alpha_{i-1}}{2} \sin \frac{\alpha_{i+1}}{2}},$$

(21) is in turn equivalent to

$$2 \sqrt{3} \sum_{1 \leq i < j \leq 3} \sin \frac{\alpha_i}{2} \sin \frac{\alpha_j}{2} \leq \sum_{i=1}^3 \sin \alpha_i. \quad (22)$$

Now

$$\sin \frac{\alpha_i}{2} \sin \frac{\alpha_j}{2} = \frac{1}{4R} \left(a_i a_j \tan \frac{\alpha_i}{2} \tan \frac{\alpha_j}{2} \right)^{1/2},$$

where a_i denotes the length of the side opposite angle α_i . And by Cauchy's inequality and (15),

$$\sum_{i < j} \left(a_i a_j \tan \frac{\alpha_i}{2} \tan \frac{\alpha_j}{2} \right)^{1/2} \leq \left(\sum_{i < j} a_i a_j \right)^{1/2};$$

as $a_i = 2 R \sin \alpha_i$, this gives us

$$\left(2 \sum_{i < j} \sin \frac{\alpha_i}{2} \sin \frac{\alpha_j}{2} \right)^2 \leq \sum_{i < j} \sin \alpha_i \sin \alpha_j,$$

a stronger inequality than (22), by (14).

7. Sines and secants

In attempting to extend the inequality $4 M_0 (\sin \alpha/2) \leq \sqrt{3} M_0 (\sec \alpha/2)$, we meet a different type of problem. We will establish this result: Assume $\alpha_i \neq \pi/3$ for some i . Then,

$$4 M_r \left(\sin \frac{\alpha}{2} \right) < \sqrt{3} M_r \left(\sec \frac{\alpha}{2} \right) \tag{23}$$

for all real r , if $\alpha_3 \geq 2\pi/3$. If $\alpha_3 < 2\pi/3$, there is a number $r_0 > 1$ such that (23) holds for $r < r_0$, and is reversed for $r > r_0$.

Proof. Firstly, if $\alpha_3 \geq 2\pi/3$, then $\alpha_1 \leq \alpha_2 < \pi/3$, whence $\sin \alpha_i \leq \sqrt{3}/2$, or $4 \sin \alpha_i/2 \leq \sqrt{3} \sec \alpha_i/2$, $i = 1, 2, 3$; and at least two of these inequalities are strict. Hence (23) holds for all real r .

Secondly, note that (23) holds for $r = 1$, whatever (α) . For $f(x) = \sin x/2$ is convex on $0 \leq x \leq \pi$, so by Jensen's inequality ([3], §3.6), $M_1 (\sin \alpha/2) \leq 1/2$. And we deduce from (7) that $\sqrt{3} M_1 (\sec \alpha/2) = \sqrt{3} [M_{-1} (\cos \alpha/2)]^{-1} \geq 2$.

Now if $(\pi/3) < \alpha_3 < 2\pi/3$, we have

$$4 \sin \frac{\alpha_3}{2} > \sqrt{3} \sec \frac{\alpha_3}{2}, \tag{24}$$

that is, $4 M_\infty (\sin \alpha/2) > \sqrt{3} M_\infty (\sec \alpha/2)$. This and (23) for $r = 1$ establish the existence of a zero, say $r = r_0$, of $4 M_r (\sin \alpha/2) - \sqrt{3} M_r (\sec \alpha/2)$, with $r_0 > 1$. Then, as usual, it follows from (2) and (24) that this function has no other zero.

8. Other inequalities

In trying to extend the inequality $3 M_0 (\tan \alpha/2) \leq M_0 (\cot \alpha/2)$, we find a situation quite different from the one which we met in §§ 3 through 7, where the functions under consideration always had at most one real zero. Indeed, since $\tan \alpha_i/2 \cot \alpha_i/2 = 1$, the inequalities

$$3 M_r \left(\tan \frac{\alpha}{2} \right) \leq M_r \left(\cot \frac{\alpha}{2} \right) \quad \text{and} \quad 3 M_{-r} \left(\tan \frac{\alpha}{2} \right) \leq M_{-r} \left(\cot \frac{\alpha}{2} \right)$$

are equivalent (for a similar argument in another setting, see Makowski [6]). Hence, $g(r) := 3 M_r (\tan \alpha/2) - M_r (\cot \alpha/2)$ has an even number of real zeros (possibly none), unless it vanishes identically. In fact, it has no zeros for certain choices of (α) , as the following proposition shows:

In order that

$$3 M_r \left(\tan \frac{\alpha}{2} \right) \leq M_r \left(\cot \frac{\alpha}{2} \right) \tag{25}$$

for all r , it is necessary and sufficient that $\tan \alpha_1/2 \tan \alpha_3/2 \leq 1/3$. If $r \neq +\infty$, equality holds when $\alpha_1 = \alpha_2 = \alpha_3 = \pi/3$, and only then.

Proof. The condition is necessary because it is equivalent to the necessary condition $3 M_\infty (\tan \alpha/2) \leq M_\infty (\cot \alpha/2)$.

To prove its sufficiency we consider $g(r)$, defined as above. By (1), $g(0) < 0$ unless $\alpha_1 = \alpha_2 = \alpha_3$. And if $\tan \alpha_1/2 \tan \alpha_3/2 < 1/3$, then $g(+\infty) < 0$. But then if g has any zeros on $(0, \infty)$, it has at least two, hence at least 4 on $(-\infty, \infty)$; by Theorem B this can occur only if $\alpha_1 = \alpha_2 = \alpha_3$. Further, if $\tan \alpha_1/2 \tan \alpha_3/2 = 1/3$ then $\tan \alpha_1/2 = (1/3) \cot \alpha_3/2$ and $\tan \alpha_3/2 = (1/3) \cot \alpha_1/2$, so that if equality held in (25) for some $r \neq +\infty$ we would also have $\tan \alpha_2/2 = \sqrt{3}/3$. And then we would have $M_0 (\tan \alpha/2) = \sqrt{3}/3$, whence $\alpha_1 = \alpha_2 = \alpha_3$, by the condition of equality in (1). This concludes the proof.

Now we shall show that in *any* triangle,

$$3 M_r \left(\tan \frac{\alpha}{2} \right) \leq M_r \left(\cot \frac{\alpha}{2} \right) \quad \text{for } |r| \leq 1, \tag{26}$$

with equality in the usual case. Indeed, it is clear from the preceding discussion that it suffices to prove (26) for $r = 1$. Since

$$\sum_{i=1}^3 \tan \frac{\alpha_i}{2} = \frac{(r + 4R)}{s} \quad \text{and} \quad \sum_{i=1}^3 \cot \frac{\alpha_i}{2} = \frac{s}{r},$$

we must show that $s^2 \geq 3r(r + 4R)$; and this inequality is known ([2], § 5.6).

In a like manner, one can show that the condition $\sin \alpha_1/2 \sin \alpha_3/2 \leq 1/4$ is necessary and sufficient in order that $4 M_r (\sin \alpha/2) \leq M_r (\operatorname{cosec} \alpha/2)$ for all r , and that $4 M_r (\cos \alpha/2) \leq 3 M_r (\sec \alpha/2)$ for all r if and only if $\cos \alpha_1/2 \cos \alpha_3/2 \leq 3/4$. One can also show that in any non-equilateral triangle,

$$4 M_r \left(\sin \frac{\alpha}{2} \right) < M_r \left(\operatorname{cosec} \frac{\alpha}{2} \right) \quad \text{for } |r| \leq 1 \tag{27}$$

and

$$4 M_r \left(\cos \frac{\alpha}{2} \right) < 3 M_r \left(\sec \frac{\alpha}{2} \right) \quad \text{for } |r| \leq 2. \tag{28}$$

For instance, (28) follows from (7): $4 M_2 (\cos \alpha/2) < 2\sqrt{3}$, whence $M_2 (\sec \alpha/2) = [M_{-2} (\cos \alpha/2)]^{-1} > [M_2 (\cos \alpha/2)]^{-1} > 2\sqrt{3}$, so that (28) holds for $r = 2$, and therefore for $|r| \leq 2$. The proof of (27) is similar.

Finally, if $\alpha_3 \neq \pi/3$, $h(r) := M_r (\cot \alpha/2) - 2\sqrt{3} M_r (\sin \alpha/2)$ is positive at $r = 0$, and can have two zeros, or one, or none, depending on (α) . For instance, if $(\alpha) = (10^\circ, 40^\circ, 130^\circ)$, $h(r)$ has exactly one zero, situated somewhere on $(-\infty, 0)$. On the other hand, if we choose $\alpha_2 = 60^\circ$ and α_1, α_3 arbitrary (but $\alpha_1 \neq \alpha_3$ and $\alpha_1 + \alpha_3 = 120^\circ$) we get a triangle such that $h(r) > 0$ for all r . And if $(\alpha) = (50^\circ, 50^\circ, 80^\circ)$, then $h(r)$ has two real zeros. By comparing (20) and (26) in the range $0 \leq r \leq 1$, we see that in any non-equilateral triangle,

$$2\sqrt{3} M_r \left(\sin \frac{\alpha}{2} \right) < M_r \left(\cot \frac{\alpha}{2} \right) \quad \text{for } 0 \leq r \leq 1.$$

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Über hebbare Unstetigkeiten

In [1] wurde die Menge $\mathcal{F}[a, b]$ derjenigen in $[a, b]$ definierten Funktionen betrachtet, die in jedem Punkt von $[a, b]$ unstetig sind.

Für $f \in \mathcal{F}[a, b]$ bezeichne wiederum $\mathcal{H}[f]$ die Menge derjenigen Punkte von $[a, b]$, in denen die Unstetigkeit von f hebbar ist, und $\mathcal{U}[f]$ die Menge derjenigen Punkte von $[a, b]$, in denen die Unstetigkeit von f nicht hebbar ist.

In [1] wurde nun einerseits gezeigt, dass für jedes $f \in \mathcal{F}[a, b]$ $\mathcal{U}[f]$ dicht in $[a, b]$ ist; andererseits wurden Funktionen $f \in \mathcal{F}[a, b]$ konstruiert, für die $\mathcal{H}[f]$ «sehr umfassend» ist. Daran anschliessend wurde die Frage aufgeworfen, ob für ein $f \in \mathcal{F}[a, b]$ $\mathcal{H}[f]$ sogar dicht sein kann. Wir werden in dieser Note beweisen, dass dies nicht möglich ist. Damit ist dann gleichzeitig gezeigt, dass die Beispiele aus [1] für Funktionen f mit «sehr umfassendem» $\mathcal{H}[f]$ «gut» sind.

Bei gegebenem $f \in \mathcal{F}[a, b]$ definieren wir noch die «abgeänderte» Funktion f^* :

$$f^*(x) = \begin{cases} \lim_{\xi \rightarrow x} f(\xi) & \text{für } x \in \mathcal{H}[f] \\ f(x) & \text{für } x \in \mathcal{U}[f]. \end{cases}$$

Die Menge derjenigen Punkte aus $[a, b]$, in denen f^* stetig ist, bezeichnen wir mit $\mathcal{S}[f^*]$.

Hilfssatz 1

Für jedes $f \in \mathcal{F}[a, b]$ ist $\mathcal{H}[f]$ abzählbar.

Beweis

Sei $f \in \mathcal{F}[a, b]$. Für beliebiges $\varepsilon > 0$ bilden wir die Punktmenge

$$\mathcal{H}_\varepsilon[f] = \{x \in \mathcal{H}[f] \mid |f^*(x) - f(x)| \geq \varepsilon\}.$$