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Some Trigonometric Inequalities

1. Introduction

Let α_1, α_2 and α_3 be the angles of some triangle with perimeter $2s$, inradius r and circumradius R. The identities

$$\prod_{i=1}^{3} \sin\frac{\alpha_i}{2} = \frac{r}{4R}, \quad \prod_{i=1}^{3} \tan\frac{\alpha_i}{2} = \frac{r}{s}, \quad \prod_{i=1}^{3} \cos\frac{\alpha_i}{2} = \frac{s}{4R},$$

together with some familiar inequalities ([2], §5), show that

$$\prod \sin\frac{\alpha_i}{2} \le \left(\frac{\sqrt{3}}{2}\right)^3 \prod \tan\frac{\alpha_i}{2} \le \left(\frac{\sqrt{3}}{3}\right)^3 \prod \cos\frac{\alpha_i}{2} \le \frac{1}{8}$$
$$\le \left(\frac{\sqrt{3}}{4}\right)^3 \prod \sec\frac{\alpha_i}{2} \le \left(\frac{\sqrt{3}}{6}\right)^3 \prod \cot\frac{\alpha_i}{2} \le \left(\frac{1}{4}\right)^3 \prod \operatorname{cosec}\frac{\alpha_i}{2};$$

in each case, equality holds if and only if $\alpha_1 = \alpha_2 = \alpha_3 = \pi/3$.

Another way of writing this chain of inequalities is

$$\left.\begin{aligned} &M_0\left(\sin\frac{\alpha}{2}\right) \le \frac{\sqrt{3}}{2} M_0\left(\tan\frac{\alpha}{2}\right) \le \frac{\sqrt{3}}{3} M_0\left(\cos\frac{\alpha}{2}\right) \le \frac{1}{2} \\ &\quad \le \frac{\sqrt{3}}{4} M_0\left(\sec\frac{\alpha}{2}\right) \le \frac{\sqrt{3}}{6} M_0\left(\cot\frac{\alpha}{2}\right) \le \frac{1}{4} M_0\left(\operatorname{cosec}\frac{\alpha}{2}\right), \end{aligned}\right\} \quad (1)$$

where $M_r(x)$ denotes the mean of order r of the positive numbers $(x) = (x_1, x_2, \ldots, x_n)$, defined by

$$M_r(x) = \begin{cases} \min(x) \quad \text{for} \quad r = -\infty, \quad \max(x) \quad \text{for} \quad r = +\infty, \\ \left(\prod_{i=1}^{n} x_i\right)^{1/n} \quad \text{for} \quad r = 0, \\ \left(\frac{1}{n}\sum_{i=1}^{n} x_i^r\right)^{1/r} \quad \text{otherwise}. \end{cases}$$

$M_r(x)$ is a continuous function of r for $-\infty < r < +\infty$, and a strictly increasing function of r on the same interval unless all the x_i are equal ([3], pp. 12, 15 and 26).

In this note, we propose to show how several of the inequalities in (1) can be extended to other values of r. Our main tool will be an inequality for convex functions due to Hardy, Littlewood and Pólya ([3], p. 45 and 89), rediscovered by Karamata [5]. Other applications of this inequality to elementary geometry may be found in [1] and [7].

2. Preliminaries

We state the inequality to which we have just alluded as

Theorem A (Hardy, Littlewood, Pólya). *Let* $(x) = (x_1, \ldots, x_n)$ *and* $(y) = (y_1, \ldots, y_n)$ *be real. A necessary and sufficient condition for the inequality*

$$\phi(x_1) + \cdots + \phi(x_n) \leq \phi(y_1) + \cdots + \phi(y_n)$$

to hold for every real function ϕ *continuous and convex in some interval containing all the numbers* (x) *and* (y) *is that*

$$x_1 \leq x_2 \leq \ldots \leq x_n, \quad y_1 \leq y_2 \leq \ldots \leq y_n,$$

$$x_\nu + \cdots + x_n \leq y_\nu + \cdots + y_n, \; \nu = 2, 3, \ldots, n,$$

$$x_1 + \cdots + x_n = y_1 + \cdots + y_n.$$

If $\phi''(t) > 0$ *for* $y_1 \leq t \leq y_2$, *equality holds if and only if* $(x) = (y)$.

We require only the case $n = 3$ of this inequality. In order to apply it, we need the following lemma.

Lemma 1. *If* $0 < \alpha_1 \leq \alpha_2 \leq \alpha_3 < \pi$ *and* $\alpha_1 + \alpha_2 + \alpha_3 = \pi$, *then*

$$\sin\frac{\alpha_1}{2} \leq \frac{\sqrt{3}}{2}\sec\frac{\alpha_1}{2}, \tag{2}$$

$$2\cos\frac{\alpha_1}{2} \leq \cot\frac{\alpha_1}{2} \quad \textit{and} \quad 2\cos\frac{\alpha_3}{2} \geq \cot\frac{\alpha_3}{2}, \tag{3}$$

$$\frac{\sqrt{3}}{3}\cos\frac{\alpha_1}{2} \leq \sin\frac{\alpha_3}{2} \leq \frac{\sqrt{3}}{2}\tan\frac{\alpha_3}{2}, \tag{4}$$

$$\frac{\sqrt{3}}{3}\cos\frac{\alpha_3}{2} \geq \sin\frac{\alpha_1}{2} \geq \frac{\sqrt{3}}{2}\tan\frac{\alpha_1}{2}, \tag{5}$$

with equality if and only if $\alpha_3 = \pi/3$.

Proof. Clearly, (2), (3) and the right-hand inequalities of (4) and (5) follow at once from $\alpha_1 \leq \pi/3 \leq \alpha_3$. For the left-hand side of (4), we first note that since $\alpha_3 \geq (\alpha_2 + \alpha_3)/2 = (\pi - \alpha_1)/2$, we have $\sin\alpha_3 > \cos\alpha_1/2$ if also $\alpha_3 \leq (\pi + \alpha_1)/2$. Then, as $\sin\alpha_3 = 2\cos\alpha_3/2 \sin\alpha_3/2 \leq \sqrt{3}\sin\alpha_3/2$, we have the desired inequality. And if $\alpha_3 > (\pi + \alpha_1)/2$, then $\pi/2 > \alpha_3/2 > \pi/4$, so that $\sin\alpha_3/2 > \sqrt{2}/2 > \sqrt{3}/3 > \sqrt{3}/3 \cos\alpha_1/2$.

The left-hand side of (5) is established similarly. The case of equality is obvious.

In § 8, we require the case $n = 3$ of the following result ([8], §4).

Theorem B. *If $(x) = (x_1, \ldots, x_n)$ and $(y) = (y_1, \ldots, y_n)$ are positive, and if the function $F(r) = M_r(x) - M_r(y)$ has more than $n - 1$ real zeros, then* $F(r) \doteq 0$.

In all that follows, we assume $0 < \alpha_1 \leq \alpha_2 \leq \alpha_3 < \pi$ and $\alpha_1 + \alpha_2 + \alpha_3 = \pi$. Any identities not proved in the paper can be deduced from those in [4], §150-154.

3. Sines and cosines

We shall extend the inequality $\sqrt{3}\, M_0\, (\sin \alpha/2) \leq M_0\, (\cos \alpha/2)$ by proving the existence of a number s, $0 < s < 2$, such that

$$\left.\begin{aligned} \sqrt{3}\, M_r \left(\sin \frac{\alpha}{2}\right) &\leq M_r \left(\cos \frac{\alpha}{2}\right) \quad \text{for} \quad r < s\,, \\ \sqrt{3}\, M_r \left(\sin \frac{\alpha}{2}\right) &\geq M_r \left(\cos \frac{\alpha}{2}\right) \quad \text{for} \quad r > s\,; \end{aligned}\right\} \tag{6}$$

the inequalities are strict unless $\alpha_1 = \alpha_2 = \alpha_3 = \pi/3$, when equality holds for all r.

Indeed, we have

$$M_2 \left(\cos \frac{\alpha}{2}\right) \leq \frac{\sqrt{3}}{2} \leq \sqrt{3}\, M_2 \left(\sin \frac{\alpha}{2}\right), \tag{7}$$

since $\sum \cos^2 \alpha_i/2 = (r + 4R)/2R$ and $R \geq 2\, r$. Now $M_r\, (\cos \alpha_2) - \sqrt{3}\, M_r\, (\sin \alpha/2)$ is a continuous function of r; since it is positive for $r = 0$ and negative for $r = 2$, there exists a number s, $0 < s < 2$, for which

$$M_s \left(\cos \frac{\alpha}{2}\right) = \sqrt{3}\, M_s \left(\sin \frac{\alpha}{2}\right). \tag{8}$$

We can write (8) as

$$\cos^s \frac{\alpha_1}{2} + \cos^s \frac{\alpha_2}{2} + \cos^s \frac{\alpha_3}{2} = (\sqrt{3})^s \left(\sin^s \frac{\alpha_1}{2} + \sin^s \frac{\alpha_2}{2} + \sin^s \frac{\alpha_3}{2}\right). \tag{9}$$

Since $0 < \alpha_1 \leq \alpha_2 \leq \alpha_3 < \pi$ and $s > 0$, we have

$$\cos^s \frac{\alpha_1}{2} \geq \cos^s \frac{\alpha_2}{2} \geq \cos^s \frac{\alpha_3}{2}, \qquad \sin^s \frac{\alpha_3}{2} \geq \sin^s \frac{\alpha_2}{2} \geq \sin^s \frac{\alpha_1}{2}. \tag{10}$$

Further, by (4),

$$\cos^s \frac{\alpha_1}{2} \leq (\sqrt{3})^s \sin^s \frac{\alpha_3}{2}, \tag{11}$$

while (5) and (9) together give

$$\cos^s \frac{\alpha_1}{2} + \cos^s \frac{\alpha_2}{2} \leq (\sqrt{3})^s \left(\sin^s \frac{\alpha_2}{2} + \sin^s \frac{\alpha_3}{2}\right). \tag{12}$$

Now (9), (10), (11) and (12) are precisely the conditions of the Hardy-Littlewood-Pólya inequality (in the case $n = 3$). We may now affirm that

$$\sum_{i=1}^{3} \phi\left(\cos^s \frac{\alpha_i}{2}\right) \le \sum_{i=1}^{3} \phi\left(3^{s/2} \sin^s \frac{\alpha_i}{2}\right) \tag{13}$$

for all real functions ϕ continuous and convex on an interval containing $(\cos^s \alpha/2)$ and $(3^{s/2} \sin^s \alpha/2)$.

In particular, taking successively for ϕ the functions defined by

$$\phi(x) = x^{r/s} \quad \text{if } r < 0 \text{ or } r > s\,,$$

$$\phi(x) = -x^{r/s} \text{ if } 0 < r < s\,,$$

$$\phi(x) = -\log x\,,$$

which are convex for $x > 0$, we obtain (6). It follows from the case of equality in Theorem A that for any r, equality holds in (6) if and only if the triangle is equilateral. In other terms, $M_r(\cos \alpha/2) - \sqrt{3}\, M_r(\sin \alpha/2)$, as function of r, is either identically zero, or has exactly one zero, situated in the interval $0 < r < 2$.

The exact value of s in (8) will depend on (α); for instance, $s > 1$ for $(\alpha) = (10°, 10°, 160°)$, but $s < 1$ when $(\alpha) = (30°, 70°, 80°)$.

4. Cosines and tangents

Since

$$(x_1 + x_2 + x_3)^2 \ge 3\,(x_1 x_2 + x_2 x_3 + x_3 x_1) \tag{14}$$

for any real numbers x_1, x_2, x_3, and since

$$\tan\frac{\alpha_1}{2} \tan\frac{\alpha_2}{2} + \tan\frac{\alpha_2}{2} \tan\frac{\alpha_3}{2} + \tan\frac{\alpha_3}{2} \tan\frac{\alpha_1}{2} = 1\,, \tag{15}$$

we have $\sqrt{3}\, M_1(\tan \alpha/2) \ge 1$. This, together with the left-hand side of (7) and the fact that $M_r(x)$ is a non-decreasing function of r, shows that

$$2\, M_1\left(\cos\frac{\alpha}{2}\right) \le \sqrt{3} \le 3\, M_1\left(\tan\frac{\alpha}{2}\right).$$

Reasoning as in § 3, we deduce the existence of a number t, $0 < t < 1$, with the property that

$$\left.\begin{aligned} 3\, M_r\left(\tan\frac{\alpha}{2}\right) &\le 2\, M_r\left(\cos\frac{\alpha}{2}\right) \quad \text{for} \quad r < t\,,\\ 3\, M_r\left(\tan\frac{\alpha}{2}\right) &\ge 2\, M_r\left(\cos\frac{\alpha}{2}\right) \quad \text{for} \quad r > t\,; \end{aligned}\right\} \tag{16}$$

the case of equality is the same as for (6).

5. Cosines and cotangents

We have

$$\tan\frac{\alpha_{i-1}}{2} + \tan\frac{\alpha_{i+1}}{2} = \frac{\cos\dfrac{\alpha_i}{2}}{\cos\dfrac{\alpha_{i-1}}{2}\cos\dfrac{\alpha_{i+1}}{2}}, \qquad i = 1, 2, 3,$$

where the indices are taken mod 3. By addition, we get

$$2\sum_{i=1}^{3}\tan\frac{\alpha_i}{2} = \sum_{i=1}^{3}\left(\frac{\cos\dfrac{\alpha_{i-1}}{2}\cos\dfrac{\alpha_{i+1}}{2}}{\cos\dfrac{\alpha_i}{2}}\right)^{-1}.$$

But

$$x_1^2(x_2 - x_3)^2 + x_2^2(x_3 - x_1)^2 + x_3^2(x_1 - x_2)^2 \geq 0,$$

whence

$$\left(\frac{x_1 x_2}{x_3}\right) + \left(\frac{x_2 x_3}{x_1}\right) + \left(\frac{x_3 x_1}{x_2}\right) \geq x_1 + x_2 + x_3$$

when $x_1 x_2 x_3 > 0$, with equality if and only if $x_1 = x_2 = x_3$.

Consequently,

$$2\sum_{i=1}^{3}\tan\frac{\alpha_i}{2} \geq \sum_{i=1}^{3}\left(\cos\frac{\alpha_i}{2}\right)^{-1},$$

or

$$M_{-1}\left(\cot\frac{\alpha}{2}\right) \leq 2\,M_{-1}\left(\cos\frac{\alpha}{2}\right). \tag{17}$$

As before we conclude from Theorem A, using (1), (3) and (17) that unless the triangle is equilateral, the function $M_r(\cot\alpha/2) - 2\,M_r(\cos\alpha/2)$ has a single zero, situated in the interval $-1 < r < 0$. In other words, there exists a u, $-1 < u < 0$, such that

$$\left.\begin{aligned} M_r\left(\cot\frac{\alpha}{2}\right) &\leq 2M_r\left(\cos\frac{\alpha}{2}\right) \quad \text{for} \quad r < u \\ M_r\left(\cot\frac{\alpha}{2}\right) &\geq 2M_r\left(\cos\frac{\alpha}{2}\right) \quad \text{for} \quad r > u, \end{aligned}\right\} \tag{18}$$

with equality as in (6).

6. Sines and tangents

From (1) we have

$$2\,M_0\left(\sin\frac{\alpha}{2}\right) \leq \sqrt{3}\,M_0\left(\tan\frac{\alpha}{2}\right),$$

and we shall show presently that

$$2\,M_{-1}\left(\sin\frac{\alpha}{2}\right) \geq \sqrt{3}\,M_{-1}\left(\tan\frac{\alpha}{2}\right). \tag{19}$$

Hence, using (4) and (5), we can prove the existence of a number v, $-1 < v < 0$, with the following property:

$$\left.\begin{aligned} \sqrt{3}\,M_r\left(\tan\frac{\alpha}{2}\right) &\leq 2\,M_r\left(\sin\frac{\alpha}{2}\right) \quad \text{for} \quad r < v, \\ \sqrt{3}\,M_r\left(\tan\frac{\alpha}{2}\right) &\geq 2\,M_r\left(\sin\frac{\alpha}{2}\right) \quad \text{for} \quad r > v; \end{aligned}\right\} \tag{20}$$

equality holds under the same condition as in (6).

To prove (19), we first observe that this inequality is equivalent to

$$\sqrt{3}\sum_{i=1}^{3}\left(\sin\frac{\alpha_i}{2}\right)^{-1} \leq 2\sum_{i=1}^{3}\cot\frac{\alpha_i}{2}. \tag{21}$$

But since

$$\cot\frac{\alpha_{i-1}}{2} + \cot\frac{\alpha_{i+1}}{2} = \frac{\cos\dfrac{\alpha_i}{2}}{\sin\dfrac{\alpha_{i-1}}{2}\sin\dfrac{\alpha_{i+1}}{2}},$$

(21) is in turn equivalent to

$$2\sqrt{3}\sum_{1\leq i<j\leq 3}\sin\frac{\alpha_i}{2}\sin\frac{\alpha_j}{2} \leq \sum_{i=1}^{3}\sin\alpha_i. \tag{22}$$

Now

$$\sin\frac{\alpha_i}{2}\sin\frac{\alpha_j}{2} = \frac{1}{4R}\left(a_i\,a_j\tan\frac{\alpha_i}{2}\tan\frac{\alpha_j}{2}\right)^{1/2},$$

where a_i denotes the length of the side opposite angle α_i. And by Cauchy's inequality and (15),

$$\sum_{i<j}\left(a_i\,a_j\tan\frac{\alpha_i}{2}\tan\frac{\alpha_j}{2}\right)^{1/2} \leq \left(\sum_{i<j}a_i\,a_j\right)^{1/2};$$

as $a_i = 2\,R\sin\alpha_i$, this gives us

$$\left(2\sum_{i<j}\sin\frac{\alpha_i}{2}\sin\frac{\alpha_j}{2}\right)^2 \leq \sum_{i<j}\sin\alpha_i\sin\alpha_j,$$

a stronger inequality than (22), by (14).

7. Sines and secants

In attempting to extend the inequality $4\, M_0\, (\sin \alpha/2) \leq \sqrt{3}\, M_0\, (\sec \alpha/2)$, we meet a different type of problem. We will establish this result: Assume $\alpha_i \neq \pi/3$ for some i. Then,

$$4\, M_r \left(\sin \frac{\alpha}{2}\right) < \sqrt{3}\, M_r \left(\sec \frac{\alpha}{2}\right) \tag{23}$$

for all real r, if $\alpha_3 \geq 2\,\pi/3$. If $\alpha_3 < 2\,\pi/3$, there is a number $r_0 > 1$ such that (23) holds for $r < r_0$, and is reversed for $r > r_0$.

Proof. Firstly, if $\alpha_3 \geq 2\,\pi/3$, then $\alpha_1 \leq \alpha_2 < \pi/3$, whence $\sin \alpha_i \leq \sqrt{3}/2$, or $4 \sin \alpha_i/2 \leq \sqrt{3} \sec \alpha_i/2$, $i = 1, 2, 3$; and at least two of these inequalities are strict. Hence (23) holds for all real r.

Secondly, note that (23) holds for $r = 1$, whatever (α). For $f(x) = \sin x/2$ is convex on $0 \leq x \leq \pi$, so by Jensen's inequality ([3], §3.6), $M_1\, (\sin \alpha/2) \leq 1/2$. And we deduce from (7) that $\sqrt{3}\, M_1\, (\sec \alpha/2) = \sqrt{3}\, [M_{-1}\, (\cos \alpha/2)]^{-1} \geq 2$.

Now if $(\pi/3) < \alpha_3 < 2\,\pi/3$, we have

$$4 \sin \frac{\alpha_3}{2} > \sqrt{3} \sec \frac{\alpha_3}{2}, \tag{24}$$

that is, $4\, M_\infty\, (\sin \alpha/2) > \sqrt{3}\, M_\infty\, (\sec \alpha/2)$. This and (23) for $r = 1$ establish the existence of a zero, say $r = r_0$, of $4\, M_r\, (\sin \alpha/2) - \sqrt{3}\, M_r\, (\sec \alpha/2)$, with $r_0 > 1$. Then, as usual, it follows from (2) and (24) that this function has no other zero.

8. Other inequalities

In trying to extend the inequality $3\, M_0\, (\tan \alpha/2) \leq M_0\, (\cot \alpha/2)$, we find a situation quite different from the one which we met in §§ 3 through 7, where the functions under consideration always had at most one real zero. Indeed, since $\tan \alpha_i/2 \cot \alpha_i/2 = 1$, the inequalities

$$3\, M_r \left(\tan \frac{\alpha}{2}\right) \leq M_r \left(\cot \frac{\alpha}{2}\right) \quad \text{and} \quad 3\, M_{-r} \left(\tan \frac{\alpha}{2}\right) \leq M_{-r} \left(\cot \frac{\alpha}{2}\right)$$

are equivalent (for a similar argument in another setting, see Makowski [6]). Hence, $g(r) := 3\, M_r\, (\tan \alpha/2) - M_r\, (\cot \alpha/2)$ has an even number of real zeros (possibly none), unless it vanishes identically. In fact, it has no zeros for certain choices of (α), as the following proposition shows:

In order that

$$3\, M_r \left(\tan \frac{\alpha}{2}\right) \leq M_r \left(\cot \frac{\alpha}{2}\right) \tag{25}$$

for all r, *it is necessary and sufficient that* $\tan \alpha_1/2 \tan \alpha_3/2 \leq 1/3$. *If* $r \neq +\infty$, *equality holds when* $\alpha_1 = \alpha_2 = \alpha_3 = \pi/3$, *and only then.*

Proof. The condition is necessary because it is equivalent to the necessary condition $3\, M_\infty\,(\tan\alpha/2) \leq M_\infty\,(\cot\alpha/2)$.

To prove its sufficiency we consider $g(r)$, defined as above. By (1), $g(0) < 0$ unless $\alpha_1 = \alpha_2 = \alpha_3$. And if $\tan\alpha_1/2\,\tan\alpha_3/2 < 1/3$, then $g(+\infty) < 0$. But then if g has any zeros on $(0, \infty)$, it has at least two, hence at least 4 on $(-\infty, \infty)$; by Theorem B this can occur only if $\alpha_1 = \alpha_2 = \alpha_3$. Further, if $\tan\alpha_1/2\,\tan\alpha_3/2 = 1/3$ then $\tan\alpha_1/2 = (1/3)\cot\alpha_3/2$ and $\tan\alpha_3/2 = (1/3)\cot\alpha_1/2$, so that if equality held in (25) for some $r \neq +\infty$ we would also have $\tan\alpha_2/2 = \sqrt{3}/3$. And then we would have $M_0\,(\tan\alpha/2) = \sqrt{3}/3$, whence $\alpha_1 = \alpha_2 = \alpha_3$, by the condition of equality in (1). This concludes the proof.

Now we shall show that in *any* triangle,

$$3\, M_r\left(\tan\frac{\alpha}{2}\right) \leq M_r\left(\cot\frac{\alpha}{2}\right) \quad \text{for} \quad |r| \leq 1\,, \tag{26}$$

with equality in the usual case. Indeed, it is clear from the preceding discussion that it suffices to prove (26) for $r = 1$. Since

$$\sum_{i=1}^{3} \tan\frac{\alpha_i}{2} = \frac{(r + 4\,R)}{s} \quad \text{and} \quad \sum_{i=1}^{3} \cot\frac{\alpha_i}{2} = \frac{s}{r}\,,$$

we must show that $s^2 \geq 3\,r\,(r + 4R)$; and this inequality is known ([2], § 5.6).

In a like manner, one can show that the condition $\sin\alpha_1/2\,\sin\alpha_3/2 \leq 1/4$ is necessary and sufficient in order that $4\, M_r\,(\sin\alpha/2) \leq M_r\,(\operatorname{cosec}\alpha/2)$ for all r, and that $4\, M_r\,(\cos\alpha/2) \leq 3\, M_r\,(\sec\alpha/2)$ for all r if and only if $\cos\alpha_1/2\,\cos\alpha_3/2 \leq 3/4$. One can also show that in any non-equilateral triangle,

$$4\, M_r\left(\sin\frac{\alpha}{2}\right) < M_r\left(\operatorname{cosec}\frac{\alpha}{2}\right) \quad \text{for} \quad |r| \leq 1 \tag{27}$$

and

$$4\, M_r\left(\cos\frac{\alpha}{2}\right) < 3\, M_r\left(\sec\frac{\alpha}{2}\right) \quad \text{for} \quad |r| \leq 2\,. \tag{28}$$

For instance, (28) follows from (7): $4\, M_2\,(\cos\alpha/2) < 2\sqrt{3}$, whence $M_2\,(\sec\alpha/2) = [M_{-2}\,(\cos\alpha/2)]^{-1} > [M_2\,(\cos\alpha/2)]^{-1} > 2\sqrt{3}$, so that (28) holds for $r = 2$, and therefore for $|r| \leq 2$. The proof of (27) is similar.

Finally, if $\alpha_3 \neq \pi/3$, $h(r) := M_r\,(\cot\alpha/2) - 2\sqrt{3}\, M_r\,(\sin\alpha/2)$ is positive at $r = 0$, and can have two zeros, or one, or none, depending on (α). For instance, if $(\alpha) = (10°, 40°, 130°)$, $h(r)$ has exactly one zero, situated somewhere on $(-\infty, 0)$. On the other hand, if we choose $\alpha_2 = 60°$ and α_1, α_3 arbitrary (but $\alpha_1 \neq \alpha_3$ and $\alpha_1 + \alpha_3 = 120°$) we get a triangle such that $h(r) > 0$ for all r. And if $(\alpha) = (50°, 50°, 80°)$, then $h(r)$ has two real zeros. By comparing (20) and (26) in the range $0 \leq r \leq 1$, we see that in any non-equilateral triangle,

$$2\sqrt{3}\, M_r\left(\sin\frac{\alpha}{2}\right) < M_r\left(\cot\frac{\alpha}{2}\right) \quad \text{for} \quad 0 \leq r \leq 1\,.$$

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REFERENCES

[1] J. Aczél and L. Fuchs, *A Minimum-Problem on Areas of Inscribed and Circumscribed Polygons of a Circle*, Compositio Math. *8*, 61–67 (1951).
[2] O. Bottema, R. Z. Djordjević, R. R. Janić, D. S. Mitrinović, and P. M. Vasić, *Geometric Inequalities* (Wolters-Noordhoff, Groningen 1969).
[3] G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, 2nd edition (Cambridge University Press 1959).
[4] E. W. Hobson, *A Treatise on Plane Trigonometry*, 7th edition (Cambridge University Press 1928).
[5] J. Karamata, *Sur une inégalité relative aux fonctions convexes*, Publ. Math. Univ. Belgrade *1* 145-148, (1932).
[6] A. Makowski, *Some Geometric Inequalities*, El. Math. *17*, 40–41 (1962).
[7] J. Steinig, *Sur quelques applications géométriques d'une inégalité relative aux fonctions convexes*, Ens. Math. (2) *11*, 281–285 (1965).
[8] J. Steinig, *On Some Rules of Laguerre's, and Systems of Equal Sums of Like Powers*, Rend. Mat., ser. 6, *4*, 629–644 (1971).

Über hebbare Unstetigkeiten

In [1] wurde die Menge $\mathfrak{F}[a, b]$ derjenigen in $[a, b]$ definierten Funktionen betrachtet, die in jedem Punkt von $[a, b]$ unstetig sind.

Für $f \in \mathfrak{F}[a, b]$ bezeichne wiederum $\mathcal{H}[f]$ die Menge derjenigen Punkte von $[a, b]$, in denen die Unstetigkeit von f hebbar ist, und $\mathcal{U}[f]$ die Menge derjenigen Punkte von $[a, b]$, in denen die Unstetigkeit von f nicht hebbar ist.

In [1] wurde nun einerseits gezeigt, dass für jedes $f \in \mathfrak{F}[a, b]$ $\mathcal{U}[f]$ dicht in $[a, b]$ ist; andererseits wurden Funktionen $f \in \mathfrak{F}[a, b]$ konstruiert, für die $\mathcal{H}[f]$ «sehr umfassend» ist. Daran anschliessend wurde die Frage aufgeworfen, ob für ein $f \in \mathfrak{F}[a, b]$ $\mathcal{H}[f]$ sogar dicht sein kann. Wir werden in dieser Note beweisen, dass dies nicht möglich ist. Damit ist dann gleichzeitig gezeigt, dass die Beispiele aus [1] für Funktionen f mit «sehr umfassendem» $\mathcal{H}[f]$ «gut» sind.

Bei gegebenem $f \in \mathfrak{F}[a, b]$ definieren wir noch die «abgeänderte» Funktion f^*:

$$f^*(x) = \begin{cases} \lim\limits_{\xi \to x} f(\xi) & \text{für} \quad x \in \mathcal{H}[f] \\ f(x) & \text{für} \quad x \in \mathcal{U}[f]\,. \end{cases}$$

Die Menge derjenigen Punkte aus $[a, b]$, in denen f^* stetig ist, bezeichnen wir mit $S[f^*]$.

Hilfssatz 1

Für jedes $f \in \mathfrak{F}[a, b]$ ist $\mathcal{H}[f]$ abzählbar.

Beweis

Sei $f \in \mathfrak{F}[a, b]$. Für beliebiges $\varepsilon > 0$ bilden wir die Punktmenge

$$\mathcal{H}_\varepsilon[f] = \{x \in \mathcal{H}[f] / \mid f^*(x) - f(x) \mid \geq \varepsilon\}\,.$$