# Kleine Mitteilungen

## On Regular Right Duo Semigroups

Let $S$ be a semigroup[1]). We shall say that $S$ is *right duo* if every right ideal $R$ of $S$ is two-sided. Analogously can be defined the left duo semigroup. A semigroup is said to be a *duo* semigroup if it is both left and right duo. $S$ is called *regular* if to every element $a$ of $S$ there exists an element $x$ in $S$ such that $a = a\,x\,a$. For example, the full transformation semigroup of a set of 2 elements is a regular right duo semigroup. Another example is given by the following multiplication table:

| | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 2 | 3 |
| 2 | 0 | 1 | 2 | 3 |
| 3 | 0 | 1 | 2 | 3 |

In this short note some ideal-theoretic characterizations of regular right (and left) duo semigroups will be proved.

**Theorem 1.** *A semigroup $S$ is a regular right duo semigroup if and only if the condition*

$$B \cap R = RB \tag{1}$$

*holds for every bi-ideal $B$ of $S$ and every right ideal $R$ of $S$.*

Proof. Let $S$ be a regular right duo semigroup. Then every bi-ideal $B$ of $S$ can be represented in the form

$$B = IL\,, \tag{2}$$

where $L$ is a left ideal and $I$ is a two-sided ideal of $S$ (cf .[3]). Therefore (1) is implied by (2) and the Kovács-Iséki regularity criterion (see [1], p. 34).

Conversely, suppose that $S$ is a semigroup admitting property (1) for every bi-ideal $B$ and every right ideal $R$ of $S$. Then, for any right ideal $R$ of $S$, (1) implies

$$R \cap S = SR\,. \tag{3}$$

Hence $S$ is right duo. But a right duo semigroup $S$ is regular if and only if $I \cap L = IL$ for every two-sided ideal $I$ and every left ideal $L$ of $S$, which is implied by (1).

The following criterion can similarly be proved.

**Theorem 2.** *A semigroup $S$ is a regular right duo semigroup if and only if the relation*

$$Q \cap R = RQ \tag{4}$$

*holds for every quasi-ideal $Q$ and every right ideal $R$ of $S$.*

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[1]) For the undefined notions and notations we refer to [1].

Next we give necessary and sufficient conditions for a right duo semigroup $S$ to be a semilattice of groups.

**Theorem 3.** *For a right duo semigroup $S$ the following conditions are pairwise equivalent:*

(A) *$S$ is a semilattice of groups.*

(B) *$B \cap I = BI$ for every bi-ideal $B$ and every two-sided ideal $I$ of $S$.*

(C) *$I \cap Q = QI$ for any quasi-ideal $Q$ and any two-sided ideal $I$ of $S$.*

(D) *$I \cap L = LI$ for every left ideal $L$ and every two-sided ideal $I$ of $S$.*

Proof. (A) implies (B). It is known (cf. [2], [5]) that for any two bi-ideals $B_1$, $B_2$ of a semigroup $S$ that is a semilattice of groups, the condition

$$B_1 \cap B_2 = B_1 B_2 \tag{5}$$

holds. This implies (B).

Evidently (B) implies (C) and (C) implies (D), because every left ideal is a quasi-ideal, and every quasi-ideal is a bi-ideal.

(D) implies (A). In case of $I = S$, (D) implies

$$L \cap S = LS \tag{6}$$

for every left ideal $L$ of $S$, i.e. $S$ is a left duo semigroup. Therefore condition (D) implies

$$L(a) \cap R(a) = R(a)\, L(a)\,, \tag{7}$$

that is,

$$J(a) = J^2(a)\,, \tag{8}$$

for every element $a$ of $S$. (8) implies $a \in a^2 \cup a\,S\,a$ ($\forall\, a \in S$), whence $S$ is regular. Thus $S$ is a regular duo semigroup, that is a semilattice of groups (cf. [4], Theorem 2).

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## A Contour for the Poisson Integral

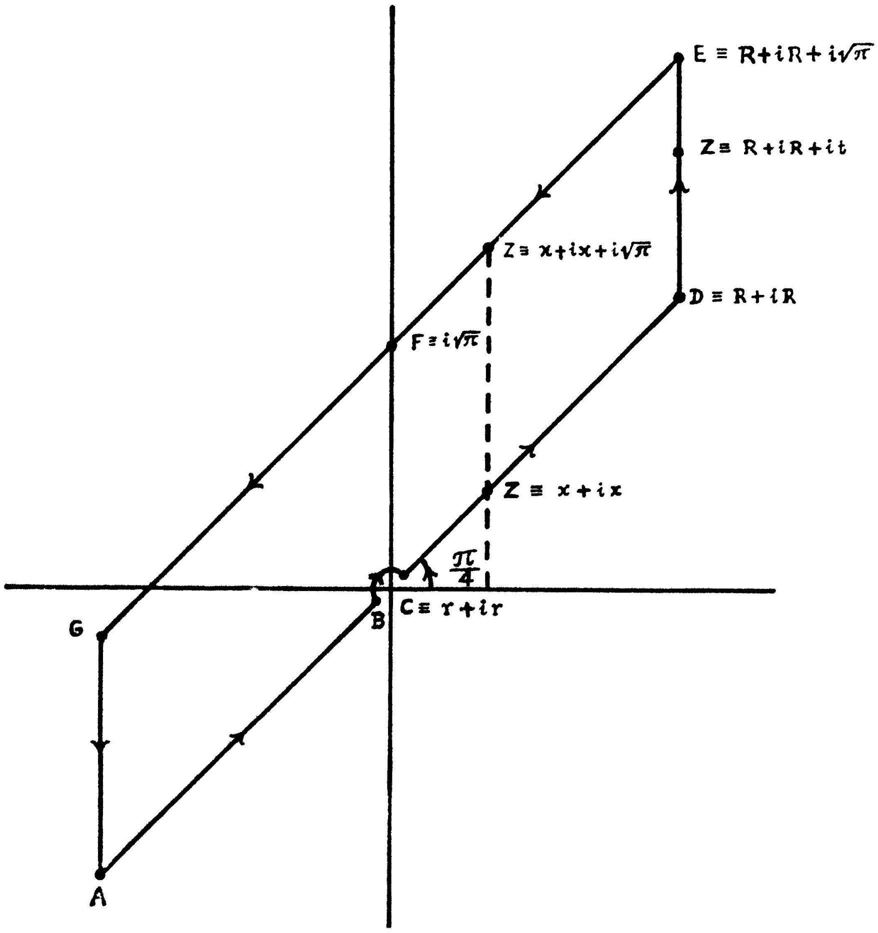
A convenient contour for the direct evaluation of the Poisson Integral

$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2} \tag{1}$$

does not appear to have been considered in the literature. On the contrary (1) is invariably borrowed from real variable theory to derive Fresnel's integrals by integrating $e^{iz^2}$ (or $e^{-z^2}$) along a sector. The Fresnel integrals, however, may be obtained independently by the procedure for evaluating Gauss's sums [2] suggesting that a suitable modification of this procedure should also yield (1). In fact, we integrate

$$f(z) = \frac{e^{\frac{iz^2}{2}}}{e^{-\sqrt{\pi}z} - 1} \tag{2}$$

along the parallelogram $AB\widehat{C}DEFGA$ as shown in the diagram.




We have

$$\left|\int_{DE} f(z)\,dz\right| \leq \frac{e^{-R^2}}{1-e^{-\sqrt{\pi}R}} \int_0^{\sqrt{\pi}} e^{-Rt}\,dt = \frac{e^{-R^2}}{R} \to 0 \quad (R\to\infty)\,. \tag{3}$$

Similarly, $\int_{GA} f(z)\,dz$ also vanishes as $R \to \infty$. Now,

$$\int_{CD} f(z)\,dz = (1+i) \int_r^R \frac{e^{-x^2}}{e^{-(1+i)\sqrt{\pi}\,x} - 1}\,dx$$

and

$$\int_{AB} f(z)\,dz = (1+i) \int_{-R}^{-r} \frac{e^{-x^2}}{e^{-(1+i)\sqrt{\pi}\,x} - 1}\,dx = (1+i) \int_r^R \frac{e^{-x^2}}{e^{(1+i)\sqrt{\pi}\,x} - 1}\,dx$$

giving

$$\int_{AB} f(z)\,dz + \int_{CD} f(z)\,dz = -(1+i) \int_r^R e^{-x^2}\,dx\,. \tag{4}$$

Likewise, we have

$$\int_{GF} f(z)\,dz + \int_{FE} f(z)\,dz = i\,(1+i) \int_0^R e^{-x^2}\,dx\,. \tag{5}$$

Equating the integrals along $AD$ and $GE$ in the limit as $r \to 0$ and $R \to \infty$ and taking into account the contribution $i\sqrt{\pi}$ from the indentation at the origin, we obtain, from (4) and (5),

$$-(1+i) \int_0^\infty e^{-x^2}\,dx + i\sqrt{\pi} = i\,(1+i) \int_0^\infty e^{-x^2}\,dx$$

yielding (1).

More generally, by integrating

$$f(z) = \frac{e^{i z^2 \cos^2\alpha}}{e^{-\sqrt{2\pi}\,z\cos\alpha} - 1} \tag{6}$$

along a parallelogram inclined at $\alpha$ to the real axis, $0 \leqq \alpha < \pi/2$ so that

$$D \equiv R + i\,R\tan\alpha\,, \quad E \equiv R + i\,R\tan\alpha + i\sqrt{\frac{\pi}{2}}\sec\alpha\,, \quad F \equiv i\sqrt{\frac{\pi}{2}}\sec\alpha$$

we obtain, in exactly the same manner and with no more effort,

$$\int_0^\infty e^{-x^2\sin 2\alpha + i x^2 \cos 2\alpha}\,dx = \frac{1+i}{2}\sqrt{\frac{\pi}{2}}\,e^{-i\alpha}\,, \quad 0 \leqq \alpha < \pi/2\,, \tag{7}$$

which evidently includes the integrals of Fresnel ($\alpha = 0$) and Poisson ($\alpha = \pi/4$) as special cases. Here again, the usual proof [1] of the generalisation (7) depends on an appeal to (1).

Changing the variable from $x$ to $\varrho\, x$ where $\varrho > 0$ is arbitrary and setting $a = \varrho^2\, e^{2i\alpha}$ (7) assumes the more compact form

$$\int_0^\infty e^{iax^2}\, dx = \frac{1+i}{2} \sqrt{\frac{\pi}{2a}}\ , \quad a \neq 0\, , \ \operatorname{Im} a \geqq 0\, , \tag{8}$$

the principal value of $\sqrt{a}$ being taken on the right.

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# Aufgaben

**Aufgabe 647.** Für eine streng monoton wachsende Folge $(a_i)$ natürlicher Zahlen seien $A(n) = \sum_{a_i<n} 1$ $(n = 1, 2, \ldots)$, $\limsup A(n)/n$ $[n \to \infty]$ die *obere Dichte* und – im Falle der Existenz – $\lim A(n)/n$ $[n \to \infty]$ die *Dichte*. Man beweise:

a) Jede streng monoton wachsende Folge natürlicher Zahlen mit oberer Dichte 1 besitzt eine unendliche Teilfolge, welche aus paarweise teilerfremden Zahlen besteht.

b) Zu jedem $\varepsilon > 0$ gibt es stets eine streng monoton wachsende Folge natürlicher Zahlen mit Dichte $> 1 - \varepsilon$ derart, dass für keine ihrer unendlichen Teilfolgen die Glieder paarweise denselben grössten gemeinsamen Teiler haben.

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Für die Lösung zu Teil a) vgl. diesen Band, p. 65.

*Lösung zu Teil b)*: Zu jedem $\varepsilon > 0$ wollen wir eine streng monoton wachsende Folge $(a_i)$ natürlicher Zahlen mit Dichte $> 1 - \varepsilon$ konstruieren derart, dass für jede natürliche Zahl $d$ nur endlich viele verschiedene $a_i$ paarweise den grössten gemeinsamen Teiler $d$ haben.

Es sei $n_0 = n_0(\varepsilon)$ genügend gross. Eine natürliche Zahl $a$ gehöre nun genau dann zur Folge $(a_i)$, wenn gilt:

1) Der kleinste Primfaktor von $a$ ist $\leq n_0$, und
2) ist $p_k | a$ und $p_k$ nicht der grösste Primfaktor von $a$, dann hat $a$ im Intervall $(p_k, e^{n_0 p_k^2})$ einen Primfaktor.

Wäre $a_{i_1} < a_{i_2} < \ldots.$ eine unendliche Teilfolge mit $(a_{i_{r_1}}, a_{i_{r_2}}) = d$ für $r_1 \neq r_2$, so würde gelten $a_{i_r} = d s_r$, $(s_{r_1}, s_{r_2}) = 1$. Also hätten die $a_{i_r}$ beliebig grosse Primfaktoren, daher hätten unendlich viele unter ihnen einen Primfaktor im Intervall