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Abkürzungen:	NZZ	Neue Zürcher Zeitung
	M.	Morgenausgabe
	Mi.	Mittagsausgabe
	A.	Abendausgabe
	S.	Sonntagsausgabe
	Gnomon	Gnomon, Kritische Zeitschrift für die gesamte Altertumswissenschaft

Gaussian Binomial Coefficients

The binomial coefficients belong to the curriculum of the secondary school, their connection with combinatorics is known since the days of Leibniz, Pascal and Jacob Bernoulli. The 'Gaussian binomial coefficients' are much less widely known, their connection with combinatorics is of a more recent date. We thought that an exposition of some of the relations between Gaussian and ordinary binomial coefficients may add some zest to a traditional secondary school subject.

1. *Definition.* We call

$$\begin{bmatrix} n \\ r \end{bmatrix} = \frac{(q^n - 1)(q^{n-1} - 1)\dots(q^{n-r+1} - 1)}{(q - 1)(q^2 - 1)\dots(q^r - 1)} \tag{1.1}$$

a *Gaussian binomial coefficient;* n and r are integers, $0 \leqslant r \leqslant n$, and q is a variable [1]. The definition (1.1) must be supplemented by an appropriate interpretation for $r = 0$, and we add an obvious consequence for $r = n$:

$$\begin{bmatrix} n \\ 0 \end{bmatrix} = 1, \quad \begin{bmatrix} n \\ n \end{bmatrix} = 1. \tag{1.2}$$

The value $q = 1$ is forbidden (for the moment). Yet each of the r factors in the numerator and also each of the r factors in the denominator on the right hand side of (1.1) is divisible by $q - 1$. Performing these divisions we see that

$$\begin{bmatrix} n \\ r \end{bmatrix} \to \binom{n}{r} \text{ as } q \to 1 \tag{1.3}$$

where

$$\binom{n}{r} = \frac{n}{1} \, \frac{n-1}{2} \cdots \frac{n-r+1}{r} \tag{1.1*}$$

is an ordinary binomial coefficient.

2. *A generalization of the binomial theorem.* We consider the polynomial in x

$$f(x) = (1 + x)(1 + q\,x) \ldots (1 + q^{n-1} x); \tag{2.1}$$

its zeros form a geometric progression of n terms whose quotient is q^{-1} and initial term -1. Our next task is to find the coefficients $Q_0, Q_1, Q_2, \ldots Q_n$ of the expansion

$$f(x) = Q_0 + Q_1 x + Q_2 x^2 + \ldots + Q_n x^n. \tag{2.2}$$

Obviously

$$Q_0 = 1, \quad Q_n = q^{n(n-1)/2}. \tag{2.3}$$

To proceed further, we observe that (2.1) implies

$$(1 + x) f(q\,x) = f(x)(1 + q^n x) \tag{2.4}$$

or, in view of (2.2),

$$(1 + x) \sum_{r=0}^{n} Q_r q^r x^r = (1 + q^n x) \sum_{r=0}^{n} Q_r x^r. \tag{2.5}$$

The comparison of like powers of x yields

$$Q_r q^r + Q_{r-1} q^{r-1} = Q_r + q^n Q_{r-1} \tag{2.6}$$

or

$$Q_r = Q_{r-1} \frac{q^{n-r+1} - 1}{q^r - 1} q^{r-1} \tag{2.7}$$

for $r = 1, 2, 3, \ldots n$. Repeated application of (2.7) yields, in view of (2.3) and (1.1), that

$$Q_r = \begin{bmatrix} n \\ r \end{bmatrix} q^{r(r-1)/2}. \tag{2.8}$$

Therefore, see (2.1) and (2.2),

$$\prod_{k=1}^{n} (1 + q^{k-1} x) = \sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix} q^{r(r-1)/2} x^r . \tag{2.9}$$

Compare this with the particular case $q = 1$:

$$(1 + x)^n = \sum_{r=0}^{n} \binom{n}{r} x^r . \tag{2.9*}$$

3. *The recursion formula.* Substituting $n + 1$ for n in (2.9) leads to the same result as multiplying (2.9) by $(1 + q^n x)$. If, in the equation so obtained, we compare like powers of x, we find that

$$\begin{bmatrix} n+1 \\ r \end{bmatrix} = \begin{bmatrix} n \\ r \end{bmatrix} + \begin{bmatrix} n \\ r-1 \end{bmatrix} q^{n-r+1} . \tag{3.1}$$

This result, which we can also directly verify from the definition (1.1), is analogous to

$$\binom{n+1}{r} = \binom{n}{r} + \binom{n}{r-1} . \tag{3.1*}$$

In starting from the 'initial condition' (1.2) and using the 'recursion formula' (3.1) to pass from n to $n + 1$ we can compute the Gaussian binomial coefficients very conveniently, in fact, only by addition and multiplication, without subtraction or division. Thus, $\left[\begin{smallmatrix} n \\ r \end{smallmatrix}\right]$ which, defined by (1.1), appeared as a rational function of q, turns out to be a polynomial in q

$$\begin{bmatrix} n \\ r \end{bmatrix} = \sum_{\alpha=0}^{r(n-r)} A_{n,r,\alpha}\, q^{\alpha} \tag{3.2}$$

whose coefficients $A_{n,r,\alpha}$ are positive integers. (Also the ordinary binomial coefficient, defined by (1.1*), appears initially as a rational number, but turns out to be an integer.) The degree $r\,(n - r)$ shown on the right hand side of (3.2) can be found as the difference of the degrees of numerator and denominator on the right hand side of (1.1), or can be verified by mathematical induction.

The principal aim of the present paper is to reveal the intuitive significance of the integers $A_{n,r,\alpha}$.

4. *A combinatorial interpretation.* We consider a rectangular coordinate system in the plane. A point whose coordinates are both integers (usually called a lattice point) will be considered as a *street corner.* A straight line that passes through a street corner and is parallel to one or the other coordinate axis will be called a *street.* We think of a pedestrian (a moving material point) walking in this network of streets. A shortest way in this network between the street corners $(0, 0)$ and $(r, n - r)$ (we assume $0 \leqslant r \leqslant n$) will be called a *zigzag path;* its length is n. The number of different zigzag paths is $\binom{n}{r}$. This is widely known [2]. We wish to add the

Theorem: *The number of those zigzag paths the area under which is* α *equals* $A_{n,r,\alpha}$ [3].

The 'area under the path' is contained between the path, the horizontal coordinate axis $y = 0$, and the line $x = r$ parallel to the vertical coordinate axis.

Figure 1 illustrates the theorem by exhibiting the particular case where $n = 6$ and $r = 2$. In examining it, bear in mind that

$$\binom{6}{2} = 15\,, \quad \binom{5}{2} = 10\,, \quad \binom{5}{1} = 5\,.$$

Figure 1 shows 15 zigzag paths starting from (0, 0) and ending at (2, 4). The corresponding Gaussian binomial coefficient is of degree $2 \times 4 = 8$; its expansion consists of 9 terms; in each term the exponent of x indicates the area under the path and the coefficient the number of different paths with such an area.

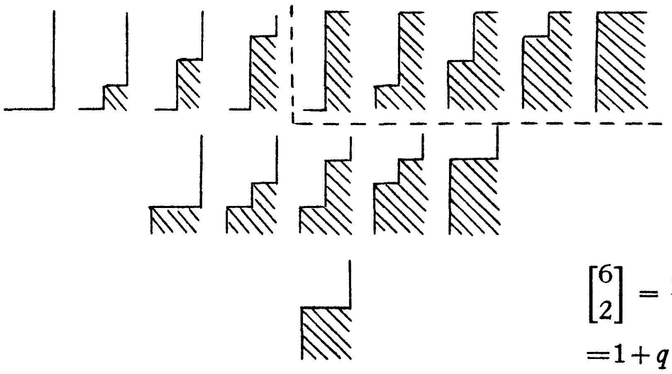

Figure 1

$$\begin{bmatrix}6\\2\end{bmatrix} = \frac{q^6-1}{q-1} \cdot \frac{q^5-1}{q^2-1} = \begin{bmatrix}5\\2\end{bmatrix} + \begin{bmatrix}5\\1\end{bmatrix} q^4$$
$$= 1 + q + 2q^2 + 2q^3 + 3q^4 + 2q^5 + 2q^6 + q^7 + q^8$$

The theorem can be proved by mathematical induction; the recursion formula (3.1) provides the bridge for the passage from n to $n + 1$. The proof is clearly indicated by Figure 1 which exhibits the passage from 5 to 6. Each of the 15 zigzag paths shown is of length 6 and should be conceived as consisting of two parts, an *initial part* of length 5 starting from (0, 0) and of a *terminal segment* of length 1 ending at (2, 4). Of the 15 zigzag paths considered, 10 have a vertical, and 5 a horizontal, terminal segment (notice the line of separation in Figure 1). The area under the whole path is the same as the one under the initial part for the 10, but it is larger by 4 units for the 5; the 10 correspond to the first, and the 5 to the second, term on the right hand side of (3.1). The reader should be able to see the general argument behind the representative particular case exhibited.

It is instructive to verify directly some points connected with the theorem whose proof we have just indicated. It is obvious, either from the figure or from the expression (1.1), that

$$A_{n,r,\alpha} = A_{n,r,r(n-r)-\alpha}\,. \tag{4.1}$$

Moreover, as the value of the Gaussian binomial coefficient for $q = 1$ is just the corresponding ordinary binomial coefficient, see (1.3), (3.2) yields

$$\sum_{\alpha=0}^{r(n-r)} A_{n,r,\alpha} = \binom{n}{r} \tag{4.2}$$

as it should be.

5. *Another combinatorial interpretation.* A zigzag path consists of consecutive segments of unit length; each segment joins two neighboring street corners. Starting

from the initial point (0, 0) we name these segments consecutively x or y according to which coordinate axis each one is parallel. We obtain so a one-to-one mapping of the set of zigzag paths onto a set of letter sequences; see the example in Figure 2 where the zigzag path ends at the street corner (5, 4) and the corresponding sequence consists of 9 letters.

Divide the area under the zigzag path with endpoint $(r, n - r)$ by equidistant parallels to the y-axis into r rectangles each of base 1. In Figure 2 there are 5 such rectangles; their heights are (we survey them from left to right)

0, 1, 3, 3, 4

respectively. Each rectangle has as top a horizontal unit segment of the zigzag path, and corresponds so to an x in the letter sequence. The height, and so the area, of the rectangle is the number of y's preceding that x, and so equals the *number of inversions determined by that* x in the letter sequence. (In a letter sequence of length n there are $n(n - 1)/2$ pairs of letters. Such a pair forms an *inversion* if, and only if, it consists of a *y preceding an x*.) Thus, the joint area of the r rectangles, that is, *the area under the zigzag path equals the number of inversions in the letter sequence* (11 in our example).

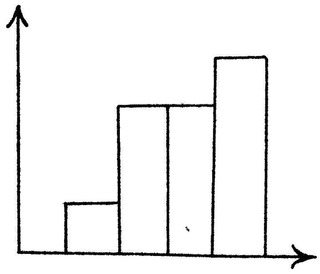

Figure 2
xyxyyxxyx

Thus, the Gaussian binomial, by enumerating the areas under the zigzag paths, enumerates *ipso facto* the inversions in the letter sequences considered. This connection of the Gaussian binomials with inversions was known [4]. What we wanted to point out is the intuitive transition from areas to inversions [5].

6. *Another approach.* In fact, the theorem of Section 4 can be obtained as a slight reinterpretation of a known particular result of a classical theory, the theory of *partitions* whose foundation was laid down by Euler [6]. We shall assume as known an essential point of this theory in the following derivation.

Remember the definition of a zigzag path given in Section 4 and set $n = r + s$. As Figure 2 suggests, we can build up a zigzag path with r juxtaposed rectangles. Each of these rectangles has a horizontal base of length 1 and its vertical altitude is measured by a non-negative integer $\leqslant s$; the altitude 0 is admissible. The bases of the r rectangles are alined along the x-axis starting from the origin, the altitudes form a non-decreasing sequence, and the sum of the areas (or altitudes) is α, the area under the zigzag path. Thus the altitudes form a *partition* of α into exactly r non-negative integers none of which exceeds s; we *define* now (in opposition to Section 3) $A_{r+s,r,\alpha}$ as the *number of such partitions*.

We can determine this number by Euler's method according to which if we set

$$(1 - x)(1 - q x)(1 - q^2 x) \dots (1 - q^s x) = g(x) \tag{6.1}$$

(compare (2.1)) we obtain the 'generating function'

$$\sum_r \sum_\alpha A_{r+s,r,\alpha}\, x^r q^\alpha = \frac{1}{g(x)} \quad . \tag{6.2}$$

If we define

$$P_0 = 1, \sum_\alpha A_{r+s,r,\alpha}\, q^\alpha = P_r \tag{6.3}$$

for $r \geqslant 1$, we can write (6.2) as

$$\frac{1}{g(x)} = P_0 + P_1 x + P_2 x^2 + \ldots + P_r x^r + \ldots . \tag{6.4}$$

Now (6.1) involves

$$\frac{1-x}{g(x)} = \frac{1-q^{s+1}x}{g(q\,x)} \tag{6.5}$$

(compare (2.4)) or, in view of (6.4),

$$(1-x)\sum_0^\infty P_r x^r = (1-q^{s+1}x)\sum_0^\infty P_r q^r x^r . \tag{6.6}$$

The comparison of like powers yields

$$P_r - P_{r-1} = q^r P_r - q^{r+s} P_{r-1} \tag{6.7}$$

or

$$P_r = P_{r-1}\frac{q^{r+s}-1}{q^r-1} \tag{6.8}$$

for $r = 1, 2, 3, \ldots$. Repeated application of (6.8) yields, in view of (6.3) and the definition (1.1),

$$P_r = \begin{bmatrix} r+s \\ r \end{bmatrix} = \sum_{\alpha=0}^{rs} A_{r+s,r,\alpha}\, q^\alpha \tag{6.9}$$

and this proves our theorem stated in Section 4.

To check (6.9) we may observe that, for $q \to 1$, (6.4) goes over into

$$(1-x)^{-s-1} = \sum_{r=0}^\infty \binom{r+s}{r} x^r . \tag{6.4*}$$

The computation of P_r in this section was very similar to the computation of Q_r in Section 2. Yet in this section we started with a combinatorial definition of $A_{n,r,\alpha}$ and proceeded hence to a formula, whereas in Section 4 we defined $A_{n,r,\alpha}$ by the formula (3.2) and verified the combinatorial interpretation afterwards.

7. *A brief outlook on 'Gaussian multinomial coefficients'*. Let n be a nonnegative integer and let us call 'Gaussian factorial' the polynomial in q

$$\begin{aligned} n!! &= \frac{(q-1)(q^2-1)(q^3-1)\ldots(q^n-1)}{(q-1)^n} \\ &= 1 \cdot (1+q) \cdot (1+q+q^2) \quad \ldots \quad (1+q+q^2+\ldots+q^{n-1}) \\ &= \sum_{i=0}^{n(n-1)/2} B_{n,i}\, q^i ; \end{aligned} \tag{7.1}$$

its coefficients $B_{n,i}$ are nonnegative integers, its degree is $n(n-1)/2$. It could be also defined by the initial condition

$$0!! = 1 \tag{7.2}$$

and the recursion formula

$$(n+1)!! = (1 + q + q^2 + \ldots + q^n)\, n!! \,. \tag{7.3}$$

Consider a 'word' formed by n given different letters of the alphabet; the number of all such words is obviously $n!$. In any one of these words there are $n(n-1)/2$ pairs of letters; such a pair forms an inversion if, and only if, the alphabetically preceding letter comes later. *The number of those among the $n!$ words (permutations) that show precisely i inversions is $B_{n,i}$* [7].

This is easy to prove by mathematical induction from the recursion formula (7.3). Add to the n different letters which form the word a new letter that precedes all of them alphabetically. According as the letter added occupies the first, the second, the third, ... or the last place, the number of inversions increases by 0, 1, 2, ... or n and that is what (7.3) expresses.

From the defining formula or from the combinatorial interpretation

$$B_{n,i} = B_{n,[n(n-1)/2]-i}\,, \tag{7.4}$$

$$\sum_{i=0}^{n(n-1)/2} B_{n,i} = n!\,. \tag{7.5}$$

The Gaussian binomial coefficient can be written in the form

$$\begin{bmatrix} n \\ r \end{bmatrix} = \frac{n!!}{r!!\,(n-r)!!} \tag{7.6}$$

Here is a Gaussian analogue to a multinomial coefficient:

$$\frac{n!!}{r!!\,s!!\,t!!} \tag{7.7}$$

where $r + s + t = n$ and r, s, t and n are nonnegative integers; it can be shown that (7.7) is a polynomial in q, of degree $r s + r t + s t$, with positive integer coefficients. These coefficients have a combinatorial significance: They count such sequences of n letters, each of which may be x, y or z, as show a given number of inversions. Such a letter sequence can be regarded as representing a zigzag path in a three dimensional lattice. The number of inversions equals the sum of three areas, each under the projection of the zigzag path onto a coordinate plane; but 'under' must be carefully interpreted.

The Gaussian analogues to multinomial coefficients count the number of certain letter sequences with a given number of inversions; the interpretation with areas is possible but becomes clumsy in several dimensions.

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[1] Cf. C.F. GAUSS, *Summatio quarundam serierum singularium*, Werke v. 2, especially p.16–17.

[2] See e.g. G. PÓLYA, *Mathematical discovery* (Wiley, 1962), v. 1, p.68–75, or the German translation, *Vom Lösen mathematischer Aufgaben* (Birkhäuser, 1966), v. 1, p.110–119. Also French, Japanese and Hungarian translations available.

[3] G. PÓLYA, *J. Combinatorial Theory*, v. 6, 1969, p.102–105; see p.105.

[4] See e.g. M.G. KENDALL and A. STUART, *The advanced theory of statistics* (London, 1961), v. 2, p.494.

[5] G. PÓLYA, *Proceedings of the Second Chapel Hill Conference on Combinatorial Mathematics and its Applications* (1970), p. 381–384, from which, with the kind permission of the Organizing Committee, extensive passages of Sections 4 and 5 are extracted.

[6] LEONHARD EULER, *Introductio in Analysin Infinitorum* (Lausanne, 1748), v. 1, p.253–275 (De Partitione Numerorum) or *Opera Omnia*, ser. 1, v. 8, p.313–338. There are several modern expositions; the reader can find what is needed in the sequel with relatively little trouble in John Riordan, *An Introduction to Combinatorial Analysis* (Wiley, 1958), p.107–123, and especially p.153, Problem 5.

[7] See, e.g., M. G. KENDALL and A. STUART, l.c.[4]), p. 479. Also E. NETTO, *Lehrbuch der Kombinatorik*, 2nd ed. (Leipzig & Berlin, 1927), p. 94–97.

Zu Formeln von Fejes Tóth und Hoppe für den Inhalt sphärischer Tetraeder

FEJES TÓTH [2] gab 1956 eine Formel für den Inhalt jener sphärischen Tetraeder an, die SCHLÄFLI [5] Orthoscheme genannt hat. An anderem Ort bemerkt FEJES TÓTH [3], dass es ihm nicht gelungen ist, die Identität dieser Formel mit einer gleichwertigen von HOPPE [4] aus dem Jahre 1882 auf direktem Wege nachzuweisen. Wie man mit einfachen Mitteln beide Formeln ineinander überführen kann, soll nachstehend gezeigt werden.

Bezeichnet man den Inhalt des Orthoschems im dreidimensionalen sphärischen Raum der Krümmung + 1 mit $S^{(4)}$, so ist nach HOPPE

$$2\,S^{(4)} = \int_{\delta}^{\beta_1} \phi(\psi)\,d\psi\,; \qquad \frac{\tan^2\phi}{\tan^2\beta_3\,\sin^2\beta_2} + \frac{\tan^2\psi}{\tan^2\beta_2} = 1 \tag{1}$$

$$\sin\delta = \sin\beta_2\,\cos\beta_1$$

und nach FEJES TÓTH

$$2\,S^{(4)} = \int_0^{\alpha_1} \left[S_1^{(2)} - \phi^*(\psi)\,\arctan\frac{\tan S_1^{(2)}}{\phi^*(\psi)} \right] d\psi \tag{2}$$

mit:

$$(\phi^*)^2 = \frac{\cos^2\alpha_2\,\cos^2\psi}{\sin^2\alpha_1 - \cos^2\alpha_2\,\sin^2\psi} \tag{2*}$$

$$\tan^2 S_1^{(2)} = \tan^2 S_1^{(2)}\,(\alpha_1, \alpha_2, \alpha_3) = \frac{\sin^2\alpha_1\,\sin^2\alpha_3 - \cos^2\alpha_2}{\sin^2\alpha_1\,\cos^2\alpha_3}\,.$$

Zur Erläuterung der auftretenden Grössen möge die Abbildung dienen.

Der dreidimensionale sphärische Raum wird als Kugelfläche in einen vierdimensionalen euklidischen Raum eingebettet. Aus dem Mittelpunkt der Kugel wird