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## Uniqueness Theorems for Power Equations

The uniqueness of the solution for the system of power equations

$$\sum_{i=1}^n x_i^k = 0, \quad k = 1, 2, \dots, n$$

is well known. FLANDERS [1] in a classroom note states that the Newton formulae are not quite so elementary and gives a more elementary proof using the Vandermonde determinant. In this note, we give an elementary proof of a generalization which uses only part of the Newton formulae. Then by use of the complete set of Newton formulae, we give further extensions.

If  $T_1, T_2, \dots$ , denote the elementary symmetric functions of  $x_1, x_2, \dots, x_n$ , i.e.,

$$P(x) = \prod_{i=1}^n (x - x_i) = x^n - T_1 x^{n-1} + T_2 x^{n-2} - \dots + (-1)^n T_n$$

and if

$$S_k = \sum_{i=1}^n x_i^k,$$

then the Newton formulae [2, 3] are

$$(A) \quad S_k - T_1 S_{k-1} + T_2 S_{k-2} - \dots + (-1)^{k-1} T_{k-1} S_1 + (-1)^k k T_k = 0 \quad (k \leq n),$$

$$(B) \quad S_k - T_1 S_{k-1} + T_2 S_{k-2} - \dots + (-1)^n T_n S_{k-n} = 0 \quad (k > n).$$

To obtain (B), we merely sum over the roots  $x_i$  of the equation

$$x^m (x^n - T_1 x^{n-1} + T_2 x^{n-2} - \dots + (-1)^n T_n) = 0 \quad (\text{here } k = m + n).$$

To obtain (A) requires considerably more work. The proof in [2] is based on knowing the formula and using properties of the elementary symmetric functions to confirm it. The proof in [3] derives the formula by an expansion of the identity

$$P'(x) = \sum_{i=1}^n \frac{P(x)}{x - x_i}.$$

We now give a considerable simplification of the latter proof and obtain both (A) and (B). We start from the different expansion

$$\frac{P'(x)}{P(x)} = \sum_{i=1}^n \frac{1}{x - x_i} = \frac{n}{x} + \frac{S_1}{x^2} + \frac{S_2}{x^3} + \dots \quad (1)$$

Changing  $x$  to  $1/x$  and simplifying, we obtain

$$\frac{T_1 - 2 T_2 x + \dots + (-1)^{n-1} n T_n x^n}{1 - T_1 x + T_2 x^2 - \dots + (-1)^n T_n x^n} = S_1 + S_2 x + S_3 x^2 + \dots$$

Now by multiplying both sides by the denominator and equating coefficients of like powers of  $x$ , we obtain both (A) and (B).

Using just (B), we can establish

**Theorem 1:** If  $S_k = 0$  ( $k = r + 1, r + 2, \dots, r + n$ ), then  $x_i = 0$  ( $i = 1, 2, \dots, n$ ).

*Proof:* From (B) with  $k = n + r$ , it follows that  $T_n S_r = 0$ . Thus either  $T_n$  or  $S_r = 0$ . If  $S_r = 0$ , then (B) with  $k = n + r - 1$  implies that  $T_n S_{r-1} = 0$ . It now follows inductively that  $T_n = 0$  or equivalently that at least one of the  $x_i$ 's is zero. Repeating the same argument for the set of the remaining  $(n - 1)$   $x_i$ 's, we get inductively that all the  $x_i$ 's are zero.

We could have also proven Theorem 1 in the manner of Flanders using the Vandermonde determinant. Another proof is based on the somewhat interesting elementary

**Lemma:** If  $P(x)$  denotes a polynomial of degree  $n > 0$ , then the power expansion of  $\exp[P(x)]$  does not have  $n$  consecutive zero coefficients.

*Proof:* Suppose otherwise that

$$e^{P(x)} = A_0 + A_1 x + \dots + A_m x^m + O(x^{m+n+1})$$

where  $A_m \neq 0$  and  $P(x) = a_0 + a_1 x + \dots + a_n x^n$ ,  $a_n \neq 0$ . Differentiating, we obtain

$$P'(x) e^{P(x)} = A_1 + 2 A_2 x + \dots + m A_m x^{m-1} + O(x^{m+n})$$

and that the coefficient of  $x^{m+n-1}$  is zero. On the other hand,

$$P'(x) e^{P(x)} = \{a_1 + 2 a_2 x + \dots + n a_n x^{n-1}\} \{A_0 + A_1 x + \dots + A_m x^m + O(x^{m+n+1})\},$$

so that the coefficient of  $x^{m+n-1}$  is  $n a_n A_m \neq 0$ . This contradiction completes the proof.

Corollary: If the expansion of  $\exp\{a_1 x + a_2 x^2 + \dots + a_n x^n\}$  has  $n$  consecutive zero coefficients, then  $a_1 = a_2 = \dots = a_n = 0$ .

For our alternate proof of Theorem 1, we integrate (1) to give

$$\log \frac{P(x)}{x^n} = C - \frac{S_1}{x} - \dots - \frac{S_r}{r x^r} + O(x^{-r-n-1})$$

so that

$$P(x) = A x^n \exp - \{S_1/x + \dots + S_r/r x^r\} + O(x^{-r-1}).$$

Since  $P(x)$  is an  $n$ th order polynomial, the terms in  $x^{-n-1}, x^{-n-2}, \dots, x^{-n-r}$ , must be absent from the expansion of

$$\exp - \{S_1/x + \dots + S_r/r x^r\}.$$

By the previous corollary,  $S_1 = S_2 = \dots = S_r = 0$ . Consequently,  $P(x) = A x^n + O(x^{-r-1})$ , and again since  $P(x)$  is a polynomial,  $P(x) = A x^n$ , which entails that  $x_i = 0$ ,  $i = 1, 2, \dots, n$ .

The proof of the next generalization uses both (A) and (B).

**Theorem 2:** If  $\sum_{i=1}^n x_i^k = \sum_{i=1}^n a_i^k$  ( $k = 1, 2, \dots, n$ ),  $a_i$ -given, then aside from permutations  $(x_1, x_2, \dots, x_n) = (a_1, a_2, \dots, a_n)$ .

*Proof:* It follows from Newton's formulae that  $S_k$  is given uniquely as a function of  $T_1, T_2, \dots, T_k$  and also that  $T_k$  is given uniquely as a function of  $S_1, S_2, \dots, S_k$ . Explicit representations in the form of determinants is given in [2]. Consequently, the

elementary symmetric functions of the  $a_i$ 's must be identical to the elementary symmetric functions of the  $x_i$ 's. Whence,

$$\prod_{i=1}^n (x - x_i) = \prod_{i=1}^n (x - a_i)$$

which gives the desired result.

Theorem 2 is not unexpected in light of Bezout's theorem [4], i.e., ' $N$  polynomial equations of degrees  $n_1, n_2, \dots, n_N$  in  $N$  variables have in general  $n_1 n_2 \dots n_N$  common solutions. When the number is greater than this, it is infinite.' If the  $a_i$ 's are distinct, then counting all the permutations there are  $1, 2, 3 \dots n$  solutions. If some of the  $a_i$ 's were equal, then the number of permutations would be less than  $n!$  solutions or else each solution could have a multiplicity. An example where there are less than  $n_1 n_2 \dots n_N$  solutions is given by the system

$$x + y = 1, \quad x^3 + y^3 = 1.$$

The only two solutions are  $(1, 0)$  and  $(0, 1)$  and these cannot have any multiplicity.

We cannot extend the validity of Theorem 2 by changing the range  $k = 1, 2, \dots, n$  to  $k = r + 1, r + 2, \dots, r + n$  as in Theorem 1. A simple counterexample is given by the system

$$x^2 + y^2 = 1, \quad x^3 + y^3 = 1.$$

The 2.3 solutions of this set are given by

$$x = 0, y = 1; \quad x = 1, y = 0;$$

$$x = \frac{1 + \lambda^2}{1 + \lambda^3}, \quad y = \frac{\lambda(1 + \lambda^2)}{1 + \lambda^3}; \quad x = \frac{\lambda(1 + \lambda^2)}{1 + \lambda^3}, \quad y = \frac{1 + \lambda^2}{1 + \lambda^3}$$

where  $3\lambda = 1 \pm 2i\sqrt{2}$ . Again this is not unexpected since by Bezout's theorem, there are  $(r + 1)(r + 2) \dots (r + n)$  solutions in general.

As an application of Theorem 2, we give a solution to Aufgabe 591 (El. Math. 24, 20 (1969): For which real values of  $\alpha$  is the implication

$$\sum_{i=1}^n x_i = \sum_{i=1}^n x_i^2 = \dots = \sum_{i=1}^n x_i^n = \alpha \rightarrow (\forall r \in N) \sum_{i=1}^n x_i^{r+1} = \alpha$$

valid if the  $x_i$  are real?

It follows, as in the proof of Theorem 2, that the set of equations

$$\sum_{i=1}^n x_i^k = b_k, \quad k = 1, 2, \dots, n$$

has a unique solution, aside from permutations.

From (A) and (B) with  $k = n$  and  $n + 1$ , we get

$$S_n - T_1 S_{n-1} + \dots + (-1)^n n T_n = 0,$$

$$S_{n+1} - T_1 S_n + \dots + (-1)^n S_1 T_n = 0.$$

Since  $S_i = \alpha$  for all  $i$ , we get by subtraction that  $T_n(n - \alpha) = 0$ . Thus, either  $\alpha = n$  or  $T_n = 0$ . If  $\alpha = n$ , then by Theorem 2,  $x_i = 1, i = 1, 2, \dots, n$ . If  $T_n = 0$ , at least one of the  $x_i$ 's is zero. We cancel this  $x_i$  out and repeat the argument. Thus,  $\alpha = 0, 1, 2, \dots, n$ . And for these  $\alpha$ 's,  $x_1 = x_2 = \dots = x_\alpha = 1, x_{\alpha+1} = x_{\alpha+2} = \dots = x_n = 0$ .

A simpler solution can be gotten by considering  $\sum x_i^{2^r}$  as  $r$  increases without bound. It then follows easily that the  $x_i$ 's can only be 0 or 1. However, in the former solution it follows without any further argument that there was no need to restrict  $\alpha$  and the  $x_i$ 's to be real.

We now consider systems of equations where there are more variables than equations.

**Theorem 3:** If  $\sum_{r=1}^n x_r^k = 0, k = 1, 2, \dots, n - 1$ , then aside from permutations,  $x_r = A e^{2\pi ir/n}$ .

*Proof:* From  $A$ , it follows that

$$T_1 = T_2 = \dots = T_{n-1} = 0.$$

Thus,  $x^n + (-1)^n T_n = 0$  and the  $x_r$ 's are proportional to the  $n$ th roots of unity.

Corollary: If  $\sum_{r=1}^n x_r^k = 0, k = 1, 2, \dots, n - 1, m \nmid n$ , then the  $x_r$ 's are zero.

**Theorem 4:** If  $\sum_{r=1}^n x_r^k = 0, k = 1, 2, \dots, n - 2, (n \geq 4)$  then  $S_p = \sum_{r=1}^n x_r^p = 0$  where

$$p = \begin{cases} n + 1, n + 2, \dots, 2n - 3, \\ 2n + 1, 2n + 2, \dots, 3n - 4, \\ \cdot \quad \quad \cdot \quad \quad \cdot \\ \cdot \quad \quad \cdot \quad \quad \cdot \\ \cdot \quad \quad \cdot \quad \quad \cdot \\ rn + 1, rn + 2, \dots, (r + 1)n - r - 2 \quad (r \leq n - 3). \end{cases}$$

*Proof:* Since  $T_1 = T_2 = \dots = T_{n-2}$ ,

$$x^n + (-1)^{n-1} (x T_{n-1} - T_n) = 0. \tag{2}$$

Multiplying (1) by  $x, x^2, \dots, x^{n-3}$  and summing over the roots  $x_i$ , we get

$$S_p = 0, \quad p = n + 1, n + 2, \dots, 2n - 3.$$

We now multiply (2) by  $x^{n+1}, x^{n+2}, \dots, x^{2n-4}$  and sum over the  $x_i$ 's to give

$$S_p = 0, \quad p = 2n + 1, 2n + 2, \dots, 3n - 4.$$

We then continue in the same manner to obtain the remaining values of  $p$ .

In a similar way, we can extend Theorem 4 to

**Theorem 5:** If  $\sum_{r=1}^n x_r^k = 0, k = 1, 2, \dots, n - m, (n \geq 2m)$ , then  $S_p = 0$  where

$$p = \begin{cases} n + 1, n + 2, \dots, 2n + 1 - 2m, \\ 2n + 1, 2n + 2, \dots, 3n + 2 - 3m, \\ \cdot \quad \quad \cdot \quad \quad \cdot \\ \cdot \quad \quad \cdot \quad \quad \cdot \\ \cdot \quad \quad \cdot \quad \quad \cdot \\ rn + 1, rn + 2, \dots, (r + 1)n + r - (r + 1)m, \end{cases}$$

and  $r(m - 1) \leq (n - 1 - m)$ .

Our last theorem will concern consecutive odd power sums. As to be expected, we cannot give comparable uniqueness results as before.

**Theorem 6:** If  $\sum_{r=1}^{2n} x_r^{2k-1} = 0, k = 1, 2, \dots, n$ , then aside from permutations, the roots occur in pairs such that  $x_i + x_j = 0$ .

*Proof:* Using the Newton formulae

$$S_{2m-1} - T_1 S_{2m-2} + T_2 S_{2m-3} - \dots + S_1 T_{2m-2} - (2m - 1) T_{2m-1} = 0$$

for  $m = 1, 2, \dots, n$  and the hypothesis, it follows by induction that

$$T_1 = T_3 = \dots = T_{2n-1} = 0.$$

Thus,  $\prod_{i=1}^{2n} (x - x_i) = x^{2n} + T_2 x^{2n-2} + T_4 x^{2n-4} + \dots = 0$ .

Since the equation is even, the theorem is proved.

As an application, the only solutions, aside from permutations, of the system of equations

$$\begin{aligned} x + y + z + w &= a + b + c + d, \\ x^3 + y^3 + z^3 + w^3 &= a^3 + b^3 + c^3 + d^3, \\ x^5 + y^5 + z^5 + w^5 &= a^5 + b^5 + c^5 + d^5, \\ x^7 + y^7 + z^7 + w^7 &= a^7 + b^7 + c^7 + d^7, \end{aligned}$$

where  $a, b, c, d$  are given and none of the sums  $a + b, a + c, a + d, b + c, b + d, c + d$  are zero, are

$$(x, y, z, w) = (a, b, c, d).$$

If  $c + d = 0 \neq a + b$ , then  $x = a, y = b, z + w = 0$ .

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