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Uniqueness Theorems for Power Equations

The uniqueness of the solution for the system of power equations

$$\sum_{i=1}^{n} x^k = 0\,, \qquad k = 1, 2, \dots, n$$

is well known. FLANDERS [1] in a classroom note states that the Newton formulae are not quite so elementary and gives a more elementary proof using the Vandermonde determinant. In this note, we give an elementary proof of a generalization which uses only part of the Newton formulae. Then by use of the complete set of Newton formulae, we give further extensions.

If $T_1, T_2, \dots,$ denote the elementary symmetric functions of $x_1, x_2, \dots, x_n$, i.e.,

$$P(x) = \prod_{i=1}^{n} (x - x_i) = x^n - T_1\, x^{n-1} + T_2\, x^{n-2} - \cdots + (-1)^n\, T_n$$

and if

$$S_k = \sum_{i=1}^{n} x_i^k\,,$$

then the Newton formulae [2, 3] are

(A) $\quad S_k - T_1\, S_{k-1} + T_2\, S_{k-2} - \cdots + (-1)^{k-1}\, T_{k-1}\, S_1 + (-1)^k\, k\, T_k = 0 \ (k \leqslant n),$

(B) $\quad S_k - T_1\, S_{k-1} + T_2\, S_{k-2} - \cdots + (-1)^n\, T_n\, S_{k-n} = 0 \ (k > n)\,.$

To obtain (B), we merely sum over the roots x_i of the equation

$$x^m\,(x^n - T_1\, x^{n-1} + T_2\, x^{n-2} - \cdots + (-1)^n\, T_n) = 0 \ (\text{here } k = m + n)\,.$$

To obtain (A) requires considerably more work. The proof in [2] is based on knowing the formula and using properties of the elementary symmetric functions to confirm it. The proof in [3] derives the formula by an expansion of the identity

$$P'(x) = \sum_{i=1}^{n} \frac{P(x)}{x - x_i}\,.$$

We now give a considerable simplification of the latter proof and obtain both (A) and (B). We start from the different expansion

$$\frac{P'(x)}{P(x)} = \sum_{i=1}^{n} \frac{1}{x - x_i} = \frac{n}{x} + \frac{S_1}{x^2} + \frac{S_2}{x^3} + \cdots. \tag{1}$$

Changing x to $1/x$ and simplifying, we obtain

$$\frac{T_1 - 2\, T_2\, x + \cdots + (-1)^{n-1}\, n\, T_n\, x^n}{1 - T_1\, x + T_2\, x^2 - \cdots + (-1)^n\, T_n\, x^n} = S_1 + S_2\, x + S_3\, x^2 + \cdots.$$

Now by multiplying both sides by the denominator and equating coefficients of like powers of x, we obtain both (A) and (B).

Using just (B), we can establish

Theorem 1: If $S_k = 0$ $(k = r+1, r+2, \ldots, r+n)$, then $x_i = 0$ $(i = 1, 2, \ldots, n)$.

Proof: From (B) with $k = n + r$, it follows that $T_n S_r = 0$. Thus either T_n or $S_r = 0$. If $S_r = 0$, then (B) with $k = n + r - 1$ implies that $T_n S_{r-1} = 0$. It now follows inductively that $T_n = 0$ or equivalently that at least one of the x_i's is zero. Repeating the same argument for the set of the remaining $(n-1)$ x_i's, we get inductively that all the x_i's are zero.

We could have also proven Theorem 1 in the manner of Flanders using the Vandermonde determinant. Another proof is based on the somewhat interesting elementary

Lemma: If $P(x)$ denotes a polynomial of degree $n > 0$, then the power expansion of $\exp[P(x)]$ does not have n consecutive zero coefficients.

Proof: Suppose otherwise that

$$e^{P(x)} = A_0 + A_1 x + \cdots + A_m x^m + O(x^{m+n+1})$$

where $A_m \neq 0$ and $P(x) = a_0 + a_1 x + \cdots + a_n x^n$, $a_n \neq 0$. Differentiating, we obtain

$$P'(x)\, e^{P(x)} = A_1 + 2 A_2 x + \cdots + m A_m x^{m-1} + O(x^{m+n})$$

and that the coefficient of x^{m+n-1} is zero. On the other hand,

$$P'(x)\, e^{P(x)} = \{a_1 + 2 a_2 x + \cdots + n a_n x^{n-1}\}\, \{A_0 + A_1 x + \cdots + A_m x^m + O(x^{m+n+1})\},$$

so that the coefficient of x^{m+n-1} is $n a_n A_m \neq 0$. This contradiction completes the proof. Corollary: If the expansion of $\exp\{a_1 x + a_2 x^2 + \cdots + a_n x^n\}$ has n consecutive zero coefficients, then $a_1 = a_2 = \cdots = a_n = 0$.

For our alternate proof of Theorem 1, we integrate (1) to give

$$\log \frac{P(x)}{x^n} = C - \frac{S_1}{x} - \cdots - \frac{S_r}{r\, x^r} + O(x^{-r-n-1})$$

so that

$$P(x) = A\, x^n \exp - \{S_1/x + \cdots + S_r/r\, x^r\} + O(x^{-r-1})\,.$$

Since $P(x)$ is an nth order polynomial, the terms in $x^{-n-1}, x^{-n-2}, \ldots, x^{-n-r}$, must be absent from the expansion of

$$\exp - \{S_1/x + \cdots + S_r/r\, x^r\}\,.$$

By the previous corollary, $S_1 = S_2 = \cdots = S_r = 0$. Consequently, $P(x) = A\,x^n + O(x^{-r-1})$, and again since $P(x)$ is a polynomial, $P(x) = A\, x^n$, which entails that $x_i = 0$, $i = 1, 2, \ldots, n$.

The proof of the next generalization uses both (A) and (B).

Theorem 2: If $\sum_{i=1}^{n} x_i^k = \sum_{i=1}^{n} a_i^k$ $(k = 1, 2, \ldots, n)$, a_i-given, then aside from permutations $(x_1, x_2, \ldots, x_n) = (a_1, a_2, \ldots, a_n)$.

Proof: It follows from Newton's formulae that S_k is given uniquely as a function of $T_1, T_2, \ldots, T_k$ and also that T_k is given uniquely as a function of $S_1, S_2, \ldots, S_k$. Explicit representations in the form of determinants is given in [2]. Consequently, the elementary symmetric functions of the a_i's must be identical to the elementary symmetric functions of the x_i's. Whence,

$$\prod_{i=1}^{n} (x - x_i) = \prod_{i=1}^{n} (x - a_i)$$

which gives the desired result.

Theorem 2 is not unexpected in light of Bezout's theorem [4], i.e., 'N polynomial equations of degrees $n_1, n_2, \ldots, n_N$ in N variables have in general $n_1 n_2 \ldots n_N$ common solutions. When the number is greater than this, it is infinite.' If the a_i's are distinct, then counting all the permutations there are $1, 2, 3 \ldots n$ solutions. If some of the a_i's were equal, then the number of permutations would be less than $n!$ solutions or else each solution could have a multiplicity. An example where there are less than $n_1 n_2 \ldots n_N$ solutions is given by the system

$$x + y = 1, \quad x^3 + y^3 = 1.$$

The only two solutions are (1, 0) and (0, 1) and these cannot have any multiplicity.

We cannot extend the validity of Theorem 2 by changing the range $k = 1, 2, \ldots, n$ to $k = r + 1, r + 2, \ldots, r + n$ as in Theorem 1. A simple counterexample is given by the system

$$x^2 + y^2 = 1, \quad x^3 + y^3 = 1.$$

The $2 \cdot 3$ solutions of this set are given by

$$x = 0, \ y = 1; \quad x = 1, \ y = 0;$$

$$x = \frac{1 + \lambda^2}{1 + \lambda^3}, \quad y = \frac{\lambda (1 + \lambda^2)}{1 + \lambda^3}; \quad x = \frac{\lambda (1 + \lambda^2)}{1 + \lambda^3}, \quad y = \frac{1 + \lambda^2}{1 + \lambda^3}$$

where $3\lambda = 1 \pm 2 i \sqrt{2}$. Again this is not unexpected since by Bezout's theorem, there are $(r + 1)(r + 2) \ldots (r + n)$ solutions in general.

As an application of Theorem 2, we give a solution to Aufgabe 591 (El. Math. *24*, 20 (1969): For which real values of α is the implication

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} x_i^2 = \cdots = \sum_{i=1}^{n} x_i^n = \alpha \rightarrow (\forall\, r \in N) \sum_{i=1}^{n} x^{+r} = \alpha$$

valid if the x_i are real?

It follows, as in the proof of Theorem 2, that the set of equations

$$\sum_{i=1}^{n} x_i^k = b_k, \quad k = 1, 2, \ldots, n$$

has a unique solution, aside from permutations.

From (A) and (B) with $k = n$ and $n + 1$, we get

$$S_n - T_1 S_{n-1} + \cdots + (-1)^n n T_n = 0,$$

$$S_{n+1} - T_1 S_n + \cdots + (-1)^n S_1 T_n = 0.$$

Since $S_i = \alpha$ for all i, we get by subtraction that $T_n(n-\alpha) = 0$. Thus, either $\alpha = n$ or $T_n = 0$. If $\alpha = n$, then by Theorem 2, $x_i = 1$, $i = 1, 2, \ldots, n$. If $T_n = 0$, at least one of the x_i's is zero. We cancel this x_i out and repeat the argument. Thus, $\alpha = 0, 1, 2, \ldots, n$. And for these α's, $x_1 = x_2 = \cdots = x_\alpha = 1$, $x_{\alpha+1} = x_{\alpha+2} = \cdots = x_n = 0$.

A simpler solution can be gotten by considering $\sum x_i^{2r}$ as r increases without bound. It then follows easily that the x_i's can only be 0 or 1. However, in the former solution it follows without any further argument that there was no need to restrict α and the x_i's to be real.

We now consider systems of equations where there are more variables than equations.

Theorem 3: If $\sum_{r=1}^{n} x_r^k = 0$, $k = 1, 2, \ldots, n-1$, then aside from permutations, $x_r = A\, e^{2\pi i r/n}$.

Proof: From A, it follows that

$$T_1 = T_2 = \cdots = T_{n-1} = 0\,.$$

Thus, $x^n + (-1)^n T_n = 0$ and the x_r's are proportional to the nth roots of unity.

Corollary: If $\sum_{r=1}^{n} x_r^k = 0$, $k = 1, 2, \ldots, n-1, m\,n$, then the x_r's are zero.

Theorem 4: If $\sum_{r=1}^{n} x_r^k = 0$, $k = 1, 2, \ldots, n-2$, $(n \geqslant 4)$ then $S_p = \sum_{r=1}^{n} x_r^p = 0$ where

$$p = \begin{cases} n+1, n+2, \ldots, 2n-3\,, \\ 2n+1, 2n+2, \ldots, 3n-4\,, \\ \quad\cdot \qquad \cdot \qquad \cdot \\ \quad\cdot \qquad \cdot \qquad \cdot \\ \quad\cdot \qquad \cdot \qquad \cdot \\ rn+1, rn+2, \ldots, (r+1)n-r-2\ (r \leqslant n-3)\,. \end{cases}$$

Proof: Since $T_1 = T_2 = \cdots = T_{n-2}$,

$$x^n + (-1)^{n-1}(x\,T_{n-1} - T_n) = 0\,. \tag{2}$$

Multiplying (1) by $x, x^2, \ldots, x^{n-3}$ and summing over the roots x_i, we get

$$S_p = 0\,, \quad p = n+1, n+2, \ldots, 2n-3\,.$$

We now multiply (2) by $x^{n+1}, x^{n+2}, \ldots, x^{2n-4}$ and sum over the x_i's to give

$$S_p = 0\,, \quad p = 2n+1, 2n+2, \ldots, 3n-4\,.$$

We then continue in the same manner to obtain the remaining values of p.

In a similar way, we can extend Theorem 4 to

Theorem 5: If $\sum_{r=1}^{n} x_r^k = 0$, $k = 1, 2, \ldots, n - m$, $(n \geqq 2m)$, then $S_p = 0$ where

$$p = \begin{cases} n+1, n+2, \ldots, 2n+1-2m\,, \\ 2n+1, 2n+2, \ldots, 3n+2-3m\,, \\ \vdots \\ rn+1, rn+2, \ldots, (r+1)n+r-(r+1)m\,, \end{cases}$$

and
$$r(m-1) \leqq (n-1-m)\,.$$

Our last theorem will concern consecutive odd power sums. As to be expected, we cannot give comparable uniqueness results as before.

Theorem 6: If $\sum_{r=1}^{2n} x_r^{2k-1} = 0$, $k = 1, 2, \ldots, n$, then aside from permutations, the roots occur in pairs such that $x_i + x_j = 0$.

Proof: Using the Newton formulae

$$S_{2m-1} - T_1 S_{2m-2} + T_2 S_{2m-3} - \cdots + S_1 T_{2m-2} - (2m-1) T_{2m-1} = 0$$

for $m = 1, 2, \ldots, n$ and the hypothesis, it follows by induction that

$$T_1 = T_3 = \ldots = T_{2n-1} = 0\,.$$

Thus,
$$\prod_{i=1}^{2n} (x - x_i) = x^{2n} + T_2 x^{2n-2} + T_4 x^{2n-4} + \cdots = 0\,.$$

Since the equation is even, the theorem is proved.

As an application, the only solutions, aside from permutations, of the system of equations

$$x + y + z + w = a + b + c + d\,,$$
$$x^3 + y^3 + z^3 + w^3 = a^3 + b^3 + c^3 + d^3\,,$$
$$x^5 + y^5 + z^5 + w^5 = a^5 + b^5 + c^5 + d^5\,,$$
$$x^7 + y^7 + z^7 + w^7 = a^7 + b^7 + c^7 + d^7\,,$$

where a, b, c, d are given and none of the sums $a+b$, $a+c$, $a+d$, $b+c$, $b+d$, $c+d$ are zero, are

$$(x, y, z, w) = (a, b, c, d)\,.$$

If $c + d = 0 \neq a + b$, then $x = a$, $y = b$, $z + w = 0$.

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REFERENCES

[1] H. FLANDERS, *An Application of the Vandermonde Determinant*, Amer. Math. Monthly, Dec. 1953, p. 708.
[2] A. MOSTOWSKI, M. STARK, *Introduction to Higher Algebra* (Pergamon, London 1964), p. 360–361.
[3] S. BARNARD, J. M. CHILD, *Higher Algebra* (MacMillan, London 1949), p. 297.
[4] J. L. COOLIDGE, A. *Treatise on Algebraic Plane Curves* (Dover, N. Y. 1959), p. 10.