

Zeitschrift: Elemente der Mathematik
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 25 (1970)
Heft: 6

Artikel: Note on a diophantine equation
Autor: Williams, H.C.
DOI: <https://doi.org/10.5169/seals-27360>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 08.08.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

a distance. Can this be done with as few as $O(n^{1/2})$ points; or with $O(n^{1/3})$ points in one dimension?

Another open problem [1] is given any n points in the plane (not necessarily lattice points) [or in d dimensions], how many can one select so that the distances which are determined are all distinct? P. ERDÖS and R. K. GUY, Budapest

REFERENCES

- [1] ERDÖS, P., *Nehany geometriai problémáról* (in Hungarian), Mat. Lapok 8, 86–92 (1957); M. R. 20, 6056 (1959).
- [2] ERDÖS, P. and TURAN, P., *On a Problem of Sidon in Additive Number Theory and Some Related Problems*, J. London Math. Soc. 16, 212–215 (1941); M. R. 3, 270 (1942).
- [3] HARDY, G. H. and WRIGHT, E. M., *Introduction to the Theory of Numbers*, 4th ed. (Oxford 1960).
- [4] LANDAU, E., *Handbuch der Lehre von der Verteilung der Primzahlen* (Leipzig 1909), II, 643.
- [5] LINDSTRÖM, B., *An Inequality for B_2 -sequences*, J. Combinatorial Theory 6, 211–212 (1969).
- [6] SINGER, J., *A Theorem in Finite Projective Geometry and Some Applications to Number Theory*, Trans. Amer. Math. Soc. 43, 377–385 (1938).

Note on a Diophantine Equation

SCHINZEL and SIERPIŃSKI [1] have given the general solution of the diophantine equation

$$(x^2 - 1)(y^2 - 1) = \left[\left(\frac{x-y}{2} \right)^2 - 1 \right]^2,$$

and SZYMICZEK [2] has given the general solution of

$$(x^2 - z^2)(y^2 - z^2) = \left[\left(\frac{y-x}{2} \right)^2 - z^2 \right]^2.$$

The purpose of this paper is to obtain a complete solution of the diophantine equation

$$(x^2 + a)(y^2 + a) = \left[a \left(\frac{y-x}{2b} \right)^2 + b^2 \right]^2, \quad (1)$$

where a and b are any two given integers.

Let $X = x - y$, $Y = x + y$; then $X \equiv Y \pmod{2}$ and (1) becomes

$$b^4(X^2 + 2XY + Y^2 + 4a)(X^2 - 2XY + Y^2 + 4a) = (aX^2 + 4b^4)^2.$$

This equation reduces to

$$b^4((Y^2 - X^2)^2 + 8a(Y^2 - X^2) + 16a^2) = (aX^2 + 4b^4)^2 - 16ab^4X^2$$

and we have

$$b^2(Y^2 - X^2 + 4a) = \pm(aX^2 - 4b^4).$$

At this point it becomes necessary to find the complete solutions of

$$b^2 (Y^2 - X^2 + 4a) = + (aX^2 - 4b^4) \quad (2)$$

and

$$b^2 (Y^2 - X^2 + 4a) = - (aX^2 - 4b^4). \quad (3)$$

We shall first solve (2).

Let $(a, b^2) = \gamma^2 \delta$, where δ has no square factors; then $a = \gamma^2 \delta \alpha$ and $b = \gamma \delta \beta$, where $(\beta, \delta, \alpha) = 1$.

From (2), we must have

$$\delta \beta^2 \mid \alpha X^2$$

or, equivalently, $X = \delta \beta Z$, for some integer Z . On substituting this value for X and re-arranging the terms in (2), we obtain the equation

$$Y^2 - (\delta \beta^2 + \alpha) \delta Z^2 = -4 \gamma^2 (\delta^2 \beta^2 + \delta \alpha). \quad (4)$$

If we let

$$c_1^2 d_1 = \delta \beta^2 + \alpha, \quad (5)$$

where d_1 has no square factors, then $(d_1, \delta) = 1$ and therefore, $Y = c_1 d_1 \delta W$, for some integer W . Equation (4) reduces to

$$Z^2 - d_1 \delta W^2 = 4 \gamma^2. \quad (6)$$

Thus the problem of obtaining all solutions of (2) reduces to the solved problem of finding all solutions of (6). It may be shown similarly that if we let

$$c_2^2 d_2 = \delta \beta^2 - \alpha,$$

where d_2 has no square factor, then the problem of obtaining all solutions of (3) reduces to the problem of obtaining all solutions of

$$z^2 - d_2 \delta w^2 = -4 \gamma^2.$$

It is evident that part of the solutions of (1) must be of the form

$$x = (\delta \beta Z + c_1 d_1 \delta W)/2, \quad y = (c_1 d_1 \delta W - \delta \beta Z)/2,$$

where (Z, W) is a solution of (6). If $2 \mid \delta$ or $2 \mid d_1$, it is clear that x and y will both be integers. If $2 \nmid \delta$ and $2 \nmid d_1$, then, by (6), Z and W will be of the same parity. If they are both even, then x and y are integers; if they are both odd, we must have, in order that x and y be integers, that $\beta \equiv c_1 \pmod{2}$. But, from (5),

$$c_1 \equiv \beta + \alpha \pmod{2};$$

thus, it is necessary that $\alpha \equiv 0 \pmod{2}$.

Hence, if $2 \mid (d_1 \delta \alpha)$, $(c_1 d_1 \delta W \pm \delta \beta Z)/2$ are integers. If $2 \nmid (d_1 \delta \alpha)$, then $(c_1 d_1 \delta W \pm \delta \beta Z)/2$ will be integers if and only if $2 \mid W$ and $2 \mid Z$.

This same discussion may be applied to what must be the forms of the remaining solutions of (1), that is,

$$x = (\delta \beta z + c_2 d_2 \delta w)/2, \quad y = (c_2 d_2 \delta w - \delta \beta z)/2.$$

Thus we have proved the following

Theorem. If $(a, b^2) = \gamma^2 \delta$, where δ has no square factors, and

$$c^2 d = \delta \beta^2 \pm \alpha,$$

where

$$\beta = \frac{b}{\gamma \delta},$$

and

$$\alpha = \frac{a}{\gamma^2 \delta},$$

and d has no square factors, then any solution of (1) must be of the form

$$x = (c d \delta W + \delta \beta Z)/2, \quad y = (c d \delta W - \delta \beta Z)/2,$$

where (Z, W) is a solution of

$$Z^2 - d \delta W^2 = \pm 4 \gamma^2$$

and $2 \mid (\alpha \delta d)$; if $2 \nmid (\alpha \delta d)$, then any solution of (1) must be of the form

$$x = c d \delta W + \delta \beta Z, \quad y = c d \delta W - \delta \beta Z,$$

where (Z, W) is a solution of

$$Z^2 - d \delta W^2 = \pm \gamma^2.$$

This theorem has the following

Corollary. The equation

$$(x^2 + a)(y^2 + a) = (az^2 + b^2)^2, \quad (7)$$

where a and b are integers, has integer solutions given by

$$x = b u + c d v, \quad y = b u - c d v, \quad z = u,$$

provided

$$b^2 \pm a = c^2 d,$$

where d has no square factors, and u and v are given by

$$u^2 - d v^2 = \pm 1.$$

It is not known to the author whether there are values of a and b for which the above result provides all the solutions to the equation (7); however, the equation (7) with $a = b = 1$, which was originally suggested to the author by L. J. MORDELL, has no solutions other than those predicted by the foregoing method for all integers x and y , where $1 \leq x \leq 300$ and $1 \leq y \leq x$.

H. C. WILLIAMS, University of Manitoba

REFERENCES

- [1] A. SCHINZEL et W. SIERPIŃSKI, *Sur l'équation diophantienne* $(x^2 - 1)(y^2 - 1) = [(y - x)/2]^2 - 1]^2$, *El. Math.* 18, 132–133 (1963).
- [2] K. SZYMICZEK, *On a Diophantine Equation*, *El. Math.* 22, 37–38 (1967).