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Autor: Williams, H.C.

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a distance. Can this be done with as few as $O(n^{1/2})$ points; or with $O(n^{1/3})$ points in one dimension?

Another open problem [1] is given any n points in the plane (not necessarily lattice points) [or in d dimensions], how many can one select so that the distances which are determined are all distinct?

P. ERDÖS and R. K. GUY, Budapest

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Note on a Diophantine Equation

SCHINZEL and SIERPIŃSKI [1] have given the general solution of the diophantine equation

$$(x^2 - 1)(y^2 - 1) = \left[\left(\frac{x-y}{2}\right)^2 - 1\right]^2,$$

and SZYMICZEK [2] has given the general solution of

$$(x^2 - z^2)(y^2 - z^2) = \left[\left(\frac{y-x}{2}\right)^2 - z^2\right]^2.$$

The purpose of this paper is to obtain a complete solution of the diophantine equation

$$(x^2 + a)(y^2 + a) = \left[a\left(\frac{y-x}{2b}\right)^2 + b^2\right]^2, \tag{1}$$

where a and b are any two given integers.

Let $X = x - y$, $Y = x + y$; then $X \equiv Y \pmod 2$ and (1) becomes

$$b^4 (X^2 + 2XY + Y^2 + 4a)(X^2 - 2XY + Y^2 + 4a) = (aX^2 + 4b^4)^2.$$

This equation reduces to

$$b^4 \left((Y^2 - X^2)^2 + 8a(Y^2 - X^2) + 16a^2\right) = (aX^2 + 4b^4)^2 - 16ab^4X^2$$

and we have

$$b^2 (Y^2 - X^2 + 4a) = \pm (aX^2 - 4b^4).$$

At this point it becomes necessary to find the complete solutions of

$$b^2 (Y^2 - X^2 + 4a) = + (a X^2 - 4 b^4) \tag{2}$$

and

$$b^2 (Y^2 - X^2 + 4a) = - (a X^2 - 4 b^4) . \tag{3}$$

We shall first solve (2).

Let $(a, b^2) = \gamma^2 \delta$, where δ has no square factors; then $a = \gamma^2 \delta\alpha$ and $b = \gamma\,\delta\beta$, where $(\beta\,\delta, \alpha) = 1$.

From (2), we must have

$$\delta\beta^2 \mid \alpha X^2$$

or, equivalently, $X = \delta\beta Z$, for some integer Z. On substituting this value for X and re-arranging the terms in (2), we obtain the equation

$$Y^2 - (\delta\beta^2 + \alpha)\,\delta Z^2 = -4\gamma^2 (\delta^2\beta^2 + \delta\alpha) . \tag{4}$$

If we let

$$c_1^2\, d_1 = \delta\beta^2 + \alpha , \tag{5}$$

where d_1 has no square factors, then $(d_1, \delta) = 1$ and therefore, $Y = c_1\, d_1\delta W$, for some integer W. Equation (4) reduces to

$$Z^2 - d_1\delta W^2 = 4\gamma^2 . \tag{6}$$

Thus the problem of obtaining all solutions of (2) reduces to the solved problem of finding all solutions of (6). It may be shown similarly that if we let

$$c_2^2\, d_2 = \delta\beta^2 - \alpha ,$$

where d_2 has no square factor, then the problem of obtaining all solutions of (3) reduces to the problem of obtaining all solutions of

$$z^2 - d_2\delta w^2 = -4\gamma^2 .$$

It is evident that part of the solutions of (1) must be of the form

$$x = (\delta\beta Z + c_1\, d_1\delta W)/2 , \qquad y = (c_1\, d_1\delta W - \delta\beta Z)/2 ,$$

where (Z, W) is a solution of (6). If $2 \mid \delta$ or $2 \mid d_1$, it is clear that x and y will both be integers. If $2 \nmid \delta$ and $2 \nmid d_1$, then, by (6), Z and W will be of the same parity. If they are both even, then x and y are integers; if they are both odd, we must have, in order that x and y be integers, that $\beta \equiv c_1 \pmod 2$. But, from (5),

$$c_1 \equiv \beta + \alpha \pmod 2 ;$$

thus, it is necessary that $\alpha \equiv 0 \pmod 2$.

Hence, if $2 \mid (d_1\delta\alpha)$, $(c_1\, d_1\delta W \pm \delta\beta\, Z)/2$ are integers. If $2 \nmid (d_1\delta\alpha)$, then $(c_1\, d_1\delta W \pm \delta\beta\, Z)/2$ will be integers if and only if $2 \mid W$ and $2 \mid Z$.

This same discussion may be applied to what must be the forms of the remaining solutions of (1), that is,

$$x = (\delta\beta z + c_2\, d_2\delta w)/2 , \qquad y = (c_2\, d_2\delta w - \delta\beta z)/2 .$$

Thus we have proved the following

Theorem. If $(a, b^2) = \gamma^2 \delta$, where δ has no square factors, and

$$c^2 d = \delta\beta^2 \pm \alpha ,$$

where

$$\beta = \frac{b}{\gamma \delta},$$

and

$$\alpha = \frac{a}{\gamma^2 \delta},$$

and d has no square factors, then any solution of (1) must be of the form

$$x = (c\, d\delta W + \delta\beta Z)/2 , \qquad y = (c\, d\delta W - \delta\beta Z)/2 ,$$

where (Z, W) is a solution of

$$Z^2 - d\delta W^2 = \pm 4\gamma^2$$

and $2 \mid (\alpha\, \delta d)$; if $2 \nmid (\alpha\, \delta d)$, then any solution of (1) must be of the form

$$x = c\, d\delta W + \delta\beta Z , \qquad y = c\, d\delta W - \delta\beta Z ,$$

where (Z, W) is a solution of

$$Z^2 - d\delta W^2 = \pm \gamma^2 .$$

This theorem has the following

Corollary. The equation

$$(x^2 + a)(y^2 + a) = (a z^2 + b^2)^2 , \qquad (7)$$

where a and b are integers, has integer solutions given by

$$x = b u + c\, dv , \qquad y = b u - c\, dv , \qquad z = u ,$$

provided

$$b^2 \pm a = c^2 d ,$$

where d has no square factors, and u and v are given by

$$u^2 - dv^2 = \pm 1 .$$

It is not known to the author whether there are values of a and b for which the above result provides all the solutions to the equation (7); however, the equation (7) with $a = b = 1$, which was originally suggested to the author by L. J. MORDELL, has no solutions other than those predicted by the foregoing method for all integers x and y, where $1 \leqslant x \leqslant 300$ and $1 \leqslant y \leqslant x$.

H. C. WILLIAMS, University of Manitoba

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