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# ELEMENTE DER MATHEMATIK

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## Distinct Distances Between Lattice Points

How many points  $(x_i, y_i)$ ,  $1 \leq i \leq k$ , with integer coordinates  $0 < x_i, y_i \leq n$ , may be chosen with all mutual distances distinct? By counting such distances, and pairs of differences of coordinates, we have

$$\binom{k}{2} \leq \binom{n+1}{2} - 1, \quad (1)$$

so that  $k \leq n$ , and for  $2 \leq n \leq 7$  such a bound can be attained; e.g. for  $2 \leq n \leq 5$ , by the points (1,1), (1,2), (3,1), (4,4) and (5,3); for  $n = 6$  by (1,1), (1,2), (2,4), (4,6), (6,3) and (6,6); and for  $n = 7$  by (1,1), (1,3), (2,3), (3,7), (4,1), (6,6) and (7,7).

However, the fact that numbers may be expressed in more than one way as the sum of two squares indicates that this bound cannot be attained for  $n > 15$ . A result of LANDAU [4] states that the number of integers less than  $x$  expressible as the sum of two squares is asymptotically  $c_1 x (\log x)^{-1/2}$ , so we can replace the right member of (1) by  $c_2 n^2 (\log n)^{-1/2}$  and we have the upper bound

$$k < c_3 n (\log n)^{-1/4}, \quad (2)$$

where  $c_i$  is in each case a positive constant.

A heuristic argument can be given to support the conjecture

$$(?) \quad k < c_4 n^{2/3} (\log n)^{1/6}, \quad (3)$$

but it lacks conviction since the corresponding argument in one dimension gives a false result.

On the other hand we can show

$$k > n^{2/3-\varepsilon} \quad (4)$$

for any  $\varepsilon > 0$  and sufficiently large  $n$ , by means of the following construction. Choose points successively; when  $k$  points have been chosen, take another so that

(a) it does not lie on any circle having one of the  $k$  points as centre and one of the  $\binom{k}{2}$  distinct distances determined by these points as radius.

(b) it does not form, with any of the first  $k$  points, a line with slope  $b/a$ ,  $(a, b) = 1$ ,  $|a| < n^{1/3}$ ,  $|b| < n^{1/3}$ . Note that in particular no two points determine a distance less than  $n^{1/3}$ .

(c) it is not equidistant from any pair of the first  $k$  points.

We may choose such a point provided that all  $n^2$  points are not excluded by these conditions.

Condition (a) excludes at most  $k \binom{k}{2} n^{c_s/\log \log n}$  points, since there are  $\binom{k}{2}$  circles round each of  $k$  points, and each circle contains at most  $n^{c_s/\log \log n}$  lattice points<sup>1)</sup>.

Condition (b) excludes at most

$$k \sum_{a=1}^{n^{1/3}} 4 \varphi(a) \frac{n}{a} < c_6 k n^{4/3}$$

points, since a line with slope  $b/a$ ,  $b < a$ ,  $(a, b) = 1$ , contains at most  $n/a$  lattice points.

Condition (c) excludes at most  $\binom{k}{2} n^{2/3}$  points, since there are  $\binom{k}{2}$  lines of equidistant points, each of which has slope  $b/a$ ,  $(a, b) = 1$ ,  $|a| \geq n^{1/3}$  and such a line contains at most  $n/|a| \leq n^{2/3}$  lattice points.

Hence, so long as

$$\frac{1}{2} k^3 n^{c_s/\log \log n} + c_6 k n^{4/3} + \frac{1}{2} k^2 n^{2/3} < n^2,$$

there remain eligible points, and this is the case if  $k \leq n^{2/3-\epsilon}$ . The lower bound (4) is thus established.

For the corresponding problem in one dimension, the existence of perfect difference sets [6] shows that for  $n$  an even power of a prime,

$$k \geq n^{1/2} + 1,$$

so that generally

$$k > n^{1/2} (1 - \epsilon). \quad (5)$$

On the other hand it is known [2, 5] that

$$k < n^{1/2} + n^{1/4} + 1. \quad (6)$$

In  $d$  dimensions,  $d \geq 3$ , we may replace Landau's theorem by the theorems on sums of three or four squares, giving an upper bound

$$k < c_7 d^{1/2} n, \quad (7)$$

while the corresponding heuristic argument suggests the conjecture

$$(?) \quad k < c_8 d^{2/3} n^{2/3} (\log n)^{1/3}. \quad (8)$$

The construction, with (hyper)spheres and (hyper)planes, corresponding to that given above, yields the same lower bound (4) as before.

One can also ask for configurations containing a *minimum* number of points, determining distinct distances, so that *no* point may be added without duplicating

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<sup>1)</sup> It is well known that the number of solutions of  $n = x^2 + y^2$  is less than or equal to  $d(n)$ , the number of divisors of  $n$  [3] and  $d(n) < n^{c/\log \log n}$  by a well known result of WIGERT [3].

a distance. Can this be done with as few as  $O(n^{1/2})$  points; or with  $O(n^{1/3})$  points in one dimension?

Another open problem [1] is given any  $n$  points in the plane (not necessarily lattice points) [or in  $d$  dimensions], how many can one select so that the distances which are determined are all distinct? P. ERDÖS and R. K. GUY, Budapest

## REFERENCES

- [1] ERDÖS, P., *Néhány geometriai problémáról* (in Hungarian), Mat. Lapok 8, 86–92 (1957); M. R. 20, 6056 (1959).
- [2] ERDÖS, P. and TURAN, P., *On a Problem of Sidon in Additive Number Theory and Some Related Problems*, J. London Math. Soc. 16, 212–215 (1941); M. R. 3, 270 (1942).
- [3] HARDY, G. H. and WRIGHT, E. M., *Introduction to the Theory of Numbers*, 4th ed. (Oxford 1960).
- [4] LANDAU, E., *Handbuch der Lehre von der Verteilung der Primzahlen* (Leipzig 1909), II, 643.
- [5] LINDSTRÖM, B., *An Inequality for  $B_2$ -sequences*, J. Combinatorial Theory 6, 211–212 (1969).
- [6] SINGER, J., *A Theorem in Finite Projective Geometry and Some Applications to Number Theory*, Trans. Amer. Math. Soc. 43, 377–385 (1938).

## Note on a Diophantine Equation

SCHINZEL and SIERPIŃSKI [1] have given the general solution of the diophantine equation

$$(x^2 - 1)(y^2 - 1) = \left[ \left( \frac{x - y}{2} \right)^2 - 1 \right]^2,$$

and SZYMICZEK [2] has given the general solution of

$$(x^2 - z^2)(y^2 - z^2) = \left[ \left( \frac{y - x}{2} \right)^2 - z^2 \right]^2.$$

The purpose of this paper is to obtain a complete solution of the diophantine equation

$$(x^2 + a)(y^2 + a) = \left[ a \left( \frac{y - x}{2b} \right)^2 + b^2 \right]^2, \quad (1)$$

where  $a$  and  $b$  are any two given integers.

Let  $X = x - y$ ,  $Y = x + y$ ; then  $X \equiv Y \pmod{2}$  and (1) becomes

$$b^4 (X^2 + 2XY + Y^2 + 4a)(X^2 - 2XY + Y^2 + 4a) = (aX^2 + 4b^4)^2.$$

This equation reduces to

$$b^4 ((Y^2 - X^2)^2 + 8a(Y^2 - X^2) + 16a^2) = (aX^2 + 4b^4)^2 - 16ab^4X^2$$

and we have

$$b^2 (Y^2 - X^2 + 4a) = \pm (aX^2 - 4b^4).$$