

Zeitschrift: Elemente der Mathematik
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 25 (1970)
Heft: 6

Artikel: Distinct distances between lattice points
Autor: Erdős, P. / Guy, R.K.
DOI: <https://doi.org/10.5169/seals-27359>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 17.04.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

ELEMENTE DER MATHEMATIK

Revue de mathématiques élémentaires – Rivista di matematica elementare

*Zeitschrift zur Pflege der Mathematik
und zur Förderung des mathematisch-physikalischen Unterrichts*

Publiziert mit Unterstützung des Schweizerischen Nationalfonds
zur Förderung der wissenschaftlichen Forschung

El. Math.

Band 25

Heft 6

Seiten 121-144

10. November 1970

Distinct Distances Between Lattice Points

How many points (x_i, y_i) , $1 \leq i \leq k$, with integer coordinates $0 < x_i, y_i \leq n$, may be chosen with all mutual distances distinct? By counting such distances, and pairs of differences of coordinates, we have

$$\binom{k}{2} \leq \binom{n+1}{2} - 1, \quad (1)$$

so that $k \leq n$, and for $2 \leq n \leq 7$ such a bound can be attained; e.g. for $2 \leq n \leq 5$, by the points $(1,1)$, $(1,2)$, $(3,1)$, $(4,4)$ and $(5,3)$; for $n = 6$ by $(1,1)$, $(1,2)$, $(2,4)$, $(4,6)$, $(6,3)$ and $(6,6)$; and for $n = 7$ by $(1,1)$, $(1,3)$, $(2,3)$, $(3,7)$, $(4,1)$, $(6,6)$ and $(7,7)$.

However, the fact that numbers may be expressed in more than one way as the sum of two squares indicates that this bound cannot be attained for $n > 15$. A result of LANDAU [4] states that the number of integers less than x expressible as the sum of two squares is asymptotically $c_1 x (\log x)^{-1/2}$, so we can replace the right member of (1) by $c_2 n^2 (\log n)^{-1/2}$ and we have the upper bound

$$k < c_3 n (\log n)^{-1/4}, \quad (2)$$

where c_i is in each case a positive constant.

A heuristic argument can be given to support the conjecture

$$(?) \quad k < c_4 n^{2/3} (\log n)^{1/6}, \quad (3)$$

but it lacks conviction since the corresponding argument in one dimension gives a false result.

On the other hand we can show

$$k > n^{2/3-\varepsilon} \quad (4)$$

for any $\varepsilon > 0$ and sufficiently large n , by means of the following construction. Choose points successively; when k points have been chosen, take another so that

(a) it does not lie on any circle having one of the k points as centre and one of the $\binom{k}{2}$

distinct distances determined by these points as radius.

(b) it does not form, with any of the first k points, a line with slope b/a , $(a, b) = 1$, $|a| < n^{1/3}$, $|b| < n^{1/3}$. Note that in particular no two points determine a distance less than $n^{1/3}$.

(c) it is not equidistant from any pair of the first k points.

We may choose such a point provided that all n^2 points are not excluded by these conditions.

Condition (a) excludes at most $k \binom{k}{2} n^{c_s/\log \log n}$ points, since there are $\binom{k}{2}$ circles round each of k points, and each circle contains at most $n^{c_s/\log \log n}$ lattice points¹).

Condition (b) excludes at most

$$k \sum_{a=1}^{n^{1/3}} 4 \varphi(a) \frac{n}{a} < c_6 k n^{4/3}$$

points, since a line with slope b/a , $b < a$, $(a, b) = 1$, contains at most n/a lattice points.

Condition (c) excludes at most $\binom{k}{2} n^{2/3}$ points, since there are $\binom{k}{2}$ lines of equidistant points, each of which has slope b/a , $(a, b) = 1$, $|a| \geq n^{1/3}$ and such a line contains at most $n/|a| \leq n^{2/3}$ lattice points.

Hence, so long as

$$\frac{1}{2} k^3 n^{c_s/\log \log n} + c_6 k n^{4/3} + \frac{1}{2} k^2 n^{2/3} < n^2,$$

there remain eligible points, and this is the case if $k \leq n^{2/3-\epsilon}$. The lower bound (4) is thus established.

For the corresponding problem in one dimension, the existence of perfect difference sets [6] shows that for n an even power of a prime,

$$k \geq n^{1/2} + 1,$$

so that generally

$$k > n^{1/2} (1 - \epsilon). \tag{5}$$

On the other hand it is known [2, 5] that

$$k < n^{1/2} + n^{1/4} + 1. \tag{6}$$

In d dimensions, $d \geq 3$, we may replace Landau's theorem by the theorems on sums of three or four squares, giving an upper bound

$$k < c_7 d^{1/2} n, \tag{7}$$

while the corresponding heuristic argument suggests the conjecture

$$(?) \quad k < c_8 d^{2/3} n^{2/3} (\log n)^{1/3}. \tag{8}$$

The construction, with (hyper)spheres and (hyper)planes, corresponding to that given above, yields the same lower bound (4) as before.

One can also ask for configurations containing a *minimum* number of points, determining distinct distances, so that *no* point may be added without duplicating

¹) It is well known that the number of solutions of $n = x^2 + y^2$ is less than or equal to $d(n)$, the number of divisors of n [3] and $d(n) < nc/\log \log n$ by a well known result of WIGERT [3].

a distance. Can this be done with as few as $O(n^{1/2})$ points; or with $O(n^{1/3})$ points in one dimension?

Another open problem [1] is given any n points in the plane (not necessarily lattice points) [or in d dimensions], how many can one select so that the distances which are determined are all distinct? P. ERDÖS and R. K. GUY, Budapest

REFERENCES

- [1] ERDÖS, P., *Néhány geometriai problémáról* (in Hungarian), Mat. Lapok 8, 86–92 (1957); M. R. 20, 6056 (1959).
- [2] ERDÖS, P. and TURAN, P., *On a Problem of Sidon in Additive Number Theory and Some Related Problems*, J. London Math. Soc. 16, 212–215 (1941); M. R. 3, 270 (1942).
- [3] HARDY, G. H. and WRIGHT, E. M., *Introduction to the Theory of Numbers*, 4th ed. (Oxford 1960).
- [4] LANDAU, E., *Handbuch der Lehre von der Verteilung der Primzahlen* (Leipzig 1909), II, 643.
- [5] LINDSTRÖM, B., *An Inequality for B_2 -sequences*, J. Combinatorial Theory 6, 211–212 (1969).
- [6] SINGER, J., *A Theorem in Finite Projective Geometry and Some Applications to Number Theory*, Trans. Amer. Math. Soc. 43, 377–385 (1938).

Note on a Diophantine Equation

SCHINZEL and SIERPIŃSKI [1] have given the general solution of the diophantine equation

$$(x^2 - 1)(y^2 - 1) = \left[\left(\frac{x - y}{2} \right)^2 - 1 \right]^2,$$

and SZYMICZEK [2] has given the general solution of

$$(x^2 - z^2)(y^2 - z^2) = \left[\left(\frac{y - x}{2} \right)^2 - z^2 \right]^2.$$

The purpose of this paper is to obtain a complete solution of the diophantine equation

$$(x^2 + a)(y^2 + a) = \left[a \left(\frac{y - x}{2b} \right)^2 + b^2 \right]^2, \quad (1)$$

where a and b are any two given integers.

Let $X = x - y$, $Y = x + y$; then $X \equiv Y \pmod{2}$ and (1) becomes

$$b^4 (X^2 + 2XY + Y^2 + 4a)(X^2 - 2XY + Y^2 + 4a) = (aX^2 + 4b^4)^2.$$

This equation reduces to

$$b^4 ((Y^2 - X^2)^2 + 8a(Y^2 - X^2) + 16a^2) = (aX^2 + 4b^4)^2 - 16ab^4X^2$$

and we have

$$b^2 (Y^2 - X^2 + 4a) = \pm (aX^2 - 4b^4).$$