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Distinct Distances Between Lattice Points

How many points (x_i, y_i) , $1 \leq i \leq k$, with integer coordinates $0 < x_i, y_i \leq n$, may be chosen with all mutual distances distinct? By counting such distances, and pairs of differences of coordinates, we have

$$\binom{k}{2} \leq \binom{n+1}{2} - 1, \quad (1)$$

so that $k \leq n$, and for $2 \leq n \leq 7$ such a bound can be attained; e.g. for $2 \leq n \leq 5$, by the points (1,1), (1,2), (3,1), (4,4) and (5,3); for $n = 6$ by (1,1), (1,2), (2,4), (4,6), (6,3) and (6,6); and for $n = 7$ by (1,1), (1,3), (2,3), (3,7), (4,1), (6,6) and (7,7).

However, the fact that numbers may be expressed in more than one way as the sum of two squares indicates that this bound cannot be attained for $n > 15$. A result of LANDAU [4] states that the number of integers less than x expressible as the sum of two squares is asymptotically $c_1 x (\log x)^{-1/2}$, so we can replace the right member of (1) by $c_2 n^2 (\log n)^{-1/2}$ and we have the upper bound

$$k < c_3 n (\log n)^{-1/4}, \quad (2)$$

where c_i is in each case a positive constant.

A heuristic argument can be given to support the conjecture

$$(?) \quad k < c_4 n^{2/3} (\log n)^{1/6}, \quad (3)$$

but it lacks conviction since the corresponding argument in one dimension gives a false result.

On the other hand we can show

$$k > n^{2/3-\varepsilon} \quad (4)$$

for any $\varepsilon > 0$ and sufficiently large n , by means of the following construction. Choose points successively; when k points have been chosen, take another so that

(a) it does not lie on any circle having one of the k points as centre and one of the $\binom{k}{2}$

distinct distances determined by these points as radius.

(b) it does not form, with any of the first k points, a line with slope b/a , $(a, b) = 1$, $|a| < n^{1/3}$, $|b| < n^{1/3}$. Note that in particular no two points determine a distance less than $n^{1/3}$.

(c) it is not equidistant from any pair of the first k points.

We may choose such a point provided that all n^2 points are not excluded by these conditions.

Condition (a) excludes at most $k \binom{k}{2} n^{c_s/\log \log n}$ points, since there are $\binom{k}{2}$ circles round each of k points, and each circle contains at most $n^{c_s/\log \log n}$ lattice points¹⁾.

Condition (b) excludes at most

$$k \sum_{a=1}^{n^{1/3}} 4 \varphi(a) \frac{n}{a} < c_6 k n^{4/3}$$

points, since a line with slope b/a , $b < a$, $(a, b) = 1$, contains at most n/a lattice points.

Condition (c) excludes at most $\binom{k}{2} n^{2/3}$ points, since there are $\binom{k}{2}$ lines of equidistant points, each of which has slope b/a , $(a, b) = 1$, $|a| \geq n^{1/3}$ and such a line contains at most $n/|a| \leq n^{2/3}$ lattice points.

Hence, so long as

$$\frac{1}{2} k^3 n^{c_s/\log \log n} + c_6 k n^{4/3} + \frac{1}{2} k^2 n^{2/3} < n^2,$$

there remain eligible points, and this is the case if $k \leq n^{2/3-\epsilon}$. The lower bound (4) is thus established.

For the corresponding problem in one dimension, the existence of perfect difference sets [6] shows that for n an even power of a prime,

$$k \geq n^{1/2} + 1,$$

so that generally

$$k > n^{1/2} (1 - \epsilon). \tag{5}$$

On the other hand it is known [2, 5] that

$$k < n^{1/2} + n^{1/4} + 1. \tag{6}$$

In d dimensions, $d \geq 3$, we may replace Landau's theorem by the theorems on sums of three or four squares, giving an upper bound

$$k < c_7 d^{1/2} n, \tag{7}$$

while the corresponding heuristic argument suggests the conjecture

$$(?) \quad k < c_8 d^{2/3} n^{2/3} (\log n)^{1/3}. \tag{8}$$

The construction, with (hyper)spheres and (hyper)planes, corresponding to that given above, yields the same lower bound (4) as before.

One can also ask for configurations containing a *minimum* number of points, determining distinct distances, so that *no* point may be added without duplicating

¹⁾ It is well known that the number of solutions of $n = x^2 + y^2$ is less than or equal to $d(n)$, the number of divisors of n [3] and $d(n) < nc/\log \log n$ by a well known result of WIGERT [3].

a distance. Can this be done with as few as $O(n^{1/2})$ points; or with $O(n^{1/3})$ points in one dimension?

Another open problem [1] is given any n points in the plane (not necessarily lattice points) [or in d dimensions], how many can one select so that the distances which are determined are all distinct? P. ERDÖS and R. K. GUY, Budapest

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Note on a Diophantine Equation

SCHINZEL and SIERPIŃSKI [1] have given the general solution of the diophantine equation

$$(x^2 - 1)(y^2 - 1) = \left[\left(\frac{x - y}{2} \right)^2 - 1 \right]^2,$$

and SZYMICZEK [2] has given the general solution of

$$(x^2 - z^2)(y^2 - z^2) = \left[\left(\frac{y - x}{2} \right)^2 - z^2 \right]^2.$$

The purpose of this paper is to obtain a complete solution of the diophantine equation

$$(x^2 + a)(y^2 + a) = \left[a \left(\frac{y - x}{2b} \right)^2 + b^2 \right]^2, \tag{1}$$

where a and b are any two given integers.

Let $X = x - y$, $Y = x + y$; then $X \equiv Y \pmod{2}$ and (1) becomes

$$b^4 (X^2 + 2XY + Y^2 + 4a)(X^2 - 2XY + Y^2 + 4a) = (aX^2 + 4b^4)^2.$$

This equation reduces to

$$b^4 ((Y^2 - X^2)^2 + 8a(Y^2 - X^2) + 16a^2) = (aX^2 + 4b^4)^2 - 16ab^4X^2$$

and we have

$$b^2 (Y^2 - X^2 + 4a) = \pm (aX^2 - 4b^4).$$