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5.1. Die Fünfecke von  $\Pi_1$  bestehen aus Würfelkanten in den Seitenflächen von  $\Pi$ , daher ist  $\bar{\Pi}_1 = \Pi$ . Auch  $\bar{\Pi}_3$  ist ein Dodekaeder, denn  $\varepsilon_{31}$  ist Verbindungsebene von Würfelkanten in 5 Schnittpunkten III und  $\varepsilon_{31}$  ist zur Seitenfläche 01234 von  $\Pi$  parallel.  $\bar{\Pi}_3$  geht aus  $\Pi$  durch  $(\sqrt{5} - 2)$ -fache Streckung aus  $M$  hervor.

5.2. In allen 9 in Figur 1 enthaltenen Schnittpunkten I', II', III' von Tetraederkanten ist deren Verbindungsebene die Ebene  $\bar{3}\bar{2}\bar{6}\bar{7}\bar{9}$ . Diese Ebene ist zugleich die Ebene  $\varepsilon_{22}$  von Figur 2b, sie enthält die Würfelkanten  $\bar{3}\bar{9}$ ,  $\bar{7}\bar{8}$ ,  $\bar{6}\bar{2}$ , die sich in 3 Ecken II von  $\Pi_2$  schneiden. Daher sind  $\bar{\Pi}_2$ ,  $\bar{\Pi}'_1$ ,  $\bar{\Pi}'_2$ ,  $\bar{\Pi}'_3$  ein und dasselbe Ikosaeder (Mittelpunkt  $M$ , Inkugelradius = Inkugelradius der Tetraeder =  $d/2\sqrt{3}$ ). *Dieses Ikosaeder ist die Durchschnittsmenge der zehn Tetraeder im Dodekaeder.*

5.3. Die Ebenen von  $\bar{\Pi}'_4$  bilden die Seitenflächen der 5 Würfel.  $\bar{\Pi}'_4$  entsteht aus  $\Pi'_4$  durch Polarisieren an der Inkugel der 5 Würfel, ist also ein Rhombentriakontaeder (Mittelpunkt  $M$ , Inkugelradius =  $d/2$ ). *Dieses Rhombentriakontaeder ist die Durchschnittsmenge der fünf Würfel im Dodekaeder.*

FRITZ HOHENBERG, Graz

## Randomly Traversable Graphs

### 1. Introduction

A graph  $G$  is *eulerian* if it possesses a circuit containing all vertices and edges of  $G$ . These graphs are named for LEONHARD EULER [1], who encountered them while giving a solution to the Königsberg Bridge Problem. It is well known that a graph is eulerian if and only if it is connected and each of its vertices is even.

Similar to the eulerian graphs are the traversable graphs. A graph  $G$  is *traversable* if it possesses an open trail containing all vertices and edges of  $G$ . Traversable graphs are characterized (see [2], p. 65) by the properties of being connected and containing exactly two odd vertices. It is an elementary fact that every graph has an even number of odd vertices. A connected graph  $G$  with odd vertices is called *n-traversable* if there exist  $n$  open trails but no fewer which partition the edge set of  $G$ . Hence the 1-traversable graphs and the traversable graphs are identical. It follows (see [2], p. 65) that a connected graph  $G$  is *n*-traversable,  $n \geq 1$ , if and only if  $G$  has exactly  $2n$  odd vertices.

In [3] ORE introduced an interesting class of eulerian graphs. An eulerian graph  $G$  is *randomly eulerian from a vertex  $v$*  of  $G$  if the following procedure always results in an eulerian circuit of  $G$ : Begin a trail at  $v$  by choosing any edge incident with  $v$ . Next (and at each step thereafter), the trail is continued by selecting any edge not already chosen which is adjacent with the edge most recently selected. The process terminates when no such edge is available. Equivalently, a graph  $G$  is randomly eulerian from  $v$  if every trail of  $G$  beginning at  $v$  can be extended to an eulerian circuit of  $G$ .

It is the object of this paper to study eulerian graphs which are randomly eulerian from one or more of their vertices and to extend this concept to traversable graphs and to *n*-traversable graphs in general.

## 2. Fundamental Terminology

In order to make this article self-contained, we present here those fundamental definitions which are most pertinent to our discussion. For basic graph theory terminology we follow [2].

For vertices  $u$  and  $v$  of a graph  $G$ , a  $u$ - $v$  *trail* of  $G$  is an alternating sequence

$$u = u_1, e_1, u_2, e_2, u_3, \dots, u_{n-1}, e_{n-1}, u_n = v \quad (1)$$

of vertices and edges of  $G$ , beginning with  $u$  and ending with  $v$ , such that each edge is incident with the two distinct vertices immediately preceding and following it and such that no edge is repeated. It should be noted that while no edge may be repeated in a trail, vertices may be repeated. Further, we may represent the trail (1) more simply as

$$u = u_1, u_2, u_3, \dots, u_{n-1}, u_n = v, \quad (2)$$

since the edges of the trail are then evident. In general, we assume that every trail contains at least one edge and, therefore, at least two vertices. A  $u$ - $v$  *path*,  $u \neq v$ , is a  $u$ - $v$  trail in which no vertices are repeated.

A graph  $G$  is *connected* if for every two distinct vertices  $u$  and  $v$  of  $G$ , there exists a  $u$ - $v$  trail (or  $u$ - $v$  path) in  $G$ . A maximal connected subgraph of a graph  $G$  is called a *component* of  $G$ .

A  $u$ - $v$  trail is *closed* if  $u = v$ ; otherwise, it is *open*. A closed trail is also referred to as a *circuit*. A circuit in which no vertex is repeated is called a *cycle*.

A circuit containing all edges of a connected graph  $G$  is an *eulerian circuit* of  $G$ , while an open trail containing all edges of  $G$  is an *eulerian trail* of  $G$ .

Finally, the *degree* of a vertex  $v$  in a graph  $G$ , denoted  $\deg v$ , is the number of edges in  $G$  incident with  $v$ ; the vertex  $v$  is *even* or *odd* depending on whether  $\deg v$  is even or odd.

## 3. Randomly Eulerian Graphs

We have already noted that an eulerian graph  $G$  is randomly eulerian from a vertex  $v$  of  $G$  if and only if every trail beginning at  $v$  can be extended to an eulerian circuit of  $G$ . In Figure 1 are shown four eulerian graphs, each of which has six vertices. The graph  $G_0$  is randomly eulerian from no vertices,  $G_1$  is randomly eulerian from exactly one vertex, namely  $u$ ,  $G_2$  is randomly eulerian from the two vertices  $v$  and  $w$ , while  $G_3$  is randomly eulerian from each of its vertices.

ORE [3] showed that an eulerian graph  $G$  is randomly eulerian from a vertex  $v$  of  $G$  if and only if every cycle of  $G$  contains  $v$ . With the aid of this result, it is easy to verify that the graphs of Figure 1 have the indicated properties. Moreover, it follows immediately that an eulerian graph is randomly eulerian from each of its vertices if and only if it is a cycle. We now show that the graphs of Figure 1 represent all possibilities regarding the number of vertices from which an eulerian graph is randomly eulerian.

*Theorem 1.* Let  $G$  be an eulerian graph with  $p$  ( $\geq 3$ ) vertices. Then the number of vertices from which  $G$  is randomly eulerian is 0, 1, 2 or  $p$ .

*Proof.* There is obviously nothing to prove if  $p = 3$ , so we assume  $p \geq 4$ . Suppose the result to be false so that there exists an eulerian graph  $H$  with  $p$  ( $\geq 4$ ) vertices such that  $H$  is randomly eulerian from three vertices, say  $u$ ,  $v$  and  $w$ , but not from all vertices. Hence  $H$  is not itself a cycle.

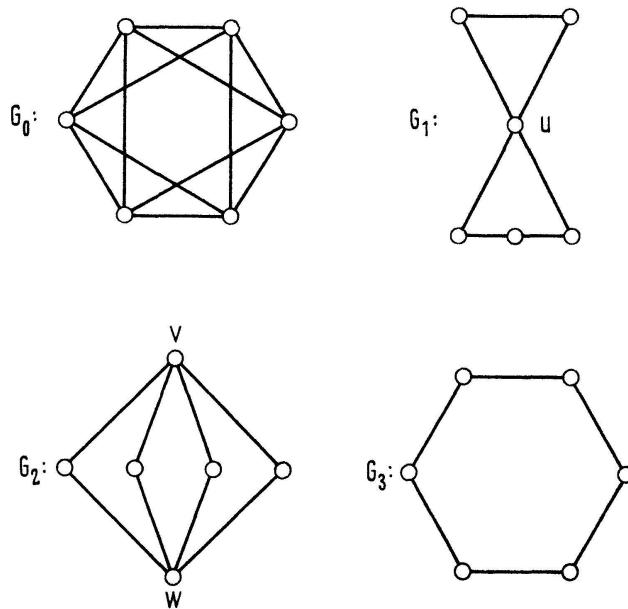


Figure 1

Since  $H$  is randomly eulerian from each of  $u$ ,  $v$  and  $w$ , it follows by Ore's theorem that every cycle of  $H$  contains  $u$ ,  $v$  and  $w$ . Furthermore, there exists a vertex  $x$  from which  $H$  is not randomly eulerian; therefore, not all cycles of  $H$  contain  $x$ . Let  $C_1$  be a cycle not containing  $x$ . Because  $H$  is eulerian, there is a circuit containing  $x$  (namely an eulerian circuit) and therefore a cycle  $C_2$  containing  $x$ . Necessarily,  $u$ ,  $v$  and  $w$  also lie on  $C_2$ . Thus the distinct cycles  $C_1$  and  $C_2$  have at least three vertices in common.

The cycle  $C_2$  determines two paths  $P_1$  and  $P_2$  connecting  $x$  with  $C_1$ . Suppose  $P_1$  is an  $x - x_1$  path while  $P_2$  is an  $x - x_2$  path, where then  $x_i$  is the only vertex of  $C_1$  on  $P_i$ , for  $i = 1, 2$ ; moreover,  $x_1 \neq x_2$ . At least one of  $u$ ,  $v$  and  $w$  is neither  $x_1$  nor  $x_2$ ; suppose  $u$  is such a vertex. Hence  $C_1$  determines two  $x_1 - x_2$  paths, only one of which contains  $u$ ; suppose  $Q$  is the  $x_1 - x_2$  path not containing  $u$ . Hence, if we begin with  $P_1$ , follow  $Q$ , and then proceed from  $x_2$  to  $x$  along  $P_2$ , we have a cycle not containing  $u$ , which produces a contradiction.

#### 4. Randomly Traversable Graphs

We define a traversable graph  $G$  to be *randomly traversable from a vertex  $v$*  if every trail in  $G$  with initial vertex  $v$  can be extended to an eulerian trail of  $G$ . Naturally, such a vertex  $v$  is necessarily an odd vertex of  $G$ , implying that a traversable graph is randomly traversable from at most two of its vertices. Figure 2 shows traversable graphs  $H_0$ ,  $H_1$ ,  $H_2$  such that  $H_k$ ,  $k = 0, 1, 2$ , is randomly traversable from  $k$  of its vertices. A traversable graph  $G$  is said to be *randomly traversable* if it is randomly traversable from both of its odd vertices.

It is possible to characterize traversable graphs which are randomly traversable from a given vertex in much the same way as ORE did for randomly eulerian graphs.

**Theorem 2.** Let  $u$  and  $v$  be the two odd vertices of a traversable graph  $G$ . Then  $G$  is randomly traversable from  $u$  if and only if every cycle of  $G$  contains  $v$ .

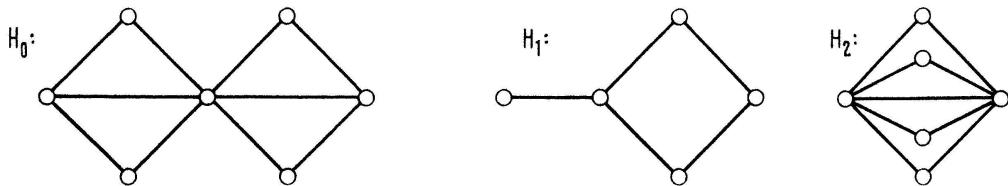


Figure 2

**Proof.** Suppose  $G$  is randomly traversable from  $u$ , and assume  $G$  has a cycle  $C$  not containing  $v$ . Denote by  $H$  the graph obtained by removing the edges of  $C$  from  $G$ . Necessarily, each vertex of  $H$  has the same parity as it does in  $G$ ; therefore,  $u$  and  $v$  are the only two odd vertices of  $H$  and thus belong to the same component  $H_1$  of  $H$ . Hence  $H_1$  is traversable and has a  $u$ - $v$  trail  $P_1$  containing all edges of  $H_1$ . Since  $P_1$  contains all edges of  $G$  incident with  $v$ , the trail  $P_1$  cannot be extended to an eulerian trail of  $G$ , contradicting the fact that  $G$  is randomly traversable from  $u$ .

Conversely, suppose every cycle of  $G$  contains  $v$ , and assume  $G$  is not randomly traversable from  $u$ . Hence there exists a maximal trail  $P$  of  $G$  beginning at  $u$  which cannot be extended to an eulerian  $u$ - $v$  trail. Thus  $P$  is a  $u$ - $v$  trail not containing all edges of  $G$ . By deleting the edges of  $P$  from  $G$ , a nonempty graph  $G'$  results in which every vertex is even and  $v$  is isolated. There exists a nontrivial component  $H'$  of  $G'$ ; thus  $H'$  is eulerian, contains an eulerian circuit, and therefore contains a cycle  $C$ . Since  $C$  does not contain  $v$ , a contradiction results.

**Corollary 2a.** Let  $u$  and  $v$  be the two odd vertices of a traversable graph  $G$ . Then  $G$  is randomly traversable if and only if every cycle of  $G$  contains both  $u$  and  $v$ .

## 5. Randomly $n$ -Traversable Graphs

Just as  $n$ -traversable graphs constitute a generalization of traversable graphs, we now introduce the concept of randomly  $n$ -traversable graphs as a generalization of randomly traversable graphs.

For a graph  $G$ , we denote its edge set by  $E(G)$ . Similarly, the edge set of a trail  $T$  of a graph  $G$  is denoted  $E(T)$ . By  $G - E(T)$  we mean the graph obtained by deleting the edges of  $T$  from  $G$ . A trail  $T$  of a graph  $G$ , having initial vertex  $v$  and terminal vertex  $w$ , is said to be *maximal from  $v$*  if every edge of  $G$  incident with  $w$  belongs to  $T$ .

An  $n$ -traversable graph  $G$  (which necessarily then has  $2n$  odd vertices) is *randomly  $n$ -traversable from an odd vertex  $v$*  if for every sequence  $v_1, v_2, \dots, v_n$  of  $n$  odd vertices of  $G$  for which  $v_1 = v$  and for every  $n$  trails  $T_1, T_2, \dots, T_n$  such that  $T_1$  is maximal from  $v_1$  in  $G$  and  $T_i$  is maximal from  $v_i$  in

$$G - \bigcup_{j=1}^{i-1} E(T_j), \quad i = 2, 3, \dots, n,$$

it follows that  $E(G) = \bigcup_{i=1}^n E(T_i)$ . A graph is *randomly n-traversable* if it is randomly  $n$ -traversable from each of its odd vertices. We note then that the randomly 1-traversable graphs coincide with the randomly traversable graphs. The following theorem gives a necessary condition for a graph to be randomly  $n$ -traversable from one of its odd vertices.

*Theorem 3.* If  $G$  is a graph which is randomly  $n$ -traversable from an odd vertex  $v$ , then every cycle of  $G$  contains an odd vertex other than  $v$ .

*Proof.* Let  $C$  be an arbitrary cycle in  $G$ , and consider  $G - E(C)$ , which has  $2n$  odd vertices. Let  $T_1$  be a trail in  $G - E(C)$  which is maximal from its initial vertex  $v$ , while for  $i = 2, 3, \dots, n$ , let  $T_i$  be a trail maximal from an odd vertex  $v_i$  in  $G - E(C) - \bigcup_{j=1}^{i-1} E(T_j)$ . Each of these trails necessarily terminates in an odd vertex of  $G$ . Either  $T_1$  is not maximal in  $G$  or  $T_i$  is not maximal in  $G - \bigcup_{j=1}^{i-1} E(T_j)$  for some  $i = 2, 3, \dots, n$ ; for if all  $n$  trails are maximal in these respective graphs, then  $\bigcup_{i=1}^n E(T_i) \neq E(G)$ , which contradicts the fact that  $G$  is randomly  $n$ -traversable from  $v$ . Thus the terminal vertex of at least one trail  $T_i$  lies on  $C$  so that  $C$  contains an odd vertex of  $G$  other than  $v$ .

The necessary condition for a graph to be randomly traversable given in Theorem 2 now follows as a corollary to Theorem 3. From Theorem 3 we may now derive a necessary condition for a graph to be randomly  $n$ -traversable.

*Corollary 3a.* If  $G$  is a randomly  $n$ -traversable graph, then every cycle of  $G$  contains at least two odd vertices.

*Proof.* Let  $C$  be a cycle of  $G$ . By Theorem 3,  $C$  contains at least one odd vertex, say  $u$ . By hypothesis,  $G$  is randomly  $n$ -traversable from  $u$  so that, again by Theorem 3,  $C$  contains an odd vertex other than  $u$ , completing the proof.

Next we present a sufficient condition for an  $n$ -traversable graph to be randomly  $n$ -traversable from one of its odd vertices.

*Theorem 4.* Let  $G$  be an  $n$ -traversable graph with odd vertex  $v$ . If every cycle of  $G$  contains at least  $n$  odd vertices other than  $v$ , then  $G$  is randomly  $n$ -traversable from  $v$ .

*Proof.* Let  $v_1, v_2, \dots, v_n$  be  $n$  odd vertices of  $G$ , where  $v_1 = v$ , and let  $T_1, T_2, \dots, T_n$  be  $n$  trails so that  $T_1$  is maximal from  $v_1$  in  $G$  and  $T_i$  is maximal from  $v_i$  in

$$G - \bigcup_{j=1}^{i-1} E(T_j), \quad i = 2, 3, \dots, n.$$

Since  $T_i$ ,  $i = 1, 2, \dots, n$ , is a trail which is maximal from  $v_i$ , it must terminate at a vertex  $w_i$  having degree zero in

$$H_i = G - \bigcup_{j=1}^i E(T_j).$$

Because every vertex which is even in  $G$  is also even in  $H_{i-1}$ ,  $w_i$  is necessarily odd in  $G$ . Furthermore,  $H_i$  has exactly  $2n - 2i$  odd vertices. Hence  $H_n$  has only even vertices.

If  $H_n$  has no edges, then  $E(G) = \bigcup_{j=1}^i E(T_j)$ , which produces the desired result. Suppose,

then, that  $H_n$  has edges. In this case,  $H_n$  contains cycles; thus let  $C$  be a cycle in  $H_n$ . By hypothesis,  $C$  contains at least  $n$  odd vertices of  $G$  other than  $v$ . Since  $G$  has exactly  $2n$  odd vertices,  $C$  must contain a vertex  $w_k$ ,  $1 \leq k \leq n$ . However,  $w_k$  has degree zero in  $H_k$  as well as in  $H_n$ . This produces a contradiction, completing the proof.

The sufficient condition for a graph to be randomly traversable given in Theorem 2 now follows as a corollary to Theorem 4.

The converse of the preceding theorem does not hold, in general. For example, the 2-traversable graph  $G$  of Figure 3 is randomly 2-traversable from  $v$ ; however, the only cycle of  $G$  contains only one odd vertex.

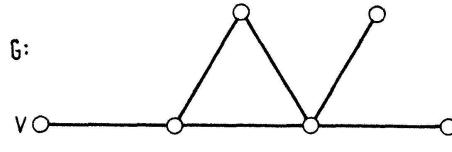


Figure 3

*Corollary 4a.* If every cycle of an  $n$ -traversable graph  $G$  contains at least  $n + 1$  odd vertices, then  $G$  is randomly  $n$ -traversable.

The number of odd vertices in the statement of Corollary 4a cannot be reduced, as the following example shows. Let  $G$  be the graph consisting of a cycle  $C: v_1, v_2, \dots, v_n, v_1$ ,  $n$  additional vertices  $u_1, u_2, \dots, u_n$ , and the edges  $u_i v_i$ ,  $i = 1, 2, \dots, n$ . Figure 4 illustrates the graph  $G$  for the case  $n = 5$ . Although  $G$  is  $n$ -traversable, and the only cycle of  $G$  contains exactly  $n$  odd vertices,  $G$  is not randomly  $n$ -traversable. For example, the  $n$  trails  $v_i, u_i$ ,  $i = 1, 2, \dots, n$  do not partition the edge set of  $G$ .

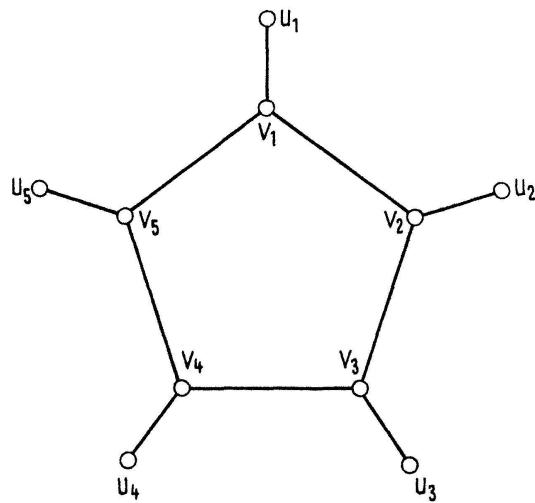


Figure 4

We conclude by verifying the converse of Corollary 4a for the case  $n = 2$ .

*Theorem 5.* If  $G$  is a randomly 2-traversable graph, then every cycle of  $G$  contains at least three odd vertices.

*Proof.* By Corollary 3a, every cycle of  $G$  contains at least two odd vertices. Suppose there exists a cycle  $C$  in  $G$  containing exactly two odd vertices, say  $u$  and  $v$ , and let  $u_1$  and  $v_1$  be the remaining odd vertices of  $G$ . We consider two cases.

*Case 1. There exists a  $u - u_1$  path in  $G - E(C)$  not containing  $v_1$  or a  $u - v_1$  path in  $G - E(C)$  not containing  $u_1$ .* Without loss of generality, we assume the former, denoting the path by  $P$ . The graph  $G - E(C) - E(P)$  has exactly two odd vertices, namely  $v$  and  $v_1$ , which necessarily belong to the same component  $G'$  of  $G - E(C) - E(P)$ . Furthermore, the degree of  $v_1$  is the same in  $G - E(C) - E(P)$  as in  $G$ . Let  $T$  be a  $v - v_1$  eulerian trail in  $G'$ ; the trail  $T$  is therefore maximal from  $v$  in  $G$ . Let  $T_1$  be a maximal trail from  $u$  in  $G - E(C) - E(T)$ , necessarily terminating at  $u_1$ . Then  $T_1$  is also maximal in  $G - E(T)$ . However,  $E(T) \cup E(T_1) \neq E(G)$ , which is contradictory.

*Case 2. There exists no  $u - u_1$  path in  $G - E(C)$ .* If there exists a  $u - v_1$  path in  $G - E(C)$ , then we are in Case 1 and a contradiction results. Hence we may assume that  $G - E(C)$  has a  $u - v$  path  $P$  containing neither  $u_1$  nor  $v_1$ . If  $P$  has a vertex of  $C$  different from  $u$  or  $v$ , then  $G$  has a cycle containing only one odd vertex, namely  $u$ , which is impossible. Now the cycle  $C$  determines two edge-disjoint  $u - v$  paths  $P_1$  and  $P_2$ . Since  $G$  is connected, there exists in  $G$  either a  $u - u_1$  path not containing  $v$  or a  $v - u_1$  path not containing  $u$ ; assume the former, denoting the  $u - u_1$  path by  $P_3$ . We further suppose that  $P_3$  does not contain  $v_1$ ; otherwise, we let  $P_3$  denote the resulting  $u - v_1$  path. The path  $P_3$  has at least one edge which is also an edge of  $C$ ; furthermore,  $P_3$  contains vertices of only one of  $P_1$  and  $P_2$  (with the exception of the vertex  $u$ ), for otherwise  $G$  has a cycle containing only one odd vertex. Suppose  $P_1$  contains a vertex of  $P_3$  different from  $u$  or  $v$  so that  $P_2$  has no such vertex. The  $u - v$  paths  $P$  and  $P_2$  combine to form a cycle  $C_1$  containing  $u$  and  $v$  but neither  $u_1$  nor  $v_1$ . However, a  $u - u_1$  path exists in  $G - E(C_1)$  returning us to Case 1 which yields a contradiction and completes the proof.

GARY CHARTRAND<sup>1)</sup> and ARTHUR T. WHITE, Western Michigan University

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## Über hebbare Unstetigkeiten

Die vorliegende Note ist als Beitrag zur Sammlung pathologischer Beispiele der Analysis gedacht, wie sie etwa in [1] gegeben wird.

Wir betrachten die Menge  $\mathfrak{F}[a, b]$  der auf dem abgeschlossenen Intervall  $[a, b]$  definierten Funktionen, die in jedem Punkt von  $[a, b]$  unstetig sind. Eine solche Funktion ist beispielsweise

$$f(x) = \begin{cases} +1 & \text{für } x \in \mathbb{Q} \\ -1 & \text{für } x \in \mathbb{R} - \mathbb{Q}, \end{cases}$$