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## Line-Coloring of Signed Graphs<sup>1)</sup>

### Introduction

A *signed graph* or *sigraph* is a graph in which some of the lines have been designated as positive and the remaining as negative. Sigraphs have been studied extensively by CARTWRIGHT and HARARY (see [2] and [5]) in their theory of balance. When drawing a sigraph it is customary to indicate positive lines by solid lines and negative lines by dashed lines. Thus, the sigraph  $S$  of Figure 1 has 3 positive and 2 negative lines.

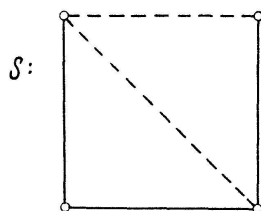


Figure 1

CARTWRIGHT and HARARY [3] have defined a sigraph  $S$  to be *colorable* if it is possible to assign colors to the points of  $S$  so that two points joined by a negative line are colored differently while two points joined by a positive line are colored the same. It was shown in [3] that a sigraph is colorable if and only if it contains no cycle with exactly one negative line. It is the purpose of this paper to define and study line-colorable sigraphs and present some of their properties. In particular, we give a characterization of line-colorable sigraphs and determine the 'line chromatic number' of special classes of sigraphs.

The *chromatic number*  $\chi(S)$  of a colorable sigraph  $S$  is the smallest number of colors needed in a coloring of  $S$ . If one were to regard an ordinary graph  $G$  as a sigraph  $S$  all of whose lines are negative, then  $\chi(G) = \chi(S)$ . Indeed, if  $S$  is a complete colorable sigraph, then the ordinary graph  $G$  obtained by converting all negative lines to ordinary lines and deleting all positive lines has the same chromatic number as  $S$ . Thus, in a certain sense, complete colorable sigraphs and ordinary graphs are related, where negative lines correspond to ordinary lines and positive lines correspond to 'no lines'.

The *line-graph*  $L(G)$  of a graph  $G$  is that graph whose points can be put in one-to-one correspondence with the lines of  $G$  so that two points of  $L(G)$  are adjacent if and only if the corresponding lines of  $G$  are adjacent. In order to propose a natural defini-

<sup>1)</sup> All definitions not given in this article may be found in the books [4, 5].

tion of the 'line-sigraph' of a sigraph, we again consider a complete sigraph  $S$ . Certainly, there must be a one-to-one correspondence between the points of  $L(S)$  and the lines of  $S$ . Since there is a strong resemblance between the negative lines of a sigraph and the lines of an ordinary graph, the sigraph  $R$  of  $S$  induced by its negative lines should have only negative lines in its line-sigraph, while all other lines in  $L(S)$  should be positive. We are thus led to the following definition. The *line-sigraph*  $L(S)$  of a sigraph  $S$  is that sigraph whose points can be put in one-to-one correspondence with the lines of  $S$  in such a way that two points of  $L(S)$  are joined by a negative line if and only if they correspond to two adjacent negative lines of  $S$  and are joined by a positive line if they correspond to some other two adjacent lines of  $S$ .

Since coloring the lines of an ordinary graph is equivalent to coloring the points of its line-graph, it seems natural to make the following definition. A sigraph  $S$  is *line-colorable* if its line-sigraph  $L(S)$  is colorable, i. e., if it is possible to assign colors to the lines of  $S$  so that two adjacent negative lines are colored differently and any other adjacent lines are colored the same.

### A Characterization of Line-Colorable Sigraphs

If  $v$  is a point of a sigraph  $S$ , then the *positive degree*  $\deg^+v$  of  $v$  is the number of positive lines of  $S$  incident with  $v$ . The *negative degree*  $\deg^-v$  of  $v$  is defined analogously. We can now present the principal result of this section.

**Theorem 1.** *A sigraph  $S$  is line-colorable if and only if the following two properties are satisfied:*

- (P1) *There exists no point  $v$  of  $S$  with  $\deg^+v \geq 1$  and  $\deg^-v \geq 2$ ,*
- (P2) *there exists no cycle having exactly two consecutive negative lines.*

*Proof.* We first show the necessity of (P1) and (P2). If a point  $v$  of  $S$  is incident with one positive line and two negative lines, then these 3 lines induce a triangle in  $L(S)$  having exactly one negative line so that  $L(S)$  is not colorable and  $S$  is not line-colorable. Similarly, if  $S$  contains a cycle  $C$  having exactly two consecutive negative lines, then the lines of  $C$  generate a cycle in  $L(S)$  having exactly one negative line, so, again,  $S$  is not line-colorable.

To prove the sufficiency of (P1) and (P2), we employ induction on the number of positive lines in a sigraph. If  $S$  has no positive lines, then  $S$  is certainly line-colorable. Assume that every sigraph having  $n$  positive lines,  $n \geq 0$ , and satisfying (P1) and (P2) is line-colorable. Let  $S$  be a sigraph with  $n + 1$  positive lines having properties (P1) and (P2). The removal of a positive line  $x = uv$  from  $S$  results in a sigraph  $S'$  having  $n$  positive lines. Since  $S'$  obviously satisfies (P1) and (P2),  $S'$  is line-colorable by the inductive hypothesis.

Assume that  $x$  is a bridge. If there are no lines other than  $x$  incident with  $u$  or  $v$ , then  $x$  may be colored arbitrarily in  $S$ . Otherwise, if necessary, the colors used for the component in  $S'$  containing  $u$  may be easily changed or permuted so that all lines incident with  $u$  are colored the same as those incident with  $v$ . Hence,  $x$  may be given that color thereby showing that  $S$  is line-colorable.

Suppose, on the other hand, that  $x$  is not a bridge. Then  $x$  belongs to a cycle  $C$  whose line-sequence is  $x, x_1, x_2, \dots, x_n = x$ . If, in a line-coloring of  $S'$ , the colors of  $x_1$  and  $x_{n-1}$  are the same, say  $\alpha$ , implying that all lines incident with  $u$  or  $v$  have color  $\alpha$ ,

then  $x$  may be replaced and colored  $\alpha$  also. If  $x_1$  and  $x_{n-1}$  are colored differently, then there must exist at least 2 consecutive negative lines in  $C$ . Thus, let  $i$  be the least integer such that  $x_i$  and  $x_{i+1}$  are negative, and let  $j$  be the largest integer such that  $x_{j-1}$  and  $x_j$  are negative. By (P2),  $x_i$  and  $x_j$  are not adjacent. Let  $\beta$  be a color not used in coloring  $S'$ , and let  $\alpha_k$ ,  $k = i, j$ , be the color of  $x_k$ . Also, let  $W_k$  be the set consisting of  $x_k$  and all lines colored  $\alpha_k$  which lie on a common path with  $x_k$ . No negative line of  $W_i$  is adjacent to a negative line of  $W_j$ , for, otherwise, there would exist a cycle with exactly two consecutive negative lines, contradicting (P2). Now if the colors of the lines in  $W_i \cup W_j$  are changed to  $\beta$ , then by replacing  $x$  and coloring it  $\beta$ , we have a line-coloring for  $S$ .

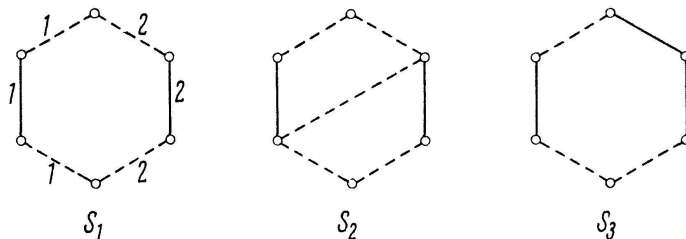


Figure 2

In Figure 2,  $S_1$  is line-colorable and can be line-colored as indicated,  $S_2$  is not line-colorable since (P1) does not hold, while  $S_3$  is not line-colorable since (P2) does not hold.

### The Line-Chromatic Number of a Sigraph

The *line-chromatic number*  $\chi'(S)$  of a line-colorable sigraph  $S$  is the minimum number of colors required in a line-coloring of  $S$ . Clearly,  $\chi'(S) = \chi(L(S))$ .

Now we present formulas for special classes of line-colorable sigraphs, beginning with trees. Since a tree contains no cycles, by (P1) a tree is line-colorable if and only if it has no point  $v$  with  $\deg^+ v \geq 1$  and  $\deg^- v \geq 2$ .

**Theorem 2.** For any line-colorable signed tree  $T$ ,  $\chi'(T) = \max \deg^- v$  if  $T$  has negative lines and  $\chi'(T) = 1$  otherwise.

The proof of this theorem is straightforward and will be omitted.

A *complete sigraph*  $S_p$  has every pair of its points joined by either a positive or negative line. For  $p \geq 2$ ,  $S_p$  is obviously line-colorable if it has no adjacent negative lines, in which case  $\chi'(S_p) = 1$ . Should  $S_p$  possess adjacent negative lines, then in order to satisfy (P1), there must be a point incident only with negative lines, but then to satisfy (P2) in addition, all lines must be negative. However, in this case, as we have seen,  $\chi'(S_p)$  has the same value as the line-chromatic number of the ordinary complete graph  $K_p$ , which is  $2\{p/2\} - 1$ , as noted in [1]. We summarize this below.

**Theorem 3.** Let  $S_p$  be a line-colorable complete sigraph with  $p \geq 2$  points. Then

$$\chi'(S_p) = \begin{cases} 1 & \text{if } S_p \text{ has no adjacent negative lines.} \\ 2\{p/2\} - 1 & \text{if } S_p \text{ is all-negative.} \end{cases}$$

We now investigate *complete bipartite sigraphs* or complete sibigraphs  $S_{m,n}$  whose point set  $V$ , where  $|V| = m + n$ , can be partitioned into subsets  $V_1$  and  $V_2$ , with  $|V_1| = m$  and  $|V_2| = n$ , such that every point of  $V_1$  is joined to a point of  $V_2$  by either a positive or negative line but no two points of the same subset  $V_i$  are adjacent.

In order to determine which of the sigraphs  $S_{m,n}$  are line-colorable, we first consider the case  $m \geq n \geq 3$ . Again, if no two negative lines are adjacent,  $S_{m,n}$  is line-colorable, and, in fact,  $\chi'(S_{m,n}) = 1$ . Otherwise,  $S_{m,n}$  has adjacent negative lines and in order to be line-colorable and thereby satisfy (P1), it must have a point  $u_1$  incident only with negative lines. If all other lines were positive, then there would exist a cycle (for example,  $u_1 v_1 u_2 v_2 u_1$ ; see Figure 3a) having exactly two consecutive negative lines. Hence,  $S_{m,n}$  must have at least one more negative line, say at  $v_1$ , but then all lines at  $v_1$  are negative (see Figure 3b). However, if all lines at  $u_1$  and  $v_1$  are negative, then  $S_{m,n}$  is all-negative, for otherwise any positive line  $u_i v_j$  implies the existence of another positive line  $u_i v_k$ , which would produce the cycle  $u_1 v_j u_i v_k u_1$  having exactly two consecutive negative lines. Therefore, if  $S_{m,n}$ ,  $m \geq n \geq 3$ , is to be line-colorable and have adjacent negative lines, it has only negative lines. In this case,  $\chi'(S_{m,n}) = \max(m, n)$  (see König [6], p.171).

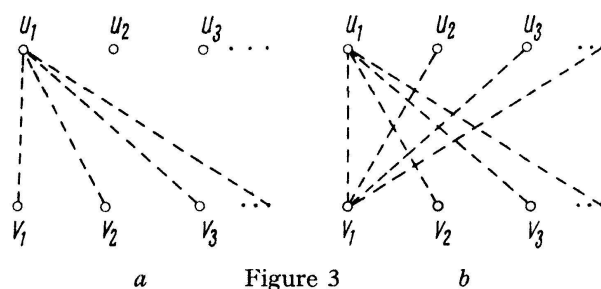


Figure 3

For  $S_{m,2}$ ,  $m \geq 3$  and  $S_{m,1}$ ,  $m \geq 1$ , the situation can be handled similarly to  $S_{m,n}$ ,  $m \geq n \geq 3$ , and identical results are obtained. This leaves the sigraph  $S_{2,2}$  to consider. If  $S_{2,2}$  contains adjacent negative lines but not all negative lines, then the only line-colorable sigraph has 3 negative lines in which case its line-chromatic number is easily seen to be 2. These results are stated in the following theorem.

**Theorem 4.** *A complete sigraph  $S_{m,n}$  is line-colorable if and only if*

- (1) *it has no two adjacent negative lines,*
- (2) *it has only negative lines, or*
- (3)  *$m = n = 2$  and it has 3 negatives lines.*

If  $S_{m,n}$  is all-positive, then  $\chi'(S_{m,n}) = 1$ , while if  $S_{m,n}$  is line-colorable but not all-positive, then  $\chi'(S_{m,n})$  is the maximum negative degree.

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