Zeitschrift: Elemente der Mathematik
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 24 (1969)
Heft: 3

Artikel: Line-coloring of signed graphs
Autor: Behzad, M.
DOI: https://doi.org/10.5169/seals-26645

Nutzungsbedingungen
Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Mehr erfahren

Conditions d'utilisation
L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. En savoir plus

Terms of use
The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. Find out more

Download PDF: 13.04.2026

ETH-Bibliothek Zürich, E-Periodica, https://www.e-periodica.ch

ELEMENTE DER MATHEMATIK
Revue de mathématiques élémentaires – Rivista di matematica elementare
Zeitschrift zur Pflege der Mathematik und zur Förderung des mathematisch-physikalischen Unterrichts
Publiziert mit Unterstützung des Schweizerischen Nationalfonds zur Förderung der wissenschaftlichen Forschung
El. Math. Band 24 Heft 3 Seiten 49–72 10. Mai 1969

Line-Coloring of Signed Graphs[1])

Introduction

A *signed graph* or *sigraph* is a graph in which some of the lines have been designated as positive and the remaining as negative. Sigraphs have been studied extensively by CARTWRIGHT and HARARY (see [2] and [5]) in their theory of balance. When drawing a sigraph it is customary to indicate positive lines by solid lines and negative lines by dashed lines. Thus, the sigraph S of Figure 1 has 3 positive and 2 negative lines.

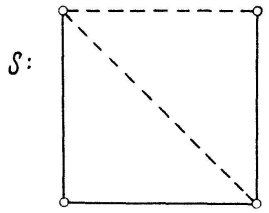


Figure 1

CARTWRIGHT and HARARY [3] have defined a sigraph S to be *colorable* if it is possible to assign colors to the points of S so that two points joined by a negative line are colored differently while two points joined by a positive line are colored the same. It was shown in [3] that a sigraph is colorable if and only if it contains no cycle with exactly one negative line. It is the purpose of this paper to define and study line-colorable sigraphs and present some of their properties. In particular, we give a characterization of line-colorable sigraphs and determine the ' line chromatic number ' of special classes of sigraphs.

The *chromatic number* $\chi(S)$ of a colorable sigraph S is the smallest number of colors needed in a coloring of S. If one were to regard an ordinary graph G as a sigraph S all of whose lines are negative, then $\chi(G) = \chi(S)$. Indeed, if S is a complete colorable sigraph, then the ordinary graph G obtained by converting all negative lines to ordinary lines and deleting all positive lines has the same chromatic number as S. Thus, in a certain sense, complete colorable sigraphs and ordinary graphs are related, where negative lines correspond to ordinary lines and positive lines correspond to ' no lines '.

The *line-graph* $L(G)$ of a graph G is that graph whose points can be put in one-to-one correspondence with the lines of G so that two points of $L(G)$ are adjacent if and only if the corresponding lines of G are adjacent. In order to propose a natural defini-

[1]) All definitions not given in this article may be found in the books [4, 5].

tion of the ' line-sigraph' of a sigraph, we again consider a complete sigraph S. Certainly, there must be a one-to-one correspondence between the points of $L(S)$ and the lines of S. Since there is a strong resemblance between the negative lines of a sigraph and the lines of an ordinary graph, the sigraph R of S induced by its negative lines should have only negative lines in its line-sigraph, while all other lines in $L(S)$ should be positive. We are thus led to the following definition. The *line-sigraph* $L(S)$ of a sigraph S is that sigraph whose points can be put in one-to-one correspondence with the lines of S in such a way that two points of $L(S)$ are joined by a negative line if and only if they correspond to two adjacent negative lines of S and are joined by a positive line if they correspond to some other two adjacent lines of S.

Since coloring the lines of an ordinary graph is equivalent to coloring the points of its line-graph, it seems natural to make the following definition. A sigraph S is *line-colorable* if its line-sigraph $L(S)$ is colorable, i.e., if it is possible to assign colors to the lines of S so that two adjacent negative lines are colored differently and any other adjacent lines are colored the same.

A Characterization of Line-Colorable Sigraphs

If v is a point of a sigraph S, then the *positive degree* $\deg^+ v$ of v is the number of positive lines of S incident with v. The *negative degree* $\deg^- v$ of v is defined analogously. We can now present the principal result of this section.

Theorem 1. *A sigraph S is line-colorable if and only if the following two properties are satisfied:*

(P1) *There exists no point v of S with $\deg^+ v \geq 1$ and $\deg^- v \geq 2$,*

(P2) *there exists no cycle having exactly two consecutive negative linies.*

Proof. We first show the necessity of (P1) and (P2). If a point v of S is incident with one positive line and two negative lines, then these 3 lines induce a triangle in $L(S)$ having exactly one negative line so that $L(S)$ is not colorable and S is not line-colorable. Similarly, if S contains a cycle C having exactly two consecutive negative lines, then the lines of C generate a cycle in $L(S)$ having exactly one negative line, so, again, S is not line-colorable.

To prove the sufficiency of (P1) and (P2), we employ induction on the number of positive lines in a sigraph. If S has no positive lines, then S is certainly line-colorable. Assume that every sigraph having n positive lines, $n \geq 0$, and satisfying (P1) and (P2) is line-colorable. Let S be a sigraph with $n + 1$ positive lines having properties (P1) and (P2). The removal of a positive line $x = u\,v$ from S results in a sigraph S' having n positive lines. Since S' obviously satisfies (P1) and (P2), S' is line-colorable by the inductive hypothesis.

Assume that x is a bridge. If there are no lines other than x incident with u or v, then x may be colored arbitrarily in S. Otherwise, if necessary, the colors used for the component in S' containing u may be easily changed or permuted so that all lines incident with u are colored the same as those incident with v. Hence, x may be given that color thereby showing that S is line-colorable.

Suppose, on the other hand, that x is not a bridge. Then x belongs to a cycle C whose line-sequence is $x, x_1, x_2, \ldots, x_n = x$. If, in a line-coloring of S', the colors of x_1 and x_{n-1} are the same, say α, implying that all lines incident with u or v have color α,

then x may be replaced and colored α also. If x_1 and x_{n-1} are colored differently, then there must exist at least 2 consecutive negative lines in C. Thus, let i be the least integer such that x_i and x_{i+1} are negative, and let j be the largest integer such that x_{j-1} and x_j are negative. By (P2), x_i and x_j are not adjacent. Let β be a color not used in coloring S', and let α_k, $k = i, j$, be the color of x_k. Also, let W_k be the set consisting of x_k and all lines colored α_k which lie on a common path with x_k. No negative line of W_i is adjacent to a negative line of W_j, for, otherwise, there would exist a cycle with exactly two consecutive negative lines, contradicting (P2). Now if the colors of the lines in $W_i \cup W_j$ are changed to β, then by replacing x and coloring it β, we have a line-coloring for S.

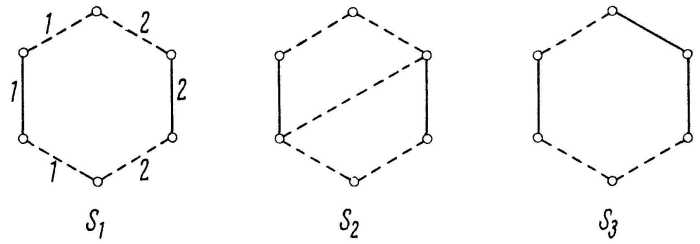


Figure 2

In Figure 2, S_1 is line-colorable and can be line-colored as indicated, S_2 is not line-colorable since (P1) does not hold, while S_3 is not line-colorable since (P2) does not hold.

The Line-Chromatic Number of a Sigraph

The *line-chromatic number* $\chi'(S)$ of a line-colorable sigraph S is the minimum number of colors required in a line-coloring of S. Clearly, $\chi'(S) = \chi(L(S))$.

Now we present formulas for special classes of line-colorable sigraphs, beginning with trees. Since a tree contains no cycles, by (P1) a tree is line-colorable if and only if it has no point v with $\deg^+ v \geq 1$ and $\deg^- v \geq 2$.

Theorem 2. *For any line-colorable signed tree T, $\chi'(T) = \max \deg^- v$ if T has negative lines and $\chi'(T) = 1$ otherwise.*

The proof of this theorem is straightforward and will be omitted.

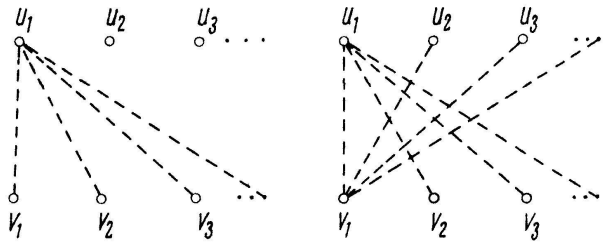
A *complete sigraph* S_p has every pair of its points joined by either a positive or negative line. For $p \geq 2$, S_p is obviously line-colorable if it has no adjacent negative lines, in which case $\chi'(S_p) = 1$. Should S_p possess adjacent negative lines, then in order to satisfy (P1), there must be a point incident only with negative lines, but then to satisfy (P2) in addition, all lines must be negative. However, in this case, as we have seen, $\chi'(S_p)$ has the same value as the line-chromatic number of the ordinary complete graph K_p, which is $2\{p/2\} - 1$, as noted in [1]. We summarize this below.

Theorem 3. *Let S_p be a line-colorable complete sigraph with $p \geqq 2$ points. Then*

$$\chi'(S_p) = \begin{cases} 1 \text{ if } S_p \text{ has no adjacent negative lines.} \\ 2\{p/2\} - 1 \text{ if } S_p \text{ is all-negative.} \end{cases}$$

We now investigate *complete bipartite sigraphs* or complete sibigraphs $S_{m,n}$ whose point set V, where $|V| = m + n$, can be partitioned into subsets V_1 and V_2, with $|V_1| = m$ and $|V_2| = n$, such that every point of V_1 is joined to a point of V_2 by either a positive or negative line but no two points of the same subset V_i are adjacent.

In order to determine which of the sigraphs $S_{m,n}$ are line-colorable, we first consider the case $m \geq n \geq 3$. Again, if no two negative lines are adjacent, $S_{m,n}$ is line-colorable, and, in fact, $\chi'(S_{m,n}) = 1$. Otherwise, $S_{m,n}$ has adjacent negative lines and in order to be line-colorable and thereby satisfy (P1), it must have a point u_1 incident only with negative lines. If all other lines were positive, then there would exist a cycle (for example, $u_1 v_1 u_2 v_2 u_1$; see Figure 3*a*) having exactly two consecutive negative lines. Hence, $S_{m,n}$ must have at least one more negative line, say at v_1, but then all lines at v_1 are negative (see Figure 3*b*). However, if all lines at u_1 and v_1 are negative, then $S_{m,n}$ is all-negative, for otherwise any positive line $u_i v_j$ implies the existence of another positive line $u_i v_k$, which would produce the cycle $u_1 v_j u_i v_k u_1$ having exactly two consecutive negative lines. Therefore, if $S_{m,n}$, $m \geq n \geq 3$, is to be line-colorable and have adjacent negative lines, it has only negative lines. In this case, $\chi'(S_{m,n}) = \max(m, n)$ (see König [6], p.171).



a Figure 3 *b*

For $S_{m,2}$, $m \geq 3$ and $S_{m,1}$, $m \geq 1$, the situation can be handled similarly to $S_{m,n}$, $m \geq n \geq 3$, and identical results are obtained. This leaves the sigraph $S_{2,2}$ to consider. If $S_{2,2}$ contains adjacent negative lines but not all negative lines, then the only line-colorable sigraph has 3 negative lines in which case its line-chromatic number is easily seen to be 2. These results are stated in the following theorem.

Theorem 4. *A complete sibigraph $S_{m,n}$ is line-colorable if and only if*

(1) *it has no two adjacent negative lines,*

(2) *it has only negative lines, or*

(3) *$m = n = 2$ and it has 3 negatives lines.*

If $S_{m,n}$ is all-positive, then $\chi'(S_{m,n}) = 1$, while if $S_{m,n}$ is line-colorable but not all-positive, then $\chi'(S_{m,n})$ is the maximum negative degree.

M. BEHZAD, Pahlavi University, Iran and G. CHARTRAND[2])
University of Michigan and Western Michigan University

REFERENCES

[1] M. BEHZAD, G. CHARTRAND, and J. K. COOPER, Jr., *The Color Numbers of Complete Graphs*, J. London Math. Soc. *42*, 226–228 (1967).

[2] D. CARTWRIGHT and F. HARARY, *Structural Balance: a Generalization of Heider's Ttheory*, Psychol. Rev. *63*, 277–293 (1956).

[3] D. CARTWRIGHT and F. HARARY, *Coloring of Signed Graphs*, El. Math., to appear.

[4] F. HARARY, *A seminar in graph theory* (New York 1967), to appear.

[5] F. HARARY, R. Z. NORMAN, and D. CARTWRIGHT, *Structural Models: an Introduction to the Theory of Directed Graphs* (New York 1965).

[6] D. KÖNIG, *Theorie der endlichen und unendlichen Graphen* (Leipzig 1936).

[2]) Research supported by grants from the U.S. Air Force Office of Scientific Research and the National Institute of Mental Health, grant MH-10834.