On the coloring of signed graphs

Autor(en): **Cartwright, D. / Harary, F.**

Objekttyp: **Article**

Zeitschrift: **Elemente der Mathematik**

Band (Jahr): **23 (1968)**

Heft 4

PDF erstellt am: **27.04.2024**

Persistenter Link: https://doi.org/10.5169/seals-26032

LITERATURVERZEICHNIS

[1] BEREIS, R., *Über das Raumbild eines ebenen Zwanglaufes (kinematische Abbildung von Blaschke und Grünwald)*, Wiss. Z. TU Dresden *13*, 7–16 (1964). – Hier findet sich auch weitere Literatur über Netzprojektion, z.B. [2] und [4].
[2] BEREIS, R. und H. BRAUNER, *Schraubung und Netzprojektion*, El. Math. *12*, 33–40 (1957).
[3] BEREIS, R. und W. D. KLIX, *Parabolische Netzprojektion*, Wiss. Z. TU Dresden *15*, 453–458 (1966).
[4] ECKHART, L., *Konstruktive Abbildungsverfahren* (Wien 1926).
[5] HERMES, O., *Über homologe Tetraeder*, Crelles Journal *56*, 222 (1859).
[6] SCHUR, F., *Über besondere Lagen zweier Tetraeder*, Math. Ann. *19*, 429–432 (1881).
[7] STRUBECKER, K., *Geometrie in einer isotropen Ebene*, Math.-nat. Unterr. *15* (1962).
[8] STRUBECKER, K., *Über Parabeln 2. bis 4. Ordnung*, Praxis d. Math. *4* (1962).
[9] STRUBECKER, K., *Über eine Kreisfigur*, Crelles Journal *169*, 79–86 (1933).

On the Coloring of Signed Graphs[1])

A *graph* G consists of a finite set of points $V(G)$ together with a prescribed subset of the collection of all lines, i.e., unordered pairs of distinct points. A *signed graph* S is obtained from a graph G when each line of G is designated either positive or negative. An *n-coloring of* G is a partition of the point set $V(G)$ into n subsets (called color sets) such that every two points joined by a line are in different color sets. An *n-coloring of* S is a partition of $V(S)$ into n subsets such that (1) every two points joined by a negative line are in different color sets and (2) every two points joined by a positive line are in the same color set. We say that S has a *coloring*, or is *colorable*, if it has an n-coloring for some n. It follows immediately from these definitions that if a signed graph S has only negative lines, the problem of coloring S is the same as that of coloring a graph. If, however, S has some positive lines, it may not be colorable. We characterize colorable signed graphs, and relate them to complete colorings of graphs.

Colorability

Let S^+ be the spanning subgraph obtained by removing all negative lines from S. By a *component* of a graph we mean a maximal connected subgraph. The *positive components* of S are the components of S^+. It follows from this definition that two distinct points of S are in the same positive component if and only if they are joined by a path consisting entirely of positive lines (called an *all-positive path*). Clearly, the positive components of S partition $V(S)$ into subsets such that each positive line joins two points in the same subset, and S has exactly one such partitioning.

We now present two equivalent conditions for a signed graph to be colorable. The equivalence of statements (1) and (3) of the theorem is given in [2].

Theorem 1. The following statements are equivalent for any signed graph S.

(1) S has a coloring.
(2) S has no negative line joining two points in the same positive component.
(3) S has no cycle with exactly one negative line.

---

[1]) Work reported here was supported by Grant MH 10834 from the National Institute of Mental Health. We wish to thank Dr. GARY CHARTRAND for the formulation of Theorem 4.

Equivalence of (1) *and* (2). If S has a coloring, any two points in the same positive component are in the same color set, for they are joined by an all-positive path. Since no two points of the same color set are joined by a negative line, none in the same positive component are. Now, if S has no negative line joining two points of the same positive component, the partition of $V(S)$ by the positive components of S satisfies the definition of a coloring.

Equivalence of (2) *and* (3). By definition, a cycle with exactly one negative line consists of an all-positive path joining two points v_i and v_j together with a negative line $v_i v_j$. The equivalence of (2) and (3) follows immediately from the observation that two points are joined by an all-positive path if and only if they are in the same positive component.

This theorem is illustrated in Figure 1 which shows a colorable signed graph S (in which negative lines are represented by dashes). Its three positive components are evident in S^+, and its point set $V(S)$ can be partitioned into the color sets $\{v_1, v_2\}$, $\{v_3\}$, $\{v_4, v_5\}$. Clearly, S has no negative line joining two points of the same positive component nor does it have a cycle with exactly one negative line.

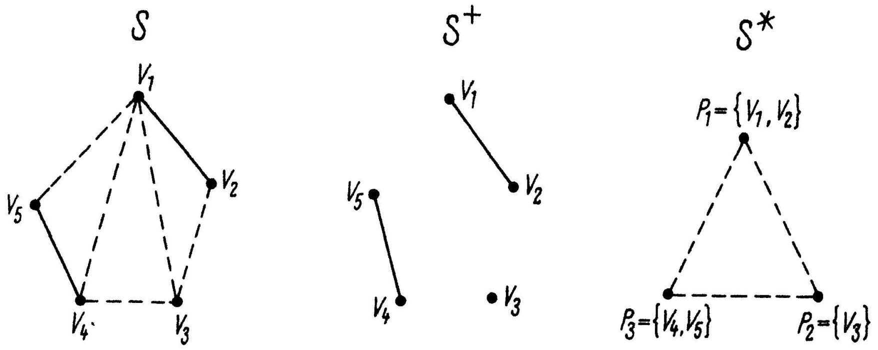

Figure 1

As an interesting and immediate special case, we have a criterion for colorability of a complete signed graph.

Corollary 1a. A complete signed graph S has a coloring if and only if S has no 3-cycle with exactly one negative line.

The *condensation of S by its positive components*, denoted S^*, is the signed graph whose points are the subsets $P_1, P_2, \ldots, P_n$ determined by the positive components of S and whose lines are determined as follows: there is a line joining points P_i and P_j of the new graph if and only if there is at least one line joining a point of P_i and a point of P_j. The construction of S^* from S is illustrated in Figure 1. It is understood that S^* may contain loops. Specifically, a point P_i of S^* will have a loop if and only if there are in S two points of P_i joined by a negative line. It follows immediately from Theorem 1 that S is colorable if and only if S^* contains no loops.

From the construction of S^* it is clear that it has only negative lines. Hence, all results on coloring a graph G apply to coloring S^*, provided, of course, that S^* has no loops.

Balanced Signed Graphs

In an attempt to formalize a psychological theory proposed by HEIDER [5], we defined a signed graph S as *balanced* if every cycle has an even number of negative lines [1, 4]. We then showed that S is balanced if and only if $V(S)$ can be partitioned into two subsets V_1 and V_2 such that every positive line joins two points in the same subset and every negative line joins a point of V_1 with one of V_2. Clearly, in terms of our present terminology V_1 and V_2 are color sets. Thus, S is balanced if and only if it has a 2-coloring (is bicolorable). Now it is readily apparent from the definition of balance that S is balanced if and only if S^* has no odd cycles. Since the problem of coloring S^* is the same as that of coloring an ordinary graph G, our theorem turns out to be the same as the characterization of bicolorable graphs, first advanced by KÖNIG [6]:

G is bicolorable if and only if it has no odd cycles.

To compare bicolorable graphs with 3-colorable graphs, we note that a bicolorable graph can have cycles only of even lengths while a 3-colorable graph may have cycles of any length $n \geqslant 3$. It remains a fiendish unsolved problem to characterize n-colorable graphs for $n > 2$; even the case $n = 3$ does not appear to be easy.

Complete Colorings and Unique Colorings

The *chromatic number* $\varkappa(G)$ of a graph G is the smallest n for which G has an n-coloring. The chromatic number $\varkappa(S)$ of a colorable signed graph S is defined similarly. By the definition of coloring S and the construction of S^* it follows that $\varkappa(S) = \varkappa(S^*)$. Since all results on coloring a graph G apply to coloring S^*, theorems about $\varkappa(G)$ are applicable to $\varkappa(S)$. In a *complete coloring* of G or S, for any two colors, there is a line joining a pair of points with these colors. It is easy to see that every $\varkappa(G)$-coloring, and hence every $\varkappa(S)$-coloring, is complete. The following theorem for graphs, given in [3], also holds for signed graphs.

Theorem 2. If G has a complete n-coloring and $\varkappa(G) < t < n$, then G also has a complete t-coloring.

We say that S has a *unique coloring* if there is only one partition of $V(S)$ into color sets. The next theorem gives a criterion for a signed graph to have a unique coloring.

Theorem 3. Let S be a signed graph with a coloring. This coloring is unique if and only if S^* is complete.

Proof of necessity. By hypothesis, S has a unique coloring. If in S^* two points are not joined by a negative line, then in S no two points from two corresponding positive components are joined by a negative line. Thus, the two point sets determined by these components can be assigned to the same color set. But these two point sets can also be assigned to different color sets since there is no positive line joining two points in different positive components. The assumption that S has a unique coloring is thus contradicted.

Proof of sufficiency. Consider a coloring of S in which each set of points determined by a positive component is assigned to a distinct color set. Clearly, there can be no other coloring in which two points from the same positive component are assigned to different color sets. The only remaining coloring is one in which the point sets of two positive components are assigned to the same color set. But since S^* is complete, for every two positive components in S there is a negative line joining a point from each. Hence, this coloring is impossible, and the given coloring is unique.

Figure 1 illustrates this theorem. It can be readily seen that S has a unique 3-coloring. And in keeping with the theorem, S^* is a complete signed graph with three points.

Theorem 3 may be rephrased in terms of color sets. A colorable signed graph S has a unique coloring if and only if the points in each color set induce a positive component and the points in any two color sets induce a connected subgraph.

Corollary 3a. Among all signed graphs with p points and a unique coloring into n color sets, the minimum number of positive lines is $p - n$, and the minimum number of negative lines is $\binom{n}{2}$.

Proof. For S to have a minimum number of positive lines, each positive component P_j must be a tree. Let p_j be the number of points in P_j so that the tree P_j has $p_j - 1$ lines. On summing from $j = 1$ to n, we find that the number of positive lines in S is $\sum (p_j - 1) = p - n$.

On the other hand, S has a minimum number of negative lines when every two positive components P_i and P_j are joined by exactly one negative line so that there are $\binom{n}{2}$ negative lines in all.

Clearly, the sum of these two expressions gives the smallest possible number of lines among all signed graphs with p points and a unique n-coloring, namely $p + n\,(n - 3)/2$.

We turn now to the concept of a uniquely colorable graph G. It is convenient to distinguish between complete and noncomplete graphs. If $G = K_p$, the complete graph with p points, then $\varkappa(G) = p$ and G has only one partition into color sets. If G is not complete, then it always has a unique partition into p color sets, one point in each. Hence we say that a graph G has a *unique coloring* if (1) G is complete or (2) G is not complete and there exists a unique partition of $V(G)$ into n color sets, where $n < p$. Note that *unique coloring* has been defined differently for graphs and for signed graphs. While there is no characterization of uniquely colorable graphs, a necessary condition is known which resembles Theorem 3 for signed graphs.

Theorem 4. If G has a unique coloring into n color sets, then the subgraph induced by the union of any two color sets is connected.

Proof. It is given that G has a unique coloring into n color sets. Assume that the subgraph induced by the union of two of these, V_i and V_j, is not connected, and consider a component G' of this subgraph whose point set is denoted V'. Now every line joining a point v_i in $V_i \cap V'$ and a point v_j in $V_j \cap V'$ must lie in G'. Thus, we may assign all points of $V_i \cap V'$ to V_j and all points of $V_j \cap V'$ to V_i to obtain another n-coloring of G, which is a contradiction.

By Theorem 4, every unique coloring of G is complete. That the converse of Theorem 4 is not true is shown by the 3-colorable graph in Figure 2. One coloring of this graph has as color sets:

$$V_1 = \{v_1, v_5\}, \quad V_2 = \{v_2, v_4\}, \quad V_3 = \{v_3, v_6\}.$$

It is readily seen that the subgraph induced by any two of these color sets is connected. But there is another coloring with color sets:

$$V_1 = \{v_1, v_4\}, \quad V_2 = \{v_2, v_6\}, \quad V_3 = \{v_3, v_5\}.$$

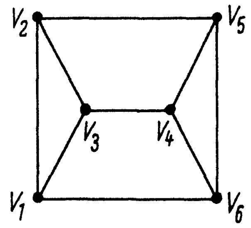

Figure 2

Corollary 4a. If G has a unique coloring into n color sets, then $n = \varkappa(G)$.

Proof. By hypothesis, G has a unique coloring into n color sets. Assume that $\varkappa(G) < n$. Then there exists a partition of $V(G)$ into $\varkappa(G)$ color sets. However, this partition has more than one refinement into n color sets, contrary to the hypothesis.

We conclude with a corollary of Theorem 4 that is analogous to Corollary 3a.

Corollary 4b. Among all graphs with p points and a unique coloring into n color sets, the minimum number of lines is $p\,(n-1) - \binom{n}{2}$.

Proof. Among all graphs with p points and a unique coloring into n color sets, a graph G has a minimum number q of lines if every pair of color sets induces a tree T_i. Let p_i be the number of points in T_i. Since the number of lines in T_i is $p_i - 1$, $q = \sum (p_i - 1)$. It can readily be seen that $\sum p_i = p\,(n-1)$. Since there are $\binom{n}{2}$ trees induced by pairs of color sets,

$$q = p\,(n-1) - \binom{n}{2} = \frac{(2\,p - n)\,(n-1)}{2}.$$

To show that such a graph exists, we construct one with $n - 1$ color sets, each containing a single point, and another color set containing $p - n + 1$ points. We join each pair of points from different singleton color sets, obtaining $\binom{n-1}{2}$ lines. We then join each point of every singleton color set to every point of the remaining color set and obtain $(p - n + 1)\,(n - 1)$ lines. The total number of lines in the graph is therefore

$$\frac{(n-1)\,(n-2)}{2} + (p - n + 1)\,(n-1) = \frac{(2\,p - n)\,(n-1)}{2}.$$

D. CARTWRIGHT and F. HARARY, Univ. of Michigan, Ann Arbor, USA

REFERENCES

[1] CARTWRIGHT, D. and F. HARARY, *Structural Balance: a Generalization of Heider's Theory*, Psychol. Review *63*, 277–293 (1956).
[2] DAVIS, J. A., *Clustering and Structural Balance in Graphs*, Human Relations *20*, 181-187 (1967).
[3] HARARY, F., S. HEDETNIEMI and G. PRINS, *An Interpolation Theorem for Graphical Homomorphisms*, to appear in Portugaliae Math.
[4] HARARY, F., R. Z. NORMAN and D. CARTWRIGHT, *Structural Models: an Introduction to the Theory of Directed Graphs* (Wiley, New York 1965).
[5] HEIDER, F., *The Psychology of Interpersonal Relations* (Wiley, New York 1958).
[6] KÖNIG, D., *Theorie der endlichen und unendlichen Graphen* (Leipzig 1936), reprinted Chelsea, New York 1950.