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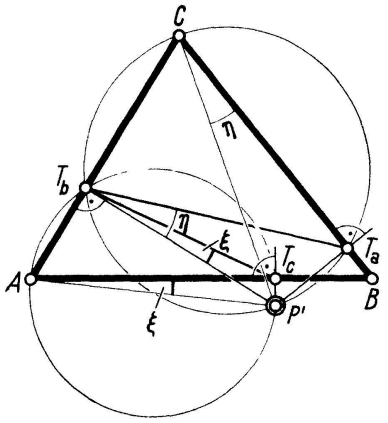
Die Punkte $P'$, $A$, $T_b$, $T_c$ und $P'$, $C$, $T_a$, $T_b$ liegen nach dem Satz von Thales je auf einem Kreis (siehe Figur 2). Bei allgemeiner Wahl von $P'$ ergeben sich aus dem Peripheriewinkelsatz folgende gleiche Winkel: $\xi = \sphericalangle P'AB = \sphericalangle P'T_bT_c$, $\eta =$




Figur 2

$\sphericalangle P'CB = \sphericalangle P'T_bT_a$. Also unterscheiden sich $\xi$ und $\eta$ um $\sphericalangle T_aT_bT_c$. Genau dann, wenn $T_a$, $T_b$, $T_c$ auf einer Geraden liegen, müssen $\xi$ und $\eta$ gleich sein, daher die vier Punkte $A$, $B$, $C$, $P'$ auf einem Kreis liegen. Also: *Dann und nur dann liegen $T_a$, $T_b$, $T_c$ auf einer Geraden, wenn $P'$ auf den Umkreis des Dreiecks $\triangle$ fällt.*

In Verbindung mit der Deutung von (4) erhält man das Ergebnis: *Die gefährliche Fläche beim räumlichen Rückwärtsschnitt ist jener Drehzylinder, der die Ebene $\varepsilon$ nach dem Umkreis des Dreiecks $\triangle$ schneidet.*

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# On a Diophantine Equation

A. SCHINZEL and W. SIERPIŃSKI [2][1]) have recently showed that all solutions of the equation

$$(x^2 - 1)(y^2 - 1) = \left(\left(\frac{y - x}{2}\right)^2 - 1\right)^2 \tag{1}$$

in natural numbers $x$, $y$, $x < y$ are of the form $x = x_n$, $y = x_{n+1}$, $n = 0, 1, \ldots$, where $x_0 = 1$, $x_1 = 3$, $x_{n+2} = 6\,x_{n+1} - x_n$. Equation (1) is a special case of the equation

$$(x^2 - 1)(y^2 - 1) = (z^2 - 1)^2 \tag{2}$$

whose all solutions are still unknown (cf. [2]–[5]). Moreover, the only known solution of (2) which is not a solution of (1) is $x = 4$, $y = 31$, $z = 11$ ([5]; cf. [2], [3]).

The purpose of this note is to give all solutions in natural numbers $t$, $x$, $y$ of the equation

$$(x^2 - t^2)(y^2 - t^2) = \left(\left(\frac{y - x}{2}\right)^2 - t^2\right)^2. \tag{3}$$

[1]) Numbers in brackets refer to References, page 38.

We shall prove the following

*Theorem.* All solutions of the Equation (3) in distinct natural numbers $t, x, y, x < y$, are of the form

$$t = | m^2 - 2n^2 | k , \quad x = (m^2 + 2n^2) k , \quad y = (3m^2 + 8mn + 6n^2) k , \tag{4}$$

where $m, n, k$ are natural numbers.

*Proof.* As in [2] we observe that

$$(x^2 - t^2)(y^2 - t^2) - \left(\left(\frac{y-x}{2}\right)^2 - t^2\right)^2 = -\frac{1}{16}(x+y)^2 (x^2 - 6xy + y^2 + 8t^2) .$$

Thus (3) is equivalent to

$$x^2 - 6xy + y^2 + 8t^2 = 0 ,$$

which may be written in the form

$$(y - 3x)^2 = 8(x^2 - t^2) . \tag{5}$$

First of all, $t < x$. Further, from (5) it follows that $4 \mid y - 3x$, so $y - 3x = 4z$, where $z > 0$, since otherwise $(y - x)/2 \leqslant x$ and from $t < x < y$ and (3) we get $(x^2 - t^2)(y^2 - t^2) \leqslant (x^2 - t^2)^2$, $y \leqslant x$, which is impossible.

Thus $z > 0$ and (5) can be written as

$$2z^2 + t^2 = x^2 . \tag{6}$$

Consequently, every solution of (5) in natural numbers $t, x, y, x < y$, gives a solution of (6) in natural numbers $t, x, z$, where $4z = y - 3x$. On the other hand, if $t, x, z$ is a solution of (6) in natural numbers, then the numbers $t, x, y = 3x + 4z$ are natural, $x < y$, and they form a solution of (5).

Thus, in order to find all solutions of (3) in natural numbers $t, x, y, x < y$ it suffices to know all solutions of (6) in natural numbers $t, x, z$ and put $y = 3x + 4z$.

But all solutions of (6) are the following (cf. [1], p. 41):

$$t = | m^2 - 2n^2 | k , \quad x = (m^2 + 2n^2) k , \quad z = 2mnk ,$$

where $m, n, k$ are natural numbers. If we put here $y = 3x + 4z$, we get the formulae (4), which completes the proof.

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