ELEMENTE DER MATHEMATIK

Revue de mathématiques élémentaires – Rivista di matematica elementare

Zeitschrift zur Pflege der Mathematik
und zur Förderung des mathematisch-physikalischen Unterrichts

Publiziert mit Unterstützung des Schweizerischen Nationalfonds
zur Förderung der wissenschaftlichen Forschung

El. Math. | Band XXII | Heft 3 | Seiten 49–72 | 10. Mai 1967

Concurrencies and Areas in a Triangle

It is astonishing but true (see e.g., [1] [1])) that many elementary results relating to the triangle are still being discovered. The following note gives a fresh derivation of some known results, along with simple extensions, some of these being applicable to various other plane configurations.

We begin with a proof of the generalized DUDENEY-STEINHAUS theorem (see [2]):

In a triangle ABC, transversals [2]) *AX, BY, CZ are drawn from the vertices cutting the opposite sides at points X, Y, Z dividing these sides internally in the respective ratios $s:(1-s), t:(1-t), u:(1-u)$. These transversals meet in pairs at L, M, N* (see Fig. 1). *Then*

$$\frac{\Delta LMN}{\Delta ABC}=\frac{\{(1-s)(1-t)(1-u)-s\,t\,u\}^2}{(s\,t+1-t)\,(t\,u+1-u)\,(u\,s+1-s)}\left(\frac{BX}{BC}=s\,,\ \frac{CY}{CA}=t\,,\ \frac{AZ}{AB}=u\right).$$

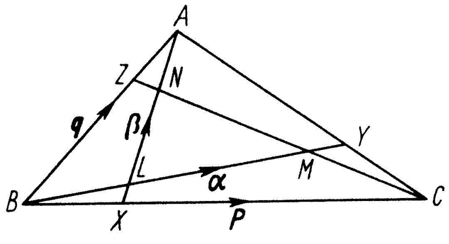

Figure 1

We put $\vec{BC}=\boldsymbol{p}$, $\vec{BA}=\boldsymbol{q}$, $\vec{LM}=\boldsymbol{\alpha}$, $\vec{LN}=\boldsymbol{\beta}$. Then, $\vec{AX}=s\,\boldsymbol{p}-\boldsymbol{q}$, $\vec{BY}=(1-t)\,\boldsymbol{p}+t\,\boldsymbol{q}$, $\vec{CZ}=(1-u)\,\boldsymbol{q}-\boldsymbol{p}$. Writing $\lambda\,\vec{BY}=\vec{BL}=s\,\boldsymbol{p}+\mu\,\vec{XA}$, we have

$$\lambda\,(1-t)\,\boldsymbol{p}+\lambda\,t\,\boldsymbol{q}=s\,(1-\mu)\,\boldsymbol{p}+\mu\,\boldsymbol{q}\,,$$

---

[1]) Numbers in brackets refer to References, page 55.

[2]) Other writers refer to these lines through the vertices as cedians, cevians, redians, and nedians; see, e.g., the references to *School Science and Mathematics* at the end of this paper.

whence $\lambda\,(1-t) = s\,(1-\mu)$ and $\lambda\,t = \mu$, so that $\lambda = s/(s\,t+1-t)$, $\mu = s\,t/(s\,t+1-t)$. From these, and similar, results it follows immediately that

$$\left.\begin{aligned} \frac{BL}{BY} &= \frac{s}{s\,t+1-t}\,, & \frac{XL}{XA} &= \frac{s\,t}{s\,t+1-t} \\ \frac{CM}{CZ} &= \frac{t}{t\,u+1-u}\,, & \frac{YM}{YB} &= \frac{t\,u}{t\,u+1-u} \\ \frac{AN}{AX} &= \frac{u}{u\,s+1-s}\,, & \frac{ZN}{ZC} &= \frac{u\,s}{u\,s+1-s} \end{aligned}\right\} \tag{1}$$

so that

$$\alpha = \vec{BY} - (\vec{BL} + \vec{MY}) = \left\{1 - \frac{BL}{BY} - \frac{MY}{BY}\right\} \vec{BY} = \phi\,\frac{t\,\boldsymbol{q} + (1-t)\,\boldsymbol{p}}{(s\,t+1-t)\,(t\,u+1-u)}\,,$$

where $\phi = (1-s)\,(1-t)\,(1-u) - s\,t\,u$; similarly

$$\beta = \phi\,\frac{\boldsymbol{q} - s\,\boldsymbol{p}}{(s\,t+1-t)\,(u\,s+1-s)}\,.$$

Thus, as $s\,t+1-t > 0$, etc.,

$$\begin{aligned} \frac{\Delta LMN}{\Delta ABC} &= \left|\frac{\alpha\times\beta}{\boldsymbol{p}\times\boldsymbol{q}}\right| = \frac{\phi^2\,|\{t\boldsymbol{q} + (1-t)\,\boldsymbol{p}\}\times\{\boldsymbol{q} - s\,\boldsymbol{p}\}\,|}{(s\,t+1-t)^2\,(t\,u+1-u)\,(u\,s+1-s)\,|\,\boldsymbol{p}\times\boldsymbol{q}\,|} \\ &= \frac{\{(1-s)\,(1-t)\,(1-u) - s\,t\,u\}^2}{(s\,t+1-t)\,(t\,u+1-u)\,(u\,s+1-s)} = f(s,t,u)\,,\ \text{say}\,. \end{aligned} \tag{2}$$

Among the consequences of the result (2) we may mention the following:

I. Taking $s = t = u = 1/3$ we obtain the DUDENEY-STEINHAUS theorem: if X, Y, Z divide the sides of ΔABC, in cyclic order, in the ratio 1:2 then $\Delta LMN = (1/7)\,\Delta ABC$.

II. Clearly, $\Delta LMN = 0$ if, and only if, $f(s, t, u) = 0$, i.e.

$$(1-s)\,(1-t)\,(1-u) = s\,t\,u\,, \tag{3}$$

$(0 \leqslant s, t, u \leqslant 1)$. As (3) holds for $s = t = u = 1/2$, it follows immediately that the medians of ΔABC are concurrent.

III. Taking $s = t = u = 1/2$, it follows readily from the relations (1) that the medians of a triangle divide each other in the ration 2:1. The converse result, that AX, BY, CZ are the medians of ΔABC if they are concurrent and divide each other in the ratio 2:1, is moderately difficult to prove by elementary methods; however, using the relations (1), a simple calculation shows that, in this case, $s = t = u = 1/2$.

IV. For $0 < s, t, u < 1$,

$$\frac{BX}{XC}\,\frac{CY}{YA}\,\frac{AZ}{ZB} = \frac{s\,t\,u}{(1-s)\,(1-t)\,(1-u)}\,.$$

As the transversals AX, BY, CZ are concurrent if, and only if, $(1-s)\,(1-t)\,(1-u) = s\,t\,u$, CEVA's theorem and its converse follow immediately.

V. Some interesting results arise from the case in which the transversals divide the sides, in cyclic order, in the same ratio. Suppose $BX : XC = CY : YA = AZ : ZB = \lambda : \mu$; then, $s = t = u = \lambda/(\lambda + \mu)$, and

$$\frac{\Delta LMN}{\Delta ABC} = \frac{(\lambda - \mu)^2}{\lambda^2 + \lambda\,\mu + \mu^2}\,. \tag{4}$$

(It will be clear that the transversals now cut, in cyclic order, in the same ratio.) We consider ΔXYZ (the MENELAIC triangle [3]): with the above notation,

$$\vec{XY} = (1-s)\,\boldsymbol{p} + t\,(\boldsymbol{q}-\boldsymbol{p})\,,\quad \vec{XZ} = (1-u)\,\boldsymbol{q} - s\,\boldsymbol{p}\,,$$

whence

$$\frac{\Delta XYZ}{\Delta ABC} = \frac{|\vec{XY}\times\vec{XZ}|}{|\boldsymbol{p}\times\boldsymbol{q}|} = (1-s)\,(1-t)\,(1-u) + s\,t\,u\,;$$

Thus, in the case considered,

$$\frac{\Delta XYZ}{\Delta ABC} = \frac{\lambda^2 - \lambda\,\mu + \mu^2}{(\lambda+\mu)^2} \tag{5}$$

and

$$\frac{\Delta LMN}{\Delta XYZ} = \frac{(\lambda^2-\mu^2)^2}{\lambda^4 + \lambda^2\,\mu^2 + \mu^4}\,. \tag{6}$$

The expressions on the right in (4), (5), (6) are symmetric in λ and μ, so that interchange of λ, μ leaves the corresponding ratios invariant; it is easily seen that, in general, the effect of this interchange is not such as to transform the triangles XYZ, LMN into identical triangles.

VI. It is easily shown that, when one or more of the points X, Y, Z divide the corresponding sides of triangle ABC *externally*, the ratio of the areas of the triangles XYZ, ABC is given by

$$\frac{\Delta XYZ}{\Delta ABC} = |\,(1-s)\,(1-t)\,(1-u) + s\,t\,u\,|\,.$$

Accordingly, $s\,t\,u/\{(1-s)\,(1-t)\,(1-u)\} = -1$ (here, we suppose X, Y, Z do not coincide with any vertex of ΔABC), if and only if $\Delta XYZ = 0$; i.e., if, and only if, X, Y, Z are collinear. Thus we obtain MENELAUS' theorem and its converse. (For this case it follows from Pasch's axiom that at least one of the points X, Y, Z lies outside ΔABC.)

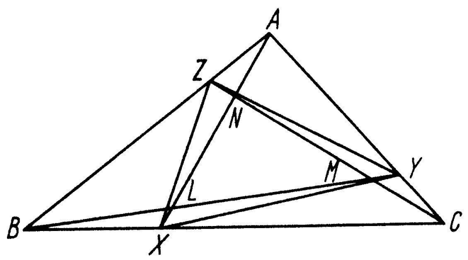

Figure 2

VII. By expressing $\vec{AN}$, $\vec{ZN}$, $\vec{MY}$, $\vec{CM}$, etc., in terms of p, q we can similarly calculate the ratios $\Delta ANZ : \Delta ABC$, $\Delta ANC : \Delta ABC$, $\Delta BXL : \Delta BCY$, etc., together with ratios of areas of quadrilaterals such as $LXCM : ANMY$, and so on. If X, Y, Z divide corresponding sides of triangle ABC in the same ratio, then $\Delta ANZ = \Delta BLX = \Delta CMY$, and the quadrilaterals $LXCM$, $MYAN$, $NZBL$ are of equal area.

VIII. If AX, BY, CZ are the interior bisectors of the corresponding angles of the triangle ABC, then, with a common notation ($BC = a$, etc.), we have $s/(1-s) = c/b$,

$t/(1-t) = a/c$, $u/(1-u) = b/a$, whence $(1-s)(1-t)(1-u) = s\,t\,u$; accordingly, these interior bisectors meet at a point I (the incentre). More generally, if AX, BY, CZ meet at an interior point J, then by (1)

$$\frac{AJ}{JX} = \frac{u}{(1-u)(1-s)}, \quad \frac{BJ}{JY} = \frac{s}{(1-s)(1-t)}, \quad \frac{CJ}{JZ} = \frac{t}{(1-t)(1-u)},$$

and $s\,t\,u = (1-s)(1-t)(1-u)$, so that

$$\frac{AJ}{JX}\,\frac{BJ}{JY}\,\frac{CJ}{JZ} = \frac{1}{s\,t\,u};$$

in particular, if J is the incentre I of triangle ABC, then we obtain

$$\frac{AI}{IX}\,\frac{BI}{IY}\,\frac{CI}{IZ} = \frac{(a+b)(b+c)(c+a)}{a\,b\,c}.$$

It will be clear that a similar result can be obtained for the case in which J lies at the centre of an escribed circle.

IX. If AX, BY, CZ are the altitudes of triangle ABC then

$$\frac{s}{1-s} = \frac{c\cos B}{b\cos C} \quad \frac{t}{1-t} = \frac{a\cos C}{c\cos A} \quad \frac{u}{1-u} = \frac{b\cos A}{a\cos B} \tag{7}$$

so that we again have $s\,t\,u = (1-s)(1-t)(1-u)$; this establishes concurrency of the altitudes. From this result and the result obtained in V we see that the area of the pedal triangle XYZ is given by

$$\frac{\Delta XYZ}{\Delta ABC} = |(1-s)(1-t)(1-u) + s\,t\,u| = 2\,|s\,t\,u|;$$

hence, by (7), $\Delta XYZ = 2\,|\cos A \cos B \cos C|\,\Delta ABC$.

X. Now let Y, Z divide CA, AB in the respective ratios $p:q$ and $q:p$. As p/q varies, the locus of the intersection of BY, CZ is the median through A. This result follows immediately from CEVA's theorem.

XI. Let a, b, c denote the lengths of the sides of triangle ABC and let m_1, m_2, m_3 denote the lengths of the transversals AX, BY, CZ. Put $\psi = (m_1^2 + m_2^2 + m_3^2)/(a^2 + b^2 + c^2)$. It is well known that, if AX, BY, CZ are medians, then

$$\psi = \frac{3}{4}. \tag{8}$$

More generally, if X, Y, Z divide BC, CA, AB in the same ratio p/q $(0 \leqslant p/q < \infty)$, it is easily shown that

$$\psi = \frac{p^2 + p\,q + q^2}{(p+q)^2} \leqslant 1;$$

For $0 \leqslant p/q < \infty$, $\inf(\psi) = 3/4$, the infimum being attained for $p = q$ (i.e. for the case in which AX, BY, CZ are medians).

Now suppose X, Y, Z divide the sides of triangle ABC in the respective ratios $s/(1-s)$, $t/(1-t)$, $u/(1-u)$. We can readily show that

$$m_1^2 + m_2^2 + m_3^2 = a^2(s^2 + 1 + u - s - t) + b^2(t^2 + 1 + s - t - u)$$
$$+ c^2(u^2 + 1 + t - u - s);$$

and from this it follows that, for $-\infty < s, t, u < +\infty$,

$$\inf(\psi) = \frac{3}{4} - \frac{1}{4\,(a^2+b^2+c^2)} \left\{ \frac{(a^2-b^2)^2}{c^2} + \frac{(b^2-c^2)^2}{a^2} + \frac{(c^2-a^2)^2}{b^2} \right\} \tag{9}$$

this infimum being attained for

$$s = \frac{c\cos B}{a}, \qquad t = \frac{a\cos C}{b}, \qquad u = \frac{b\cos A}{c}.$$

(This last result is immediate when we note that ψ attains its least value when AX, BY, CZ are altitudes of triangle ABC.)

Restricting $0 \leqslant s, t, u \leqslant 1$, i.e. all transversals internal, we have

Theorem 1. If m_1, m_2, m_3 are the lengths of transversals drawn from the vertices A, B, C of an acute-angled triangle, to the opposite sides of lengths a, b and c and if

$$\psi = \frac{m_1^2 + m_2^2 + m_3^2}{a^2+b^2+c^2},$$

then $1/2 < \text{Min}\,\{\psi\} \leqslant 3/4$, the upper bound being attained in an equilateral triangle.

Proof:

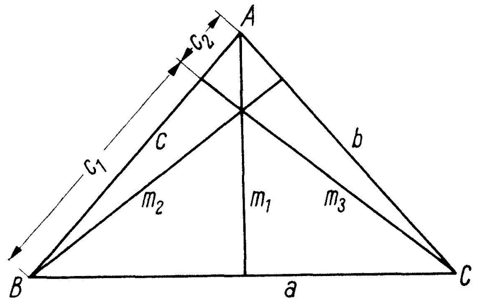

Figure 3

If all angles $A, B, C \leqslant 90°$ set $a \geqslant b \geqslant c$. Therefore $90° \geqslant A \geqslant B \geqslant C$ and $A \geqslant B \geqslant 45°$, m_1, m_2, m_3 are altitudes.

$$m_3^2 = a^2 - c_1^2, \qquad m_3^2 = b^2 - c_2^2$$

therefore

$$2\,m_3^2 = a^2 + b^2 - (c_1^2 + c_2^2)$$

but $c^2 > c_1^2 + c_2^2$ and $2\,m_3^2 > a^2 + b^2 - c^2$.

$$m_2 = c\sin A \geqslant \frac{c}{\sqrt{2}}, \qquad \text{also } m_1 \geqslant \frac{c}{\sqrt{2}}$$

therefore

$$2\,m_1^2 + 2\,m_2^2 + 2\,m_3^2 > a^2 + b^2 + c^2,$$

thus

$$\text{Min}\,\{\psi\} > \frac{1}{2}.$$

Min $\{\psi\}$ may be made to differ from 1/2 by as little as we please as may be seen in an acute-angled isosceles triangle whose base is arbitrarily small.

Again, from (9) we see that Min $\{\psi\} \ngtr 3/4$ and if $a = b = c$, Min $\{\psi\} = 3/4$.

Theorem 2. With the previous notation, if ABC is an acute-angled triangle,

$$1 \leqslant \operatorname{Max}\{\psi\} < 3/2.$$

Proof:

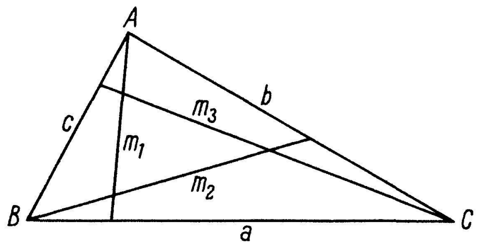

Figure 4

Let $a \geqslant b \geqslant c$, normalize side a to unity, thus $1 \geqslant b \geqslant c$. Then

$$m_1 \leqslant b\,, \quad m_2 \leqslant 1\,, \quad m_3 \leqslant 1\,,$$

and

$$\psi \leqslant \frac{2 + b^2}{1 + b^2 + c^2} \text{ for fixed } b \text{ and } c.$$

Assume

$$\frac{2 + b^2}{1 + b^2 + c^2} \geqslant \frac{3}{2}$$

then we have $1 \geqslant b^2 + 3\,c^2 > b^2 + c^2$, but by hypothesis $1 < b^2 + c^2$, thus the contradiction yields

$$\operatorname{Max}\{\psi\} < \frac{3}{2}.$$

However, the maximum may differ from 3/2 by as little as we please, as may be seen in an acute-angled isosceles triangle whose base is arbitrarily small.

Again, keeping b fixed, the denominator of $(2 + b^2)/(1 + b^2 + c^2)$ is largest when $c = b$, thus we require to minimize $f(b) = (2 + b^2)/(1 + 2\,b^2)$ under the restriction $0 \leqslant b \leqslant 1$.

$f(b)$ being a decreasing function has its minimum value at the maximum value of b, thus $f(b)_{\min} = 1$ and $\operatorname{Max}\{\psi\} \geqslant 1$ exhibiting equality in an equilateral triangle.

Theorem 3. With the previous notation, if ABC is an obtuse-angled triangle,

$$\frac{1}{3} < \psi < \frac{3 + \sqrt{3}}{3}.$$

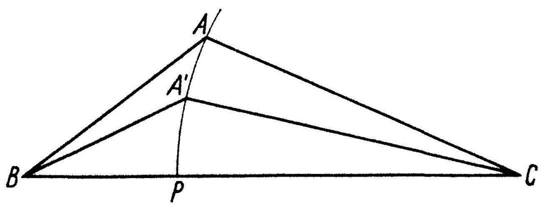

Figure 5

Normalizing a to unity, set $1 \geqslant b \geqslant c$ and then

$$m_1 \leqslant b\,, \quad m_2 \leqslant 1\,, \quad m_3 \leqslant 1\,,$$

and $\psi \leqslant (2 + b^2)/(1 + b^2 + c^2)$ for fixed b and c.

With centre C and radius b, describe an arc of a circle through vertex A cutting BC at P. Join C and B to A', any point on the arc AP inside the triangle ABC.

In $\Delta A'BC$ angle $A' >$ angle A, thus $BA' < BA = c$ and

$$\frac{2+b^2}{1+b^2+BA'^2} > \frac{2+b^2}{1+b^2+c^2}.$$

Thus we may obtain obtuse-angled triangles $A'BC$ in which the quantity $(2+b^2)/(1+b^2+c^2)$ becomes progressively larger. Sup $\{(2+b^2)/(1+b^2+c^2)\}$ occurs for $c = 1-b$, and $(2+b^2)/(1+b^2+(1-b)^2)$ becomes largest for $b = \sqrt{3}-1$. Hence

$$\operatorname{Sup}\left\{\frac{2+b^2}{1+b^2+c^2}\right\} = \frac{3+\sqrt{3}}{3}$$

and

$$\psi < \frac{3+\sqrt{3}}{3},$$

the difference $|\psi - (3+\sqrt{3})/3|$ being arbitrarily small in the triangle whose sides approach 1, $\sqrt{3}-1$, $2-\sqrt{3}$ respectively.

Again, in ΔABC

$$m_2 \geqslant c, \quad m_3 \geqslant b,$$

and $a < b + c$ (reverting to side $BC = a$).

Thus

$$m_1^2 + m_2^2 + m_3^2 > b^2 + c^2. \tag{10}$$

Further, $a^2 < 2\,b^2 + 2\,c^2$ as $2\,b\,c < b^2 + c^2$, therefore

$$a^2 + b^2 + c^2 < 3\,(b^2 + c^2). \tag{11}$$

From (10) and (11) therefore $(m_1^2 + m_2^2 + m_3^2)/(a^2 + b^2 + c^2) > 1/3$, the difference $|\psi - 1/3|$ being arbitrarily small in an obtuse-triangle whose sides approach a, a, $2\,a$ respectively.

I wish to thank Professors R. L. FORBES and J. W. REED for many helpful suggestions.

A. S. B. HOLLAND, University of Alberta, Calgary, Canada

REFERENCES

[1] J. GARFUNKEL and S. STAHL, *The Triangle Reinvestigated*, Amer. Math. Monthly *72*, 12–20 (1965).

[2] Also called Routh's Theorem, see H. S. M. COXETER, *Introduction to Geometry*, Wiley, New York 1963, p. 211.

[3] Mathematics Teacher, *44*, 496 (1951).

FURTHER REFERENCES

Amer. Math. Monthly, *56*, 269–270 (1949).

School Science and Mathematics, Vol. *38*, 935–936; Vol. *39*, 282; Vol. *40*, 483–485; Vol. *41*, 765–767, 788–789; Vol. *42*, 325–330; Vol. *43*, 684–685; Vol. *50*, 581.

DUDENEY, *Amusements in Mathematics*, London 1917, p. 27.

MIKUSINSKI, *Sur quelques propriétés du triangle*, Ann. Univ. Mariae Curie-Sklodowska, Sect. A, *1*, 45–50 (1946).

COXETER, *Contributions of Geometry to the Mainstream of Mathematics*, Dept. of Math., Oklahoma Agricultural and Mechanical College, Stillwater 1955, p. 82.

STEINHAUS, *Mathematical Snapshots*, Oxford University Press, New York 1960, p. 11.

Nabla, Vol. *8*, p. 114; Vol. *9*, p. 35.