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On Some Ternary Quartic Diophantine Equations

There are not known many instances of Diophantine equations

$$f(x, y, z) = 0 \tag{1}$$

representing a non-degenerate quartic surface for which an infinity of integer solutions exist. It may therefore be of interest to give a few.

Theorem 1

The equation

$$z^2 = U_1^2 + U_2 U_3 , \tag{2}$$

where

$$U_r = a_r x^2 + h_r x y + b_r y^2 + f_r y + g_r x \qquad (r = 1, 2, 3) ,$$

and the coefficients are integers, has an infinity of integer solutions if for either $r = 2$ or 3, $h_r^2 - 4 a_r b_r > 0$, and is not a perfect square and U_2 or U_3 is absolutely irreducible.

From (2), we have

$$z + U_1 = \frac{p}{q} U_2 , \quad z - U_1 = \frac{q}{p} U_3 ,$$

where p, q are integers and $(p, q) = 1$. Then

$$2 U_1 = \frac{p}{q} U_2 - \frac{q}{p} U_3 . \tag{3}$$

For integer solutions of (3), $U_2 \equiv 0 \pmod q$, $U_3 \equiv 0 \pmod p$, and then z is also an integer.

Write (3) as

$$P(x, y) = a x^2 + h x y + b y^2 + f y + g x = 0 , \tag{4}$$

where

$$a = p^2 a_2 - 2 p q a_1 - q^2 a_3 , \; h = p^2 h_2 - 2 p q h_1 - q^2 h_3 ,$$
$$b = p^2 b_2 - 2 p q b_1 - q^2 b_3 , \; f = p^2 f_2 - 2 p q f_1 - q^2 f_3 ,$$
$$g = p^2 g_2 - 2 p q g_1 - q^2 g_3 .$$

The equation (4) has a solution $x = 0$, $y = 0$. GAUSS has shown from a Pellian equation that (4) will have an infinity of integer solutions if $P(x, y)$ is algebraically irreducible and $h^2 - 4 a b > 0$ and is not a perfect square. The condition for reducibility is

$$\Delta = \frac{1}{2} \begin{vmatrix} 2a, & h, & g, \\ h, & 2b, & f \\ g, & f, & 2c \end{vmatrix} = 0 .$$

This is a binary sextic in p, q and is not identically zero since the coefficient of p^6 is obtained by replacing a, b, etc., in Δ by a_2, b_2, etc. Hence there will be only a finite number of values of p and q, if either U_2 or U_3 is irreducible, for which $P(x, y)$ is reducible.

Next

$$h^2 - 4ab = (p^2 h_2 - 2pq h_1 - q^2 h_3)^2 - 4(p^2 a_2 - 2pq a_1 - q^2 a_3)(p^2 b_2 - pq b_1 - q^2 b_3)$$

If $h_2^2 - 4 a_2 b_2 > 0$ and is not a perfect square, this holds for $h^2 - 4ab$ if p is large compared with q, and for an infinity of p. This proves Theorem (1).

There are many special cases not included in the theorem. We need only mention

Theorem 2

The equation

$$z^2 = k^2 + x^2 (a x^2 + b y^2), \quad a b k \neq 0,$$

has an infinity of integer solutions if k, a, b are integers and either $b > 0$, or $b < 0$, $4 a k^2 > b^2$.

We have

$$z + k = \frac{q}{p} (a x^2 + b y^2), \quad z - k = \frac{p}{q} x^2,$$

where p, q are integers and $(p, q) = 1$. Then

$$(a q^2 - p^2) x^2 + b q^2 y^2 = 2 k p q .$$

This will have the solution $x = 0$, $y = t$, where t is an arbitrary integer, if $b q t^2 = 2 k p$, and so if $\delta = (b, 2k)$, we can take

$$\lambda p = \frac{b}{\delta} t^2, \quad \lambda q = \frac{2k}{\delta}, \quad \lambda = \left(t^2, \frac{2k}{\delta}\right)$$

Hence there will be an infinity of integer solutions for x, y if $b(p^2 - a q^2) > 0$ and is not a perfect square, i.e. $b(b^2 t^4 - 4 a k^2) > 0$ and is not a perfect square. This is possible if $b > 0$ for an infinity of values of t, and also if $b < 0$, $4 a k^2 > b^2$ for $t = 1$.

The case $4 a k^2 < b^2$ seems difficult. Of course if $a < 0$, $b < 0$, there are only a finite number of solutions.

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Ungelöste Probleme

Bemerkung zu Nr. 14 (El. Math. *11*, 134–135 (1956)). A. a. O. wurde gezeigt, dass die Gleichung $x_1 + x_2 + \cdots + x_s = x_1 x_2 \ldots x_s$ für jedes natürliche s mindestens eine Lösung in natürlichen Zahlen besitzt. Nach einer Mitteilung von Herrn A. SCHINZEL (Warschau) hat M. MISIUREWICZ vor kurzem bewiesen, dass $s = 2, 3, 4, 6, 24, 144, 174, 444$ die einzigen $s \leqq 1000$ sind, für die genau eine Lösung existiert.