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## On Some Ternary Quartic Diophantine Equations

There are not known many instances of Diophantine equations

$$f(x, y, z) = 0 \quad (1)$$

representing a non-degenerate quartic surface for which an infinity of integer solutions exist. It may therefore be of interest to give a few.

### Theorem 1

The equation

$$z^2 = U_1^2 + U_2 U_3, \quad (2)$$

where

$$U_r = a_r x^2 + h_r x y + b_r y^2 + f_r y + g_r x \quad (r = 1, 2, 3),$$

and the coefficients are integers, has an infinity of integer solutions if for either  $r = 2$  or  $3$ ,  $h_r^2 - 4 a_r b_r > 0$ , and is not a perfect square and  $U_2$  or  $U_3$  is absolutely irreducible.

From (2), we have

$$z + U_1 = \frac{p}{q} U_2, \quad z - U_1 = \frac{q}{p} U_3,$$

where  $p, q$  are integers and  $(p, q) = 1$ . Then

$$2 U_1 = \frac{p}{q} U_2 - \frac{q}{p} U_3. \quad (3)$$

For integer solutions of (3),  $U_2 \equiv 0 \pmod{q}$ ,  $U_3 \equiv 0 \pmod{p}$ , and then  $z$  is also an integer.

Write (3) as

$$P(x, y) = a x^2 + h x y + b y^2 + f y + g x = 0, \quad (4)$$

where

$$a = p^2 a_2 - 2 p q a_1 - q^2 a_3, \quad h = p^2 h_2 - 2 p q h_1 - q^2 h_3,$$

$$b = p^2 b_2 - 2 p q b_1 - q^2 b_3, \quad f = p^2 f_2 - 2 p q f_1 - q^2 f_3,$$

$$g = p^2 g_2 - 2 p q g_1 - q^2 g_3.$$

The equation (4) has a solution  $x = 0, y = 0$ . GAUSS has shown from a Pellian equation that (4) will have an infinity of integer solutions if  $P(x, y)$  is algebraically irreducible and  $h^2 - 4 a b > 0$  and is not a perfect square. The condition for reducibility is

$$\Delta = \frac{1}{2} \begin{vmatrix} 2a & h & g \\ h & 2b & f \\ g & f & 2c \end{vmatrix} = 0.$$

This is a binary sextic in  $p, q$  and is not identically zero since the coefficient of  $p^6$  is obtained by replacing  $a, b$ , etc., in  $\Delta$  by  $a_2, b_2$ , etc. Hence there will be only a finite

number of values of  $p$  and  $q$ , if either  $U_2$  or  $U_3$  is irreducible, for which  $P(x, y)$  is reducible.

Next

$$h^2 - 4ab = (p^2 h_2 - 2pqh_1 - q^2 h_3)^2 - 4(p^2 a_2 - 2pqa_1 - q^2 a_3)(p^2 b_2 - pqb_1 - q^2 b_3)$$

If  $h_2^2 - 4a_2 b_2 > 0$  and is not a perfect square, this holds for  $h^2 - 4ab$  if  $p$  is large compared with  $q$ , and for an infinity of  $p$ . This proves Theorem (1).

There are many special cases not included in the theorem. We need only mention

### Theorem 2

The equation

$$z^2 = k^2 + x^2(a x^2 + b y^2), \quad a b k \neq 0,$$

has an infinity of integer solutions if  $k, a, b$  are integers and either  $b > 0$ , or  $b < 0, 4ak^2 > b^2$ .

We have

$$z + k = \frac{q}{p}(a x^2 + b y^2), \quad z - k = \frac{p}{q} x^2,$$

where  $p, q$  are integers and  $(p, q) = 1$ . Then

$$(a q^2 - p^2) x^2 + b q^2 y^2 = 2 k p q.$$

This will have the solution  $x = 0, y = t$ , where  $t$  is an arbitrary integer, if  $b q t^2 = 2 k p$ , and so if  $\delta = (b, 2k)$ , we can take

$$\lambda p = \frac{b}{\delta} t^2, \quad \lambda q = \frac{2k}{\delta}, \quad \lambda = \left( t^2, \frac{2k}{\delta} \right)$$

Hence there will be an infinity of integer solutions for  $x, y$  if  $b(p^2 - a q^2) > 0$  and is not a perfect square, i.e.  $b(b^2 t^4 - 4 a k^2) > 0$  and is not a perfect square. This is possible if  $b > 0$  for an infinity of values of  $t$ , and also if  $b < 0, 4 a k^2 > b^2$  for  $t = 1$ .

The case  $4 a k^2 < b^2$  seems difficult. Of course if  $a < 0, b < 0$ , there are only a finite number of solutions.

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## Ungelöste Probleme

**Bemerkung** zu Nr. 14 (El. Math. 11, 134–135 (1956)). A. a. O. wurde gezeigt, dass die Gleichung  $x_1 + x_2 + \dots + x_s = x_1 x_2 \dots x_s$  für jedes natürliche  $s$  mindestens eine Lösung in natürlichen Zahlen besitzt. Nach einer Mitteilung von Herrn A. SCHINZEL (Warschau) hat M. MISIUREWICZ vor kurzem bewiesen, dass  $s = 2, 3, 4, 6, 24, 144, 174, 444$  die einzigen  $s \leq 1000$  sind, für die genau eine Lösung existiert.