Packing of 33 equal circles on a sphere

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Packing of 33 equal Circles on a Sphere

Let a_n be the largest possible angular diameter of n equal circles (or spherical caps) which can be packed on the surface of a sphere without overlapping. The value of a_n has been determined [1, 2, 3] [1]) for $n = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$ and 24, based on arrangements of the circles which have been proven to be the best. For other values of n, conjectured values have been computed, based on likely arrangements of the circles, but which have not yet been proven to be the best. Summaries of the results are given by Fejes Tóth [4] and Coxeter [5]. A recent paper by Jucovič [6] gives (for $n = 17$, 25 and 33) better values than those previously published.

The arrangement for 33 circles given by Jucovič can be symbolized by 3, 9, 9, 9, 3. This means that there are five rings or bands of circles containing 3 or 9 circles in the indicated order. It is presumed that the circles are equally spaced in each band. He derived the value 34° 47′ for the diameter of the circles.

This note describes two modified arrangements which give better values. The first, shown in Figure 1, uses an unequally spaced band of circles of diameter 35° 22′ around the equator. Three touching pairs of circles alternate with single circles. The spaces between them are equal. Hence, each space has the value 6° 57′. The latitude L, of the center of the circle marked A, can be computed from the spherical right triangle of hypotenuse 35° 22′ and base $(35°22' + 6°57')/2 = 21°09'30''$. This gives $L = 29°01'30''$. Hence, the polar distance PA of this center equals 60° 58′ 30″. The polar distance PB of the center of the circle marked B can be computed from the cluster of circles surrounding the pole. This gives $PB = 41°45'$. The angle between these arcs

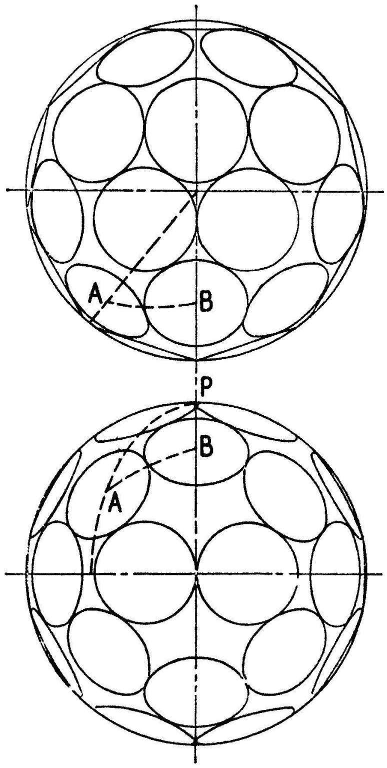

Figure 1

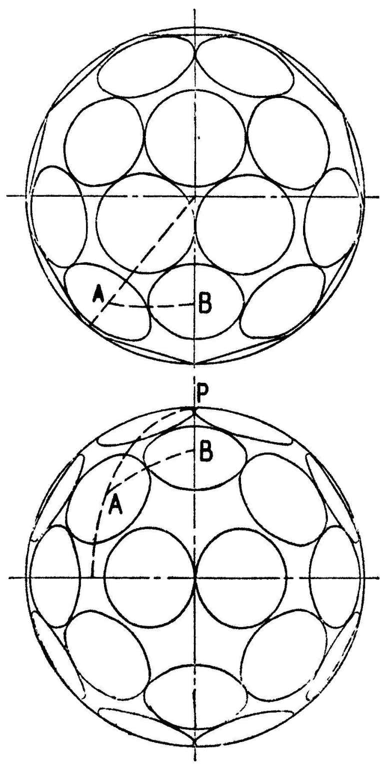

Figure 2

[1]) Numbers in brackets refer to Bibliography, page 100.

is $35°22' + (6°57')/2 = 38°50'30''$. Hence, the distance between the centers A and B is computed from the spherical triangle PAB by the cosine formula. The distance $35°22'$ is verified.

In the second arrangement, shown in Figure 2, the circles are of diameter $35°25'$. In the bands adjacent to the equatorial band, the circles are not completely separated; in each band there are three touching pairs of circles. The polar distance of the center A is $60°27'$. The polar distance of the center B is now $41°49'$. The angle between these arcs is $39°32'$. Hence, the distance between the centers of A and B is verified as $35°25'$.

The quantity D_n, which represents the density of packing on the sphere, is given by $D_n = n\,(1 - \cos 0{,}5\, a_n)/2$. If the circles have unit diameter, the sphere has the radius $R_n = 1/\sqrt{2 - 2\cos a_n}$. In the symbol for the arrangement, a number in parenthesis indicates that the circles in that band are not equally spaced.

Arrangement	a	D	R	
3, 9, 9, 9, 3	34°47′	0.755	1.673	JUCOVIČ
3, 3, (6), (9), (6), 3, 3	35°25′	0.782	1.644	GOLDBERG

The configurations of the circles can be considered as results of a problem in mechanics. The circles may be considered as being subjected to forces which push them together. These may be gravitational forces or forces due to applied pressures at the poles. Then each circle is subjected to normal forces at its points of contact with its neighbors. At equilibrium, the vector sum of the forces on a circle must be zero.

In Figure 1, each circle has at least three points of contact and there is no unbalanced force which would tend to move a circle. Therefore, the arrangement is stable.

In Figure 2, the circle pairs on the equator are free to move without changing the positions of the other circles. The remaining 27 circles make a stable arrangement.

The original arrangement proposed by JUCOVIČ was not stable. Applied forces would produce either Figure 1 or Figure 2.

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BIBLIOGRAPHY

[1] SCHÜTTE, K., VAN DER WAERDEN, B. L., *Auf welcher Kugel haben 5, 6, 7, 8 oder 9 Punkte mit Mindestabstand Eins Platz?* Math. Annalen *123* (1951) 96–124.

[2] DANZER, L., *Proofs for n = 10 and 11 have been derived but not yet published.*

[3] ROBINSON, R. M., *Arrangement of 24 points on a sphere,* Math. Annalen *144* (1961) 17–48.

[4] FEJES TÓTH, L., *Lagerungen in der Ebene auf der Kugel und im Raum* (Berlin–Göttingen–Heidelberg, 1953).

[5] COXETER, H. S. M., *The problem of packing a number of equal nonoverlapping circles on a sphere,* Trans. New York Academy of Sciences, Ser. II, *24* (1962) 320–331.

[6] JUCOVIČ, E., *Lagerung von 17, 25 und 33 Punkten auf der Kugel,* (Slovak, Russian and German summaries) Mat.-Fyz. Časopis. Slovensk Akad. Vied. *9* (1959) 173–176 [Math. Rev. *23* (1962) 533].