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Packing of 33 equal Circles on a Sphere

Let a_n be the largest possible angular diameter of n equal circles (or spherical caps) which can be packed on the surface of a sphere without overlapping. The value of a_n has been determined [1, 2, 3]¹⁾ for $n = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$ and 24, based on arrangements of the circles which have been proven to be the best. For other values of n , conjectured values have been computed, based on likely arrangements of the circles, but which have not yet been proven to be the best. Summaries of the results are given by FEJES TÓTH [4] and COXETER [5]. A recent paper by JUCOVIČ [6] gives (for $n = 17, 25$ and 33) better values than those previously published.

The arrangement for 33 circles given by JUCOVIČ can be symbolized by 3, 9, 9, 9, 3. This means that there are five rings or bands of circles containing 3 or 9 circles in the indicated order. It is presumed that the circles are equally spaced in each band. He derived the value $34^\circ 47'$ for the diameter of the circles.

This note describes two modified arrangements which give better values. The first, shown in Figure 1, uses an unequally spaced band of circles of diameter $35^\circ 22'$ around the equator. Three touching pairs of circles alternate with single circles. The spaces between them are equal. Hence, each space has the value $6^\circ 57'$. The latitude L , of the center of the circle marked A , can be computed from the spherical right triangle of hypotenuse $35^\circ 22'$ and base $(35^\circ 22' + 6^\circ 57')/2 = 21^\circ 09' 30''$. This gives $L = 29^\circ 01' 30''$. Hence, the polar distance PA of this center equals $60^\circ 58' 30''$. The polar distance PB of the center of the circle marked B can be computed from the cluster of circles surrounding the pole. This gives $PB = 41^\circ 45'$. The angle between these arcs

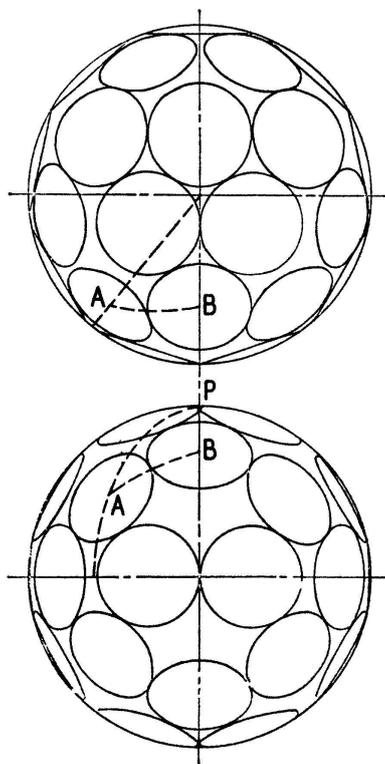


Figure 1

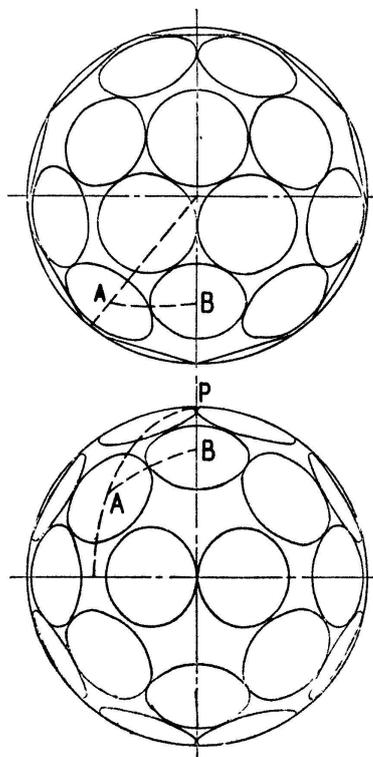


Figure 2

¹⁾ Numbers in brackets refer to Bibliography, page 100.

is $35^\circ 22' + (6^\circ 57')/2 = 38^\circ 50' 30''$. Hence, the distance between the centers A and B is computed from the spherical triangle PAB by the cosine formula. The distance $35^\circ 22'$ is verified.

In the second arrangement, shown in Figure 2, the circles are of diameter $35^\circ 25'$. In the bands adjacent to the equatorial band, the circles are not completely separated; in each band there are three touching pairs of circles. The polar distance of the center A is $60^\circ 27'$. The polar distance of the center B is now $41^\circ 49'$. The angle between these arcs is $39^\circ 32'$. Hence, the distance between the centers of A and B is verified as $35^\circ 25'$.

The quantity D_n , which represents the density of packing on the sphere, is given by $D_n = n(1 - \cos 0,5 a_n)/2$. If the circles have unit diameter, the sphere has the radius $R_n = 1/\sqrt{2 - 2 \cos a_n}$. In the symbol for the arrangement, a number in parenthesis indicates that the circles in that band are not equally spaced.

Arrangement	a	D	R	
3, 9, 9, 9, 3	$34^\circ 47'$	0.755	1.673	JUCOVIČ
3, 3, (6), (9), (6), 3, 3	$35^\circ 25'$	0.782	1.644	GOLDBERG

The configurations of the circles can be considered as results of a problem in mechanics. The circles may be considered as being subjected to forces which push them together. These may be gravitational forces or forces due to applied pressures at the poles. Then each circle is subjected to normal forces at its points of contact with its neighbors. At equilibrium, the vector sum of the forces on a circle must be zero.

In Figure 1, each circle has at least three points of contact and there is no unbalanced force which would tend to move a circle. Therefore, the arrangement is stable.

In Figure 2, the circle pairs on the equator are free to move without changing the positions of the other circles. The remaining 27 circles make a stable arrangement.

The original arrangement proposed by JUCOVIČ was not stable. Applied forces would produce either Figure 1 or Figure 2.

M. GOLDBERG, Washington

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