

Zeitschrift: Elemente der Mathematik
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 18 (1963)
Heft: 3

Artikel: On the diameter and triameter of a convex body
Autor: Chakerian, G.D.
DOI: <https://doi.org/10.5169/seals-22637>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 04.07.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

Posons maintenant

$$u_n = 3 b_n - a_n, \quad v_n = 4 b_n, \quad w_n = 3 b_n + a_n \text{ pour } n = 1, 2, \dots \quad (4)$$

D'après (1) et (2) on a $u_n < v_n < w_n$ pour $n = 1, 2, \dots$ et $u_1 = 1, v_1 = 4, w_1 = 5$, $u_{n+1} = 3 b_{n+1} - a_{n+1} = 35 a_n + 71 b_n$ pour $n = 1, 2, \dots$, donc les nombres u_n, v_n, w_n sont, pour $n = 2, 3, \dots$, des entiers > 1 et, les nombres b_n croissant avec n , on a $v_{n+1} > v_n$ pour $n = 1, 2, \dots$

Or, d'après (4), on trouve pour $n = 1, 2, \dots$

$$u_n^3 + w_n^3 - 2 v_n^3 - u_n - w_n + 2 v_n = 2 b_n [(3 a_n)^2 - 37 b_n^2 + 1],$$

donc, d'après (3):

$$u_n^3 + w_n^3 - 2 v_n^3 - u_n - w_n + 2 v_n = 0 \text{ pour } n = 1, 2, \dots,$$

d'où

$$\frac{u_n^3 - u_n}{6} + \frac{w_n^3 - w_n}{6} = 2 \frac{v_n^3 - v_n}{6} \text{ pour } n = 1, 2, \dots,$$

donc

$$T_{u_{n-1}} + T_{w_{n-1}} = 2 T_{v_{n-1}} \text{ pour } n = 1, 2, \dots, \quad (5)$$

ce qui prouve que, pour $n = 1, 2, \dots$, les trois nombres tétraédraux

$$T_{u_{n-1}}, \quad T_{v_{n-1}} \text{ et } T_{w_{n-1}}$$

forment une progression arithmétique. Comme $u_n < v_n < w_n$ et $v_{n+1} > v_n$ pour $n = 1, 2, \dots$, il en résulte notre théorème. Pour $n = 2$ on obtient $T_{140} + T_{728} = 2 T_{579}$.

Pour $n = 1$ on a $u_1 = 1, v_1 = 4, w_1 = 5$ et la formule (5) donne $T_4 = 2 T_3$. Dans ma note citée j'ai posé le problème s'il existe d'autres solutions en nombres naturels m et n de l'équation $T_m = 2 T_n$. M. S. L. SEGAL a démontré récemment qu'il n'existe pas d'autres solutions²⁾. Or, il mentionne aussi (l.c., p. 638) que M. S. CHOWLA a démontré récemment qu'il existe une infinité de nombres tétraédraux qui sont sommes de deux nombres tétraédraux (ce que j'ai démontré dans ma note citée des Elemente der Math.). La démonstration de M. CHOWLA m'est inconnue.

Il est encore à remarquer qu'il existent d'autres solutions de l'équation $T_x + T_y = 2 T_z$ autre celles que nous avons trouvées, par exemple $T_4 + T_{10} = 2 T_8$.

Or, M. A. MAKOWSKI a posé le problème suivant, dont la solution me semble être difficile: *Existe-t-il pour tout nombre naturel k une infinité de solutions de l'équation $T_x + T_y = k T_z$ en entiers positifs x, y et z?*

Je sais démontrer (ce que je ferai ailleurs) qu'il existe une infinité de nombres naturels k pour lesquels cela est vrai. W. SIERPIŃSKI (Varsovie)

On the Diameter and Triameter of a Convex Body

1. By a *convex body* in Euclidean n -dimensional space E_n we shall mean a compact, convex subset with interior points. One phase of the theory of convex bodies seeks to establish inequalities between the geometrical invariants associated with these bodies.

²⁾ S. L. SEGAL, *A note on pyramidal numbers*, Amer. Math. Monthly 69 (1962), p. 637.

The most famous of such inequalities, for example, is the so-called *isoperimetric inequality*: *Let the convex body K have volume V and surface area A . Then*

$$A^n \geq n^n \pi_n V^{n-1}, \quad (1)$$

where equality can hold if and only if K is spherical. The constant π_n is the volume of the unit ball in E_n . For a proof of (1) see [2]¹), p. 109.

The inequality (1) is quite powerful and requires some effort to prove. In this note we wish to show how an interesting technique from the theory of convex bodies can be used to establish similar, though weaker, inequalities. The technique to which we refer relies on the following formula, due to CAUCHY: *Let the convex body K have surface area A . Given a point ω on the unit sphere centered at the origin, let E_ω be the hyperplane through the origin orthogonal to the segment joining the origin to ω . The orthogonal projection, K_ω , of K onto E_ω is a convex body with respect to E_ω . Let V_ω be the ($n-1$ -dimensional) volume of K with respect to E_ω . Then*

$$A = \frac{1}{\pi_{n-1}} \int V_\omega d\omega, \quad (2)$$

where the integration is over the entire surface of the unit sphere. A proof of this remarkable relation is given in [2], p. 48.

It is evident that mathematical induction, used in conjunction with (2), might be useful in establishing geometric inequalities involving surface area. Indeed, in §2 we use this technique in deriving a well known inequality of BIEBERBACH (see formula (3)). The §3 is devoted to proving a similar inequality involving the volume and «triometer» of a convex body (see formula (9)). Our proofs are simple, but they suffer from the defect of relying on (1) or equally difficult inequalities. This is because formula (2) enables us to apply induction simply where surface area is involved, but no such useful formula exists for the volume.

2. By the *diameter* of the convex body K we shall mean the length of the longest segment contained in K . We then have the *inequality of BIEBERBACH*: *Let K have diameter D and volume V . Then*

$$V \leq \pi_n \left(\frac{D}{2} \right)^n, \quad (3)$$

with equality holding if and only if K is spherical. As usual, π_n denotes the volume of the unit ball in E_n . The theorem can be restated in the form: *Among all convex bodies of the same diameter, the sphere has the largest volume.* Proofs are given in [2], p. 76 and p. 107. A direct geometrical proof without relying on (1) is given in [3], p. 173. Our proof is as follows.

Proof. Let D_ω be the diameter of the orthogonal projection K_ω of K onto hyperplane E_ω . Then evidently $D_\omega \leq D$. Then if the theorem is true in $n-1$ -dimensional space, we have

$$V_\omega \leq \pi_{n-1} \left(\frac{D_\omega}{2} \right)^{n-1} \leq \pi_{n-1} \left(\frac{D}{2} \right)^{n-1} \quad (4)$$

Applying (2), we have, by (4),

$$A \leq n \pi_n \left(\frac{D}{2} \right)^{n-1}. \quad (5)$$

¹⁾ Numbers in brackets refer to References, page 57.

Thus, by (1) and (5),

$$n^n \pi_n V^{n-1} \leq A^n \leq n^n \pi_n^n \left(\frac{D}{2}\right)^{n(n-1)}, \quad (6)$$

and (3) follows. If equality holds in (3), then, by (6), it must hold in (1), and the uniqueness clause of the isoperimetric theorem yields that K is spherical. To complete the induction, we establish the theorem for $n = 2$, but this is just a repetition of steps (4), (5), and (6), with $n = 2$, equality holding in the first inequality in (4).

3. Let S be the (2-dimensional) area of the largest (in the sense of area) triangle contained in the convex body K . Then by the *triometer* of K we shall mean

$$T = \left(\frac{4S}{3\sqrt{3}} \right)^{1/2}. \quad (7)$$

The following inequality is due to BLASCHKE [1], p. 49: *Let K be a convex body in E_2 with triameter T and (2-dimensional) volume V . Then*

$$V \leq \pi T^2, \quad (8)$$

with equality holding if and only if K is an ellipse. An elegant proof of a generalization of this theorem to convex n -gons is given in [4], p. 36. We can generalize the theorem in another direction as follows: *Let K be a convex body in E_n with triameter T and volume V . Then*

$$V \leq \pi_n T^n, \quad (9)$$

and equality holds if and only if 1), $n = 2$ and K is an ellipse, or 2), $n > 2$ and K is spherical.

Proof. Let $n > 2$ and let T_ω be the triameter of the orthogonal projection K_ω of K onto hyperplane E_ω . Then it is immediate that $T_\omega \leq T$. Assuming the theorem for $n - 1$, we have

$$V_\omega \leq \pi_{n-1} T_\omega^{n-1} \leq \pi_{n-1} T^{n-1}. \quad (10)$$

By (2) and (10),

$$A \leq n \pi_n T^{n-1}. \quad (11)$$

Then by (1) and (11),

$$n^n \pi_n V^{n-1} \leq A^n \leq n^n \pi_n^n T^{n(n-1)}. \quad (12)$$

If equality holds in (9), then it must hold in (1), because of (12), so that K is spherical. The induction is completed by using (8).

G. D. CHAKERIAN, Pasadena USA
California Institute of Technology

REFERENCES

- [1] W. BLASCHKE, *Vorlesungen über Differentialgeometrie II*, (Springer-Verlag, Berlin 1923).
- [2] T. BONNESEN and W. FENCHEL, *Theorie der Konvexen Körper*, (Springer-Verlag, Berlin 1935).
- [3] H. HADWIGER, *Vorlesungen über Inhalt, Oberfläche und Isoperimetrie*, (Springer-Verlag, Berlin 1957).
- [4] L. FEJES TÓTH, *Lagerungen in der Ebene, auf der Kugel und im Raum*, (Springer-Verlag, Berlin 1953).