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Rubrik: Ungelöste Probleme

Ungelöste Probleme

Nr. 32. For real t let $\|t\|$ denote the positive distance between t and the integer nearest to t, and let f be a nonincreasing function such that $0 < f(n) < 1/2$ for $n = 1, 2, \ldots$. It is a classical result of KHINTSCHINE that the inequality $\|n\,\alpha\| < f(n)$ has infinitely many solutions for almost all α or almost no α, according as the series $\sum f(n)$ diverges or converges. KHINTSCHINE's theorem can also be stated in this form: Let S_n be the interval symmetric about 0, of length $f(n)$, and let T_n be the union of all the translations of S_n through integer distances. Then the assertion '$n\,\alpha \in T_n$ for infinitely many n' is true for almost all α or almost no α, according as $\sum f(n)$ diverges or converges.

What can be said if S_n is a more complicated set in $[0,1]$, of total length (or measure) $f(n)$? On the basis of a crude probability argument, it is to be expected that the nature of S_n is immaterial, and that only its measure is of significance.

In case S_n consists of the intervals of length $f(n)/n^k$ symmetric about the points $0, 1/n^k, 2/n^k, \ldots, (n^k - 1)/n^k$, where k is a fixed nonnegative integer, the relation $n\,\alpha \in T_n$ is equivalent to

$$\left| n\,\alpha - \frac{j}{n^k} - m \right| < \frac{f(n)}{n^k} \qquad \text{for some integers } j,\ m$$

or

$$\|n^{k+1}\,\alpha\| < f(n). \tag{*}$$

It has recently been proved by ERDÖS[1]) in the case $k = 0$, and LE VEQUE[2]) in the case $k > 0$, that if $\sum f(n)$ converges, inequality (*) has only finitely many solutions for almost all α, while if $\sum f(n)$ diverges, the number of solutions $n < N$ of (*) is, for almost all α, asymptotically equal to $\sum_1^N f(n)$. It is also shown[2]) that the corresponding theorem holds with certain other sequences $\{a_n\}$ in place of $\{n^{k+1}\}$ in (*), but there appears to be nothing known about the situation when S_n is not composed of finitely many intervals.

A large part of the theory of diophantine approximations is susceptible of the same kind of generalization, in which inequalities are replaced by set inclusions. The proofs will require many new ideas.

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Aufgaben

Aufgabe 322. Es sei $D_n = |a_{ik}|$ die Determinante n-ten Grades, in der das allgemeine Element a_{ik} das kleinste gemeinsame Vielfache von i und k ist. Man beweise, dass

$$D_n = n! \prod_{p \leq n} (1 - p)^{[n/p]},$$

wo das Produkt über alle Primzahlen $p \leq n$ erstreckt wird und $[\alpha]$ die grösste ganze Zahl $\leq \alpha$ bedeutet.

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[1]) P. ERDÖS, *Two Theorems on Diophantine Approximation*, to appear in Acta Arithmetica.

[2]) W. J. LE VEQUE, *On the Frequency of Small Fractional Parts in Certain Real Sequences*, III, to appear in J. reine angew. Math.