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**Abstract.** As a generalization of a classical result on the Alexander polynomial for fibered knots, we show in this paper that the Reidemeister torsion associated to a certain representation detects fiberedness of knots in the three sphere.

Mathematics Subject Classification (2000). Primary 57M25; Secondary 57M05.

Keywords. Reidemeister torsion, twisted Alexander polynomial, fibered knot, representation.

#### 1. Introduction

As is well-known, the Alexander polynomial of a fibered knot is monic (see [13], [14], [16]). That is, the coefficient of the highest degree term of the normalized Alexander polynomial is a unit  $1 \in \mathbb{Z}$ . By the symmetry (or duality) of the Alexander polynomial, its lowest degree term is also one. This criterion is sufficient for alternating knots [12] and prime knots up to 10 crossings [4] for instance. However, in general, the converse is not true. In fact, there are infinitely many non-fibered knots having monic Alexander polynomials. If we remember here Milnor's result [9], we have to remark that these claims on the Alexander polynomial can be restated by the Reidemeister torsion.

The purpose of this paper is to give a necessary condition that a knot in  $S^3$  is fibered by virtue of the Reidemeister torsion associated to a certain linear representation. More precisely, we show that the Reidemeister torsion of a fibered knot defined for a certain tensor representation is expressed as a rational function of monic polynomials. This Reidemeister torsion is nothing but Wada's twisted Alexander polynomial (see [7] for details), so that our result can be regarded as a natural generalization of the property on the classical Alexander polynomial mentioned above.

This paper is organized as follows. In the next section, we review the definition of Reidemeister torsion over a field  $\mathbb{F}$ . Further we describe how to compute it in the case

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of knot exteriors. The point of our method here is that the notion of monic makes sense for the Reidemeister torsion associated to a tensor representation of a unimodular representation over  $\mathbb{F}$  and the abelianization homomorphism. In Section 3, we state and prove the main theorem of this paper. The final section is devoted to compute some examples.

We should note here that there is a similar work by Cha [1]. The notion of Alexander polynomials twisted by a representation and its applications have appeared in several papers (see [2], [6], [8], [18]).

#### 2. Reidemeister torsion

In this section, we review the definition of Reidemeister torsion over a field  $\mathbb{F}$  (see [3] and [10] for details).

Let V be an n-dimensional vector space over  $\mathbb{F}$ , and let  $\boldsymbol{b} = (b_1, \ldots, b_n)$  and  $\boldsymbol{c} = (c_1, \ldots, c_n)$  be two bases for V. If we put  $c_i = \sum_{j=1}^n a_{ij}b_j$ , we have a nonsingular matrix  $A = (a_{ij})$  with coefficients in  $\mathbb{F}$ . Further let  $[\boldsymbol{b}/\boldsymbol{c}]$  denote the determinant of A.

Now let us consider an acyclic chain complex of finite dimensional vector spaces over  $\mathbb{F}$ :

$$C_*: 0 \longrightarrow C_m \xrightarrow{\partial_m} C_{m-1} \xrightarrow{\partial_{m-1}} \cdots \longrightarrow C_1 \xrightarrow{\partial_1} C_0 \longrightarrow 0.$$

We assume that a preferred basis  $c_q$  for  $C_q(C_*)$  is given for any q. Choose any basis  $b_q$  of  $B_q(C_*)$  and take its lift in  $C_{q+1}(C_*)$ , which we denote by  $\tilde{b}_q$ .

Since the natural inclusion map

$$B_q(C_*) \to Z_q(C_*)$$

is an isomorphism, the basis  $b_q$  can serve as a basis for  $Z_q(C_*)$ . Similarly the sequence

$$0 \longrightarrow Z_q(C_*) \longrightarrow C_q(C_*) \longrightarrow B_{q-1}(C_*) \longrightarrow 0$$

is exact and the vectors  $(\boldsymbol{b}_q, \tilde{\boldsymbol{b}}_{q-1})$  form a basis for  $C_q(C_*)$ . It is easily shown that  $[\boldsymbol{b}_q, \tilde{\boldsymbol{b}}_{q-1}/c_q]$  is independent of the choices of  $\tilde{\boldsymbol{b}}_{q-1}$ . Hence we may simply denote it by  $[\boldsymbol{b}_q, \boldsymbol{b}_{q-1}/c_q]$ .

**Definition 2.1.** The torsion of the chain complex  $C_*$  is defined by the alternating product

$$\prod_{q=0}^{m} [\bm{b}_{q}, \bm{b}_{q-1}/\bm{c}_{q}]^{(-1)^{q+1}}$$

and we denote it by  $\tau(C_*)$ .

**Remark 2.2.** The torsion  $\tau(C_*)$  depends only on the bases  $c_0, \ldots, c_m$ .

Now let us apply the above torsion to the following geometric situations. Let X be a finite cell complex and  $\tilde{X}$  the universal covering of X with the right action of  $\pi_1 X$  as deck transformations. Then the chain complex  $C_*(\tilde{X}, \mathbb{Z})$  has a structure of right free  $\mathbb{Z}[\pi_1 X]$ -modules. Let

$$\rho: \pi_1 X \to \operatorname{GL}(n, \mathbb{F})$$

be a linear representation. We may regard V as a  $\pi_1 X$ -module by using  $\rho$  and denote it by  $V_{\rho}$ . Define the chain complex  $C_*(X, V_{\rho})$  by  $C_*(\tilde{X}, \mathbb{Z}) \otimes_{\mathbb{Z}[\pi_1 X]} V_{\rho}$  and choose a preferred basis

$$\{\sigma_1 \otimes e_1, \sigma_1 \otimes e_2, \ldots, \sigma_1 \otimes e_n, \ldots, \sigma_{k_a} \otimes e_1, \ldots, \sigma_{k_a} \otimes e_n\}$$

of  $C_q(X, V_\rho)$ , where  $\{e_1, \ldots, e_n\}$  is a basis of V and  $\sigma_1, \ldots, \sigma_{k_q}$  are q-cells giving the preferred basis of  $C_q(\tilde{X}, \mathbb{Z})$ .

Now we assume that  $C_*(X, V_\rho)$  is acyclic, in other words, all homology groups  $H_*(X, V_\rho)$  vanish. In this case, we call  $\rho$  an acyclic representation.

**Definition 2.3.** Let  $\rho: \pi_1 X \to \operatorname{GL}(n, \mathbb{F})$  be an acyclic representation. Then Reidemeister torsion of X with  $V_{\rho}$ -coefficients is defined by the torsion of the chain complex  $C_*(X, V_{\rho})$ . We denote it by  $\tau_{\rho}(X)$ .

**Remark 2.4.** It is known that  $\tau_{\rho}(X)$  is well-defined as a PL-invariant, for an acyclic representation  $\rho : \pi_1 X \to \operatorname{GL}(n, \mathbb{F})$ , up to a factor  $\pm d$  where  $d \in \operatorname{Im}(\det \circ \rho) \subset \mathbb{F}^*$ . As a reference, see [10], Section 8. We can easily make a refinement of the above argument for our situation.

Here let us consider a knot K in  $S^3$  and its exterior E. For the knot group  $\pi K = \pi_1 E$  we choose and fix a Wirtinger presentation

$$P(\pi K) = \langle x_1, \ldots, x_u \mid r_1, \ldots, r_{u-1} \rangle.$$

Then we can construct a 2-dimensional cell complex X from  $P(\pi K)$  such that E collapses to X. The abelianization homomorphism

$$\alpha \colon \pi K \to H_1(E, \mathbb{Z}) \cong \mathbb{Z} = \langle t \rangle$$

is given by

$$\alpha(x_1) = \cdots = \alpha(x_u) = t.$$

Furthermore, we always suppose that the image of a representation  $\rho: \pi K \to GL(n, \mathbb{F})$  is included in  $SL(n, \mathbb{F})$ .

These maps naturally induce the ring homomorphisms  $\tilde{\rho}$  and  $\tilde{\alpha}$  from  $\mathbb{Z}[\pi K]$  to  $M(n, \mathbb{F})$  and  $\mathbb{Z}[t^{\pm 1}]$  respectively, where  $M(n, \mathbb{F})$  denotes the matrix algebra of degree n over  $\mathbb{F}$ . Then  $\tilde{\rho} \otimes \tilde{\alpha}$  defines a ring homomorphism

$$\mathbb{Z}[\pi K] \to M(n, \mathbb{F}[t^{\pm 1}]).$$

Let  $F_u$  denote the free group with generators  $x_1, \ldots, x_u$  and denote by

$$\Phi \colon \mathbb{Z}[F_u] \to M\left(n, \mathbb{F}[t^{\pm 1}]\right)$$

the composite of the surjection  $\mathbb{Z}[F_u] \to \mathbb{Z}[\pi K]$  induced by the presentation and the map  $\mathbb{Z}[\pi K] \to M(n, \mathbb{F}[t^{\pm 1}])$  given by  $\tilde{\rho} \otimes \tilde{\alpha}$ .

Let us consider the  $(u - 1) \times u$  matrix M whose (i, j)th component is the  $n \times n$  matrix

$$\Phi\left(\frac{\partial r_i}{\partial x_j}\right) \in M\left(n, \mathbb{F}[t^{\pm 1}]\right),$$

where  $\frac{\partial}{\partial x}$  denotes the free differential calculus. This matrix *M* is called the Alexander matrix of the presentation  $P(\pi K)$  associated to the representation  $\rho$ .

For  $1 \le j \le u$ , let us denote by  $M_j$  the  $(u-1) \times (u-1)$  matrix obtained from M by removing the *j*th column. We regard  $M_j$  as a  $(u-1)n \times (u-1)n$  matrix with coefficients in  $\mathbb{F}[t^{\pm 1}]$ .

Now let us recall that the tensor representation

$$\rho \otimes \alpha : \pi K \to \operatorname{GL}(n, \mathbb{F}(t))$$

is defined by  $(\rho \otimes \alpha)(x) = \rho(x)\alpha(x)$  for  $x \in \pi K$ . Here  $\mathbb{F}(t)$  denotes the rational function field over  $\mathbb{F}$  and let *V* be the *n*-dimensional vector space over  $\mathbb{F}(t)$ . Hereafter, we denote the Reidemeister torsion  $\tau_{\rho \otimes \alpha}(E)$  by  $\tau_{\rho \otimes \alpha} K$ .

**Theorem 2.5.** All homology groups  $H_*(E, V_{\rho \otimes \alpha})$  vanish (namely,  $\rho \otimes \alpha$  is an acyclic representation) if and only if det  $M_j \neq 0$  for some j. In this case, we have

$$\tau_{\rho \otimes \alpha} K = \frac{\det M_j}{\det \Phi(x_j - 1)}$$

for any j  $(1 \le j \le u)$ . Moreover,  $\tau_{\rho \otimes \alpha}$  is well-dended up to a factor  $\pm t^{nk}$   $(k \in \mathbb{Z})$  if n is odd and up to only  $t^{nk}$  if n is even.

*Proof.* The first two assertions are nothing but [7], Proposition 3.1. The independence on j follows from [7], Lemma 1.2.

Next, if we consider well-definedness up to  $\pm t^{nk}$   $(k \in \mathbb{Z})$ , we only have to recall Remark 2.4. The image of

$$\det \circ (\rho \otimes \alpha) \colon \pi K \to \operatorname{GL}(n, \mathbb{F}(t)) \to \mathbb{F}(t)^*$$

is just  $\{t^{nk}|k \in \mathbb{Z}\}\)$ , because Im  $\rho$  is included in SL $(n, \mathbb{F})$ . Therefore the claim immediately follows. Here if we take an even dimensional unimodular representation,  $\tau_{\rho\otimes\alpha}$  is well-defined up to  $t^{nk}$  (see also [3] for details).

**Remark 2.6.** This theorem asserts that the twisted Alexander polynomial [18] of a knot is the Reidemeister torsion of its knot exterior (see [7] for details). This is a generalization of Milnor's theorem in [9]. Recently this framework extended to more general situations by Kirk–Livingston in [6].

**Remark 2.7.** Assume that  $\rho$  is a homomorphism to SL(n, R) over a unique factorization domain R and the knot group  $\pi K$  has a presentation which is strongly Tietze equivalent to a Wirtinger presentation of the knot. Then Wada shows in [18] that the twisted Alexander polynomial of the knot associated to  $\rho$  is well-defined up to a factor  $\pm t^{nk}$  ( $k \in \mathbb{Z}$ ) if n is odd and up to only  $t^{nk}$  if n is even.

**Remark 2.8.** If there is an element  $\gamma$  of the commutator subgroup of  $\pi K$  such that 1 is not an eigenvalue of  $\rho(\gamma)$ , then  $\tau_{\rho\otimes\alpha}$  becomes a "polynomial" (see [18]). Namely det  $M_j$  is divided by det  $\Phi(x_j - 1)$ .

#### 3. Main theorem

In this section, we give a necessary condition that a knot K in  $S^3$  is fibered. A polynomial  $a_m t^m + \cdots + a_1 t + a_0 \in \mathbb{F}[t]$  is called monic if the coefficient  $a_m$  is one. We then see from Theorem 2.5 that the notion of monic polynomial makes sense for the Reidemeister torsion.

**Theorem 3.1.** For a **D**bered knot K in  $S^3$  and a unimodular representation  $\rho : \pi K \to SL(2n, \mathbb{F})$ , the Reidemeister torsion  $\tau_{\rho \otimes \alpha} K$  is expressed as a rational function of monic polynomials.

*Proof.* By using the fiber bundle structure, we can take the following presentation of  $\pi K$ :

$$P(\pi K) = \langle x_1, \dots, x_{2g}, h \mid r_i = h x_i h^{-1} \varphi_*(x_i)^{-1}, \ 1 \le i \le 2g \rangle,$$

where  $x_1, \ldots, x_{2g}$  is a generating system of the fundamental group of the fiber surface of genus g, h is a generator for  $S^1$ -direction corresponding to the meridian of K and  $\varphi_*$  denotes the automorphism of the surface group induced by the monodromy map  $\varphi$ . Here the abelianization homomorphism  $\alpha : \pi K \to \mathbb{Z} = \langle t \rangle$  is given by

$$\alpha(x_1) = \cdots = \alpha(x_{2g}) = 1$$
 and  $\alpha(h) = t$ .

This presentation of  $\pi K$  allows us to define another 2-dimensional cell complex *Y* instead of a cell complex *X* constructed from a Wirtinger presentation of  $\pi K$ .

Namely, it has only one vertex, 2g + 1 edges, and 2g 2-cells attached by the relations of  $P(\pi K)$ . It is easy to see that there exists a homotopy equivalence  $f: E \to Y$ . From the result of Waldhausen [19], the Whitehead group  $Wh(\pi K)$  is trivial for a knot group in general. Thus the Whitehead torsion of f is also the trivial element in  $Wh(\pi K)$ . This implies that the homotopy equivalence map f induces a simple homotopy equivalence from E to Y. Since the Reidemeister torsion is a simple homotopy invariant, we can compute the Reidemeister torsion of E as the one of Yas follows. That is, we may use the previous presentation  $P(\pi K)$  to compute  $\tau_{\rho \otimes \alpha} K$ by means of Theorem 2.5.

Let us consider the "big"  $2g \times 2g$  matrix M whose (i, j)th component is the  $2n \times 2n$  matrix

$$\Phi\left(\frac{\partial r_i}{\partial x_j}\right) \in M(2n, \mathbb{F}[t^{\pm 1}]).$$

We then see that the diagonal component of M is

$$\Phi\left(\frac{\partial r_i}{\partial x_i}\right) = \Phi\left(h - \frac{\partial \varphi_*(x_i)}{\partial x_i}\right)$$
$$= t\rho(h) - \tilde{\rho}\left(\frac{\partial \varphi_*(x_i)}{\partial x_i}\right)$$

and the coefficient of the highest degree term of det  $\Phi(\partial r_i/\partial x_i)$  is just det  $\rho(h) = 1$ . Further other components  $\Phi(\partial r_i/\partial x_j)$   $(i \neq j)$  do not contain *t*, so that the coefficient of the highest degree term of det *M* is also one.

On the other hand, the denominator is given by

$$\det \Phi(h-1) = \det(t\rho(h) - I)$$
  
=  $(\det \rho(h))t^{2n} - (\operatorname{tr} \rho(h))t^{2n-1} + \dots + 1$   
=  $t^{2n} + \dots + 1$ ,

where *I* denotes the identity matrix. Moreover,  $\rho$  is an even dimensional representation, so we see that  $\tau_{\rho\otimes\alpha}K$  is well-defined up to a factor  $t^{2nk}$  ( $k \in \mathbb{Z}$ ). This completes the proof.

**Remark 3.2.** If we can show directly that the presentation  $P(\pi K)$  in Theorem 3.1 is strongly Tietze equivalent to a Wirtinger presentation of K, then Theorem 3.1 follows without using the result of Waldhausen (see Remark 2.7).

**Remark 3.3.** If  $\mathbb{F}$  is a subfield of the real number field, then the Reidemeister torsion  $\tau_{\rho\otimes\alpha}K$  for any knot K and any representation  $\rho: \pi K \to SL(2, \mathbb{F})$  is symmetric. Namely,  $\tau_{\rho\otimes\alpha}K$  is invariant under the transformation  $t \mapsto t^{-1}$  up to a factor  $t^k$   $(k \in \mathbb{Z})$ . Such a duality theorem appears originally in [9]. See also [6], [7] and [11] for related works.

# 4. Examples

**Example 4.1.** Let K be the figure eight knot  $4_1$ . This is one of the well-known genus one fibered knots in  $S^3$ . The fundamental group of the exterior has a presentation

$$\pi K = \langle x, y \mid z x z^{-1} y^{-1} \rangle,$$

where  $z = x^{-1}yxy^{-1}x^{-1}$ . Let  $\rho : \pi K \to SL(2, \mathbb{C})$  be a noncommutative representation defined by

$$\rho(x) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } \rho(y) = \begin{pmatrix} 1 & 0 \\ -\omega & 1 \end{pmatrix},$$

where  $\omega$  is a complex number satisfying  $\omega^2 + \omega + 1 = 0$ . As pointed out by Wada in [18], it is convenient to use relations instead of relators for the computation of the Alexander matrix. Thereby applying free differential calculus to the relation r: zx = yz, we obtain

$$\frac{\partial r}{\partial x} = -x^{-1} + x^{-1}y + yx^{-1} - yx^{-1}y + yx^{-1}yxy^{-1}x^{-1}.$$

Thus we have the matrix

$$M_2 = \left(\Phi\left(\frac{\partial r}{\partial x}\right)\right) = \begin{pmatrix} -(\omega+1)t + \omega + 2 - t^{-1} & t + \omega - 2 + t^{-1} \\ (\omega-1)t - \omega + 1 & -(\omega+1)t + 3 - t^{-1} \end{pmatrix}.$$

Then the numerator of  $\tau_{\rho\otimes\alpha}$  is given by

det 
$$M_2 = t^{-2}(t^4 - 6t^3 + \omega^4 t^2 + \omega^2 t^2 + 11t^2 - 6t + 1)$$
  
=  $t^{-2}(t-1)^2(t^2 - 4t + 1)$ .

On the other hand, the denominator of  $\tau_{\rho\otimes\alpha}$  is given by

$$\det \Phi(y-1) = \det(t\rho(y) - I)$$
$$= \det \begin{pmatrix} t-1 & 0 \\ -\omega t & t-1 \end{pmatrix}$$
$$= (t-1)^2.$$

Therefore the Reidemeister torsion of the figure eight knot K is

$$\tau_{\rho\otimes\alpha}K = t^2 - 4t + 1,$$

and this is in fact a monic polynomial.

**Example 4.2.** Let KT be the Kinoshita–Terasaka knot [5]. It is well-known that KT is one of the classical examples of knots with the trivial Alexander polynomial. The knot group  $\pi$ KT has a presentation with four generators  $x_1, \ldots, x_4$  and three relations [18]:

$$r_{1}: x_{1}x_{2}x_{1}^{-1} = x_{4}x_{2}x_{4}x_{2}^{-1}x_{4}^{-1},$$
  

$$r_{2}: x_{4}x_{2}x_{4}^{-1} = x_{2}^{-1}x_{3}x_{1}x_{3}^{-1}x_{2}x_{1}x_{2}^{-1}x_{3}x_{1}^{-1}x_{3}^{-1}x_{2},$$
  

$$r_{3}: x_{1}x_{3}x_{1}^{-1} = x_{4}x_{3}x_{4}x_{3}^{-1}x_{4}^{-1}.$$

Applying free differential calculus, we have

$$\begin{aligned} \frac{\partial r_1}{\partial x_1} &= 1 - x_1 x_2 x_1^{-1}, \quad \frac{\partial r_1}{\partial x_2} = x_1 - x_4 + x_4 x_2 x_4 x_2^{-1}, \quad \frac{\partial r_1}{\partial x_3} = 0, \\ \frac{\partial r_2}{\partial x_1} &= -x_2^{-1} x_3 - x_2^{-1} x_3 x_1 x_3^{-1} x_2 + x_2^{-1} x_3 x_1 x_3^{-1} x_2 x_1 x_2^{-1} x_3 x_1^{-1}, \\ \frac{\partial r_2}{\partial x_2} &= x_4 + x_2^{-1} - x_2^{-1} x_3 x_1 x_3^{-1} + x_2^{-1} x_3 x_1 x_3^{-1} x_2 x_1 x_2^{-1} \\ &\quad - x_2^{-1} x_3 x_1 x_3^{-1} x_2 x_1 x_2^{-1} x_3 x_1^{-1} x_3^{-1}, \\ \frac{\partial r_2}{\partial x_3} &= -x_2^{-1} + x_2^{-1} x_3 x_1 x_3^{-1} - x_2^{-1} x_3 x_1 x_3^{-1} x_2 x_1 x_2^{-1} \\ &\quad + x_2^{-1} x_3 x_1 x_3^{-1} x_2 x_1 x_2^{-1} x_3 x_1^{-1} x_3^{-1}, \\ \frac{\partial r_3}{\partial x_1} &= 1 - x_1 x_3 x_1^{-1}, \quad \frac{\partial r_3}{\partial x_2} = 0, \quad \frac{\partial r_3}{\partial x_3} = x_1 - x_4 + x_4 x_3 x_4 x_3^{-1}. \end{aligned}$$

Let  $\rho: \pi KT \to SL(2, \mathbb{F}_5)$  be a noncommutative representation over the finite field  $\mathbb{F}_5$  defined as follows:

$$\rho(x_1) = \begin{pmatrix} 0 & 1 \\ 4 & 1 \end{pmatrix}, \ \rho(x_2) = \begin{pmatrix} 0 & 4 \\ 1 & 1 \end{pmatrix}, \ \rho(x_3) = \begin{pmatrix} 0 & 1 \\ 4 & 1 \end{pmatrix} \text{ and } \rho(x_4) = \begin{pmatrix} 4 & 4 \\ 3 & 2 \end{pmatrix}.$$

Then we obtain

$$M_4 = \begin{pmatrix} 3t+1 & t & t & t^2+2t & 0 & 0\\ 2t & t+1 & 4t^2+t & 4t & 0 & 0\\ 1 & 4t+3 & t+2+t^{-1} & t+t^{-1} & 3t+3+4t^{-1} & 3t+4t^{-1}\\ 4t & 2t+1 & t+1+4t^{-1} & 3t+4 & 2t+4+t^{-1} & 4t+1\\ 1 & 4t & 0 & 0 & 3t^2+t & 2t^2+2t\\ t & 4t+1 & 0 & 0 & 4t^2+t & 3t^2+4t \end{pmatrix}.$$

Therefore the Reidemeister torsion of KT is given by

$$\tau_{\rho\otimes\alpha} \mathrm{KT} = \frac{\det M_4}{\det \Phi(x_4 - 1)}$$
  
=  $\frac{t^2(4t^8 + t^7 + t^6 + 4t^5 + 3t^4 + 4t^3 + t^2 + t + 4)}{t^2 + 4t + 1}$   
=  $4t^6 + 2t^4 + t^3 + 2t^2 + 4.$ 

This is well-defined up to a factor  $t^{2k}$   $(k \in \mathbb{Z})$ , so that we may conclude that the Kinoshita–Terasaka knot KT is not fibered.

**Example 4.3.** Let K be the knot illustrated in Figure 1. The normalized Alexander polynomial of K is equal to the monic polynomial  $t^4 - t^3 + t^2 - t + 1$ . The knot group  $\pi K$  has a presentation with seven generators  $x_1, \ldots, x_7$  and six relations:

$$r_{1}: x_{2}x_{1} = x_{3}x_{2}x_{1}x_{2}x_{1}^{-1}x_{2}^{-1},$$

$$r_{2}: x_{6}x_{5}x_{6}^{-1} = x_{4}x_{3}x_{1}^{-1}x_{3}x_{1}x_{3}^{-1}x_{1}x_{3}^{-1}x_{1}x_{3}^{-1}x_{4}^{-1},$$

$$r_{3}: x_{6}x_{7}x_{6}^{-1} = x_{4}x_{3}x_{1}^{-1}x_{3}x_{1}^{-1}x_{3}x_{1}x_{3}^{-1}x_{1}x_{3}^{-1}x_{4}^{-1},$$

$$r_{4}: x_{5}x_{6}x_{5}^{-1} = x_{7}x_{2}x_{7}^{-1},$$

$$r_{5}: x_{2}x_{6}x_{2}^{-1} = x_{3}x_{2}x_{1}x_{2}x_{1}^{-1}x_{2}^{-1}x_{3}^{-1}x_{7}x_{3}x_{2}x_{1}x_{2}^{-1}x_{1}^{-1}x_{2}^{-1}x_{3}^{-1},$$

$$r_{6}: x_{5}x_{4}x_{5}^{-1}x_{7} = x_{7}x_{3}x_{2}x_{1}x_{2}x_{1}^{-1}x_{2}^{-1}x_{3}^{-1}.$$



Figure 1

Let  $\rho : \pi K \to SL(2, \mathbb{F}_5)$  be a noncommutative representation over  $\mathbb{F}_5$  defined as follows:

$$\rho(x_1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \rho(x_2) = \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}, \quad \rho(x_3) = \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}, \quad \rho(x_4) = \begin{pmatrix} 2 & 1 \\ 4 & 0 \end{pmatrix},$$

$$\rho(x_5) = \begin{pmatrix} 2 & 4 \\ 1 & 0 \end{pmatrix}, \quad \rho(x_6) = \begin{pmatrix} 3 & 1 \\ 1 & 4 \end{pmatrix} \quad \text{and} \quad \rho(x_7) = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}.$$

By the same method as in previous examples, we have the following Reidemeister torsion of K:

$$\tau_{\rho\otimes\alpha}K = \frac{\det M_7}{\det \Phi(x_7 - 1)}$$
$$= \frac{t^{12}(3t^4 + 4t^3 + t^2 + 4t + 3)}{t^2 + 3t + 1}$$
$$= 3t^2 + 3.$$

Hence this knot K is not fibered.

We use Kodama's program "KNOT" and Wada's program to compute these examples. The former is to obtain N-data (see [17]) from a knot projection, which serve as necessary input data for Wada's program. The latter one is to compute unimodular representations over finite fields of knot groups from N-data. Here we should remark that Kodama's program works on Linux while the program of Wada works on Macintosh.

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### References

- J. C. Cha, Fibred knots and twisted Alexander invariants. *Trans. Amer. Math. Soc.* 355 (2003), 4187–4200. Zbl 1028.57004 MR 1990582
- [2] B. Jiang and S. Wang, Twisted topological invariants associated with representations. In *Topics in knot theory*, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. 399, Kluwer Academic Publishers, Dordrecht 1993, 211–227. Zbl 0815.55001 MR 1257911
- [3] D. Johnson, A geometric form of Casson's invariant and its connection to Reidemeister torsion. Unpublished lecture notes.
- [4] T. Kanenobu, The augmentation subgroup of a pretzel link. Math. Sem. Notes Kobe Univ. 7 (1979), 363–384. Zbl 0456.57002 MR 0557309
- [5] S. Kinoshita and H. Terasaka, On unions of knots. Osaka Math. J. 9 (1957), 131–153.
   Zbl 0080.17001 MR 0098386
- [6] P. Kirk and C. Livingston, Twisted Alexander invariants, Reidemeister torsion, and Casson-Gordon invariants. *Topology* 38 (1999), 635–661. Zb1 0928.57005 MR 1670420

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- [7] T. Kitano, Twisted Alexander polynomial and Reidemeister torsion. Paciac J. Math. 174 (1996), 431–442. Zbl 0863.57001 MR 1405595
- [8] X. S. Lin, Representations of knot groups and twisted Alexander polynomials. Acta Math. Sin. (Engl. Ser.) 17 (2001), 361–380. Zbl 0986.57003 MR 1852950
- [9] J. Milnor, A duality theorem for Reidemeister torsion. Ann. of Math. (2) 76 (1962), 137–147. Zbl 0108.36502 MR 0141115
- [10] J. Milnor, Whitehead torsion. Bull. Amer. Math. Soc. 72 (1966), 358–426. Zbl 0147.23104 MR 0196736
- T. Morifuji, Twisted Alexander polynomial for the braid group. Bull. Austral. Math. Soc. 64 (2001), 1–13. Zb1 1008.20028 MR 1848073
- [12] K. Murasugi, On a certain subgroup of the group of an alternating link. Amer. J. Math. 85 (1963), 544–550. Zbl 0117.17201 MR 0157375
- [13] L. Neuwirth, Knot Groups. Ann. of Math. Stud. 56, Princeton University Press, Princeton, N.J., 1965. Zbl 0184.48903 MR 0176462
- [14] E. Rapaport, On the commutator subgroup of a knot group. Ann. of Math. (2) 71 (1960), 157–162. Zbl 0102.26103 MR 0116047
- [15] L. C. Siebenmann, A total Whitehead torsion obstruction to fibering over the circle. Comment. Math. Helv. 45 (1970), 1–48. Zbl 0215.24603 MR 0287564
- [16] J. Stallings, On fibering certain 3-manifolds. In *Topology of 3-manifolds and related topics* (Proc. The Univ. of Georgia Institute, 1961), Prentice-Hall, Englewood Cliffs, N.J., 1962, 95–100. Zbl 0132.20102 MR 0158375
- [17] M. Wada, Coding link diagrams. J. Knot Theory Ramiacations 2 (1993), 233–237.
   Zbl 0808.57001 MR 1227012
- [18] M. Wada, Twisted Alexander polynomial for finitely presentable groups. *Topology* 33 (1994), 241–256. Zbl 0822.57006 MR 1273784
- [19] F. Waldhausen, Algebraic K-theory of generalized free products. I, II. Ann. of Math. (2) 108 (1978), 135–204. Zbl 0397.18012 MR 0498807

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