Vector Cross Products.

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1. Introduction

Let V denote an n-dimensional vector space over the real numbers and $(\,,\,)$ the ordinary (positive definite) inner product. ECKMANN [3] has defined a *vector cross product* on V to be a continuous map

$$X: V^r \to V \qquad (1 \leqslant r \leqslant n)$$

satisfying the following axioms

$$(X(a_1, \ldots, a_r), a_i) = 0, \qquad 1 \leqslant i \leqslant r. \tag{1.1}$$

$$(X(a_1, \ldots, a_r), X(a_1, \ldots, a_r)) = \det((a_i, a_j)). \tag{1.2}$$

The following theorem has been proved by ECKMANN and WHITEHEAD ([3], [5]):

THEOREM A: *A vector cross product exists in precisely the following cases*:

$$n \text{ is even}, \qquad r = 1. \tag{1.3}$$

$$n \text{ is arbitrary}, \qquad r = n - 1. \tag{1.4}$$

$$n = 3 \text{ or } 7, \qquad r = 2. \tag{1.5}$$

$$n = 4 \text{ or } 8, \qquad r = 3. \tag{1.6}$$

ECKMANN and WHITEHEAD proved theorem A be means of algebraic topology. In the present paper we consider multilinear vector cross products on vector spaces over arbitrary fields of characteristic not two and with respect to an arbitrary non-degenerate symmetric bilinear form $\langle\,,\,\rangle$. We classify all such vector cross products. Here classification means the following:

the determination of the bilinear forms and the values of n and r for which vector cross products exist, and (1.7)

for a fixed bilinear form and fixed n and r, the determination of all isomorphism classes of vector cross products. (1.8)

For (1.7) since it is impossible to classify all quadratic forms, we give necessary and

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sufficient conditions that a bilinear form have a vector cross product associated with it. In (1.8) we say that two vector cross products X and X' defined with respect to the same bilinear form are *isomorphic* if and only if there exists a linear map $\Phi: V \to V$ satisfying

$$\langle \Phi a, \Phi b \rangle = \langle a, b \rangle \qquad \text{for all} \qquad a, b \in V. \tag{1.9}$$

$$\Phi(X(a_1, \ldots, a_r)) = X'(\Phi a_1, \ldots, \Phi a_r). \tag{1.10}$$

We find that vector cross products exist only in the cases listed in theorem A. However, there are many more bilinear forms besides the positive definite one (which was the only one considered by Eckmann and Whitehead) which possess a vector cross product. In contrast to Eckmann's and Whitehead's method, our technique is completely algebraic.

In the next four sections we classify the vector cross products falling into each of the cases (1.3)–(1.6). Then in section six we show that no other vector cross products exist. Finally in section seven we determine the vector cross products explicitly for certain fields for which the classification of quadratic forms is known.

2. Almost Complex Structures

In this section we consider the case $r=1$. An *almost complex structure* on V is a linear transformation $J: V \to V$ satisfying $J^2 = -1_V$.

Proposition (2.1): *A one-fold vector cross product on V is an almost complex structure which is an isometry relative to $\langle\ ,\ \rangle$. The converse is also true.*

Proof: A one-fold vector cross product is a linear transformation satisfying

$$\langle X a, b \rangle + \langle a, X b \rangle = 0 \qquad \text{and} \qquad \langle X a, X b \rangle = \langle a, b \rangle \tag{2.1}$$

for all $a, b \in V$; this follows from linearization of (1.1) and (1.2). From (2.1) it follows easily that $X^2 = -1_V$.

Conversely let $J: V \to V$ be an almost complex structure satisfying $\langle Ja, Jb \rangle = \langle a, b \rangle$. Then (1.2) is automatically satisfied, and furthermore

$$\langle J a, a \rangle = \langle J^2 a, J a \rangle = -\langle a, J a \rangle$$

and so $\langle Ja, a \rangle = 0$; thus (1.1) is also satisfied.

Theorem (2.2): (i) *The non-degenerate symmetric bilinear form $\langle\ ,\ \rangle$ possesses a one-fold vector cross product X if and only if the quadratic form of $\langle\ ,\ \rangle$ has the form*

$$\alpha_1 (x_1^2 + x_2^2) + \cdots + \alpha_m (x_{2m-1}^2 + x_{2m}^2), \tag{2.2}$$

where $\alpha_1, \ldots, \alpha_m \in F$, the ground field.

(ii) *Any two one-fold vector cross products X and X' defined with respect to $\langle\, ,\rangle$ are isomorphic.*

Proof: That the dimension of V is even follows from the fact that an orthogonal basis of V of the form $\{e_1, Xe_1, e_2, Xe_2, \ldots, e_m, Xe_m\}$ ($2m=n=$dimension V) can be chosen. With respect to such a basis the quadratic form of $\langle\, ,\rangle$ has the form (2.2.).

Conversely let $\{e_1, f_1, e_2, f_2, \ldots, e_m, f_m\}$ be an orthogonal basis of V such that

$$\langle e_i, e_i\rangle = \langle f_i, f_i\rangle = \alpha_i \quad \text{for} \quad i = 1, \ldots, m.$$

Define X by $Xe_i = f_i$ and $Xf_i = -e_i$. Then X is almost complex structure and hence a one-fold vector cross product.

For (ii) we choose orthogonal bases of V of the form

$$\{e_1, X e_1, \ldots, e_m, X e_m\} \quad \text{and} \quad \{f_1, X' f_1, \ldots, f_m, X' f_m\}$$

Define $\Phi\colon V \to V$ by $\Phi(e_i) = f_i$ and $\Phi(Xe_i) = X' f_i$ for $i = 1, \ldots, m$. Then Φ satisfies (1.9) and (1.10).

3. Star Operators

We show in this section that $(n-1)$-fold vector cross products are essentially determined by the star operator of $\langle\, ,\rangle$. The star operator, an important object of differential geometry, is usually defined for bilinear forms on real vector spaces. Therefore we first give the general definition for vector spaces over arbitrary fields and establish some elementary properties of star operators that we shall need.

Let ΛV denote the exterior algebra over V and write $\Lambda V = \sum_{p=0}^{n} \Lambda^p V$ where $\Lambda^p V$ consists of the elements in ΛV of degree p. As usual we extend $\langle\, ,\rangle$ to ΛV by linearity and the formula

$$\langle a_1 \wedge \ldots \wedge a_p, b_1 \wedge \ldots \wedge b_q\rangle = \begin{cases} \det(\langle a_i, b_j\rangle), & \text{if} \quad p = q \\ 0 \quad , & \text{if} \quad p \neq q, \end{cases}$$

for $a_1, \ldots, a_p, b_1, \ldots, b_q \in V$.

This extension is symmetric and non-degenerate. The vector space $\Lambda^n V$ is 1-dimensional, and we pick once and for all a basis element $\omega \in \Lambda^n V$. The *star operator* $*\colon \Lambda V \to \Lambda V$ (corresponding to $\langle\, ,\rangle$ and ω) is defined as follows. We require that $*(\Lambda^p V) \subseteq \Lambda^{n-p} V$; if $a \in \Lambda^p V$ then $*a \in \Lambda^{n-p} V$ is given by the formula

$$\langle *a, b\rangle = \langle a \wedge b, \omega\rangle \quad \text{for all} \quad b \in \Lambda^{n-p} V.$$

Finally $*$ is defined on all of ΛV by linearity.

PROPOSTIONI (3.1): *Let $a \in \Lambda^p V$; then*

$$**a = (-1)^{p(n-p)} \langle \omega, \omega\rangle\, a. \tag{3.1}$$

$$\langle *a, *b\rangle = \langle \omega, \omega\rangle \langle a, b\rangle. \tag{3.2}$$

(3.3) *Let $\dot{F}$ denote the multiplicative group of F. The element of $\dot{F}/\dot{F}^2$ represented by $\langle\omega,\omega\rangle$ is the discriminant of $\langle\ ,\ \rangle$.*

Proof: To prove (3.1) we may assume $a=\sigma e_1\wedge\cdots\wedge e_p$ where $\{e_1,\ldots,e_n\}$ is an orthogonal basis of V such that $\omega=e_1\wedge\cdots\wedge e_n$. Write $\alpha_i=\langle e_i,e_i\rangle$; then $\langle\omega,\omega\rangle=\alpha_1\ldots\alpha_n$. It is not hard to see that

$$^*a=\sigma\alpha_1\ldots\alpha_p e_{p+1}\wedge\cdots\wedge e_n$$

so that

$$^{**}a=\sigma\alpha_1\ldots\alpha_p^*(e_{p+1}\wedge\cdots\wedge e_n)=\sigma\alpha_1\ldots\alpha_n(-1)^{p(n-p)}e_1\wedge\cdots\wedge e_p$$
$$=(-1)^{p(n-p)}\langle\omega,\omega\rangle a\,.$$

For (3.2) we have

$$\langle{}^*a,{}^*b\rangle=\langle a\wedge{}^*b,\omega\rangle=(-1)^{p(n-p)}\langle{}^*b\wedge a,\omega\rangle$$
$$=(-1)^{p(n-p)}\langle{}^{**}b,a\rangle=\langle\omega,\omega\rangle\langle a,b\rangle\,.$$

Finally $\langle\omega,\omega\rangle$ represents the discriminant of $\langle\ ,\ \rangle$ since $\langle\omega,\omega\rangle=\alpha_1\ldots\alpha_n$.

Before classifying $(n-1)$-fold vector cross products we note the following useful lemma.

LEMMA (3.2): *Let X be an r-fold vector cross product and define $\varphi\colon V^{r+1}\to F$ by $\varphi(a_1,\ldots,a_{r+1})=\langle X(a_1,\ldots,a_r),a_{r+1}\rangle$. Then both φ and X are skew symmetric*; *hence we may regard φ as a map $\varphi\colon\Lambda^{r+1}V\to F$ and X as a map $X\colon\Lambda^r V\to V$.*

Proof: This follows from linearization of (1.1).

THEOREM (3.3): (i) *A necessary and sufficient condition that $\langle\ ,\ \rangle$ possess an $(n-1)$-fold vector cross product is that the discriminant of $\langle\ ,\ \rangle$ be* 1.

(ii) *If the discriminant of $\langle\ ,\ \rangle$ is* 1, *then any $(n-1)$-fold vector cross product is given as follows*: *there exists $\omega\in\Lambda^n V$ with $\langle\omega,\omega\rangle=1$ such that*

$$X(a_1,\ldots,a_{n-1})={}^*(a_1\wedge\cdots\wedge a_{n-1}) \tag{3.4}$$

for all $a_1,\ldots,a_{n-1}\in V$.

(iii) *There are exactly two distinct $(n-1)$-fold vector cross products on V and they are isomorphic to each other.*

Proof: For (i) we only prove the necessity, since the sufficiency is proved in (ii). Choose an orthogonal basis $\{e_1,\ldots,e_n\}$ such that $\omega=e_1\wedge\cdots\wedge e_n$. We may write

$$X(e_1,\ldots,e_{n-1})=\varrho\,e_n$$

for some $\varrho\in F$. Then

$$\langle\omega,\omega\rangle=\alpha_1\ldots\alpha_n=\langle X(e_1,\ldots,e_{n-1}),X(e_1,\ldots,e_{n-1})\rangle\,\alpha_n=\varrho^2\alpha_n^2\,.$$

For (ii) it is obvious that if the discriminant of $\langle\ ,\ \rangle$ is 1 we may choose $\omega\in\Lambda^n V$ with $\langle\omega,\omega\rangle=1$. Moreover only ω and $-\omega$ have this property. We may write

$$X(e_1,\ldots,\hat{e}_i,\ldots,e_n)=\xi_i^*(e_1\wedge\cdots\wedge\hat{e}_i\wedge\cdots\wedge e_n)=(-1)^{n+i}\xi_i\alpha_1\ldots\hat{\alpha}_i\ldots\alpha_n e_i\,.$$

$$\begin{aligned}\text{Then } 1 &= (\alpha_1 \ldots \hat{\alpha}_i \ldots \alpha_n)^{-1} \langle X(e_1, \ldots, \hat{e}_i, \ldots, e_n), X(e_1, \ldots, \hat{e}_i, \ldots, e_n)\rangle \\ &= \xi_i(-1)^{n+i} \langle X(e_1, \ldots, \hat{e}_i, \ldots, e_n), e_i\rangle = \xi_i \varphi(e_1, \ldots, e_n) \\ &= \xi_i(-1)^{n+j} \langle X(e_1, \ldots, \hat{e}_j, \ldots, e_n), e_j\rangle = \xi_i \xi_j, \quad \text{for all } i \text{ and } j.\end{aligned}$$

Thus $\xi_1 = \cdots = \xi_n = \pm 1$. If all the ξ_i's are $+1$, we get (3.4) with the star operator corresponding to ω; if the ξ_i's are -1, we get (3.4) with the star operator corresponding to $-\omega$.

For (iii) we see from the proof of (ii) that if X is an $(n-1)$-fold vector cross product the only other one is $X' = -X$. If $\{e_1, \ldots, e_n\}$ is an orthogonal basis of V we define an isomorphism Φ between X and X' by $\Phi(e_i) = e_i$ $(1 \le i < n)$ and $\Phi(e_n) = -e_n$.

4. Two-Fold Vector Cross Products

Two-fold and three-fold vector cross products are intimately related to composition algebras and so we first note a few facts about them. We omit most of the proofs since they are easily accessible in JACOBSON [4, pp. 55–62].

DEFINITION. *A composition algebra W* is an algebra equipped with a quadratic form N such that

(i) the bilinear form $\langle x, y\rangle = \frac{1}{2}(N(x+y) - N(x) - N(y))$ is non-degenerate;

(ii) there is an element $e \in W$ such that $ex = xe = x$ for all $x \in W$;

(iii) for all $x, y \in W$, $N(xy) = N(x) N(y)$.

It is known that any composition algebra W satisfies the alternative laws $x^2 y = x(xy)$ and $xy^2 = (xy) y$ and possesses an involution $x \to \bar{x}$ satisfying

$$\bar{\bar{x}} = x, \quad \overline{xy} = \bar{y}\bar{x}, \quad x\bar{x} = \bar{x}x = N(x)\, e, \quad 2\langle x, y\rangle = x\bar{y} + y\bar{x}. \tag{4.1}$$

From (4.1) it follows that

$$x^2 - 2\langle x, e\rangle x + \langle x, x\rangle e = 0. \tag{4.2}$$

Furthermore

$$\langle wy, zx\rangle + \langle wx, zy\rangle = 2\langle w, z\rangle \langle x, y\rangle \quad \text{and} \tag{4.3}$$

$$\langle wx, y\rangle = \langle x, \bar{w} y\rangle = \langle w, y\bar{x}\rangle. \tag{4.4}$$

From (4.3) and (4.4) it follows that

$$\langle \bar{x}, \bar{y}\rangle = \langle x, y\rangle \quad \text{and} \tag{4.5}$$

$$2\langle x, y\rangle \langle z, y\rangle = \langle z, x\rangle \langle y, y\rangle + \langle \bar{y} z, \bar{x} y\rangle. \tag{4.6}$$

The dimension of coumposition algebra W is 1, 2, 4, or 8, and W is not associative if $\dim W = 8$. Finally, it is known that two composition algebras are isomorphic if and only if their bilinear forms are equivalent.

Now we study two-fold vector cross products.

THEOREM (4.1): (i) *Let W be a composition algebra and let $V \subset W$ be the orthogonal complement of the identity e. Define X: $V \times V \to V$ by*

$$X(a, b) = a\,b + \langle a, b\rangle\, e\,. \tag{4.7}$$

Then X is a two-fold vector cross product on V.

(ii) *Conversely if X is a two-fold vector cross product on V, let W be a vector space containing V as a subspace of codimension 1. Let $W = V \oplus \{e\}$ for some $e \in W \setminus V$ and extend $\langle\ ,\ \rangle$ to W by requiring $\langle e, e\rangle = 1$ and $\langle e, V\rangle = 0$. Define a multiplication in W by* (4.7) *and set $e x = x e = e$ for $x \in W$. Finally let $N(x) = \langle x, x\rangle$ for $x \in W$. Then W is a composition algebra and the dimension of the original space V is either* 3 *or* 7.

(iii) *Two two-fold vector cross products are isomorphic if and only if their corresponding composition algebras are isomorphic. The converse is also true.*

Proof: To prove (i) we first note that it follows from (4.4) that the range of X (defined by (4.7)) is contained in V. From (4.4) and the fact that $\bar{a} = -a$ for $a \in V$ we also find that $\langle X(a, b), a\rangle = \langle X(a, b), b\rangle = 0$ for $a, b \in V$. Finally

$$\begin{aligned}\langle X(a, b), X(a, b)\rangle &= \langle a\,b, a\,b\rangle + 2\langle a, b\rangle\langle a\,b, e\rangle + \langle a, b\rangle^2 \\ &= \langle a, a\rangle\langle b, b\rangle - \langle a, b\rangle^2,\end{aligned}$$

again by (4.4).

To prove (ii) it suffices to show $N(xy) = N(x)\,N(y)$ for $x, y \in W$. Write $x = a + \alpha e$, $y = b + \beta e$, where $a, b \in V$ and $\alpha, \beta \in F$. Then

$$\begin{aligned}N((a + \alpha e)(b + \beta e)) &= N(X(a, b) + \beta a + \alpha b + (\alpha\beta - \langle a, b\rangle)\, e) \\ &= (N(a)\,N(b) - \langle a, b\rangle)^2 + \beta^2 N(a) + \alpha^2 N(b) + 2\alpha\beta\langle a, b\rangle - (\alpha\beta - \langle a, b\rangle^2) \\ &= N(a)\,N(b) + \beta^2 N(a) + \alpha^2 N(b) + \alpha^2\beta^2 \\ &= N(a + \alpha e)\,N(b + \beta e).\end{aligned}$$

For (iii) let X and X' be isomorphic two-fold vector cross products and W and W' the associated composition algebras. The bilinear forms on W and W' are equivalent and so W and W' are isomorphic. The converse is also easily proved.

5. Three-Fold Vector Cross Products

In this section we show that three-fold vector cross products arise from composition algebras in a fashion similar to that of the two-fold case. We first exhibit two kinds of three-fold products. Later on we show that each three-fold product is one of these kinds.

THEOREM 5.1: *Let V be a composition algebra with bilinear form $\langle\ ,\ \rangle_1$ and let $\alpha \neq 0$ be a field element. Then*

$$X(a, b, c) = \alpha(-a(\bar{b}c) + \langle a, b\rangle_1 c + \langle b, c\rangle_1 a - \langle c, a\rangle_1 b) \tag{5.1}$$

and $$X(a,b,c)=\alpha(-(a\bar{b})c+\langle a,b\rangle_1 c+\langle b,c\rangle_1 a-\langle c,a\rangle_1 b) \tag{5.2}$$

are both three-fold vector cross products with respect to the bilinear form $\langle\ ,\ \rangle=\alpha\langle\ ,\ \rangle_1$.

Proof. Using properties (4.4), (4.5), (4.6), we easily check that (5.1) and (5.2) are vector cross-products with respect to $\langle\ ,\ \rangle$ (*cf.* [6]).

THEOREM 5.2: *Let X be a three-fold vector cross product on a vector space V with respect to a bilinear form $\langle\ ,\ \rangle$. If e is any element of V with $\langle e,e\rangle\neq 0$, then on V there is a multiplication (depending on X) such that*

(i) *e is a two-sided identity,*

(ii) *V is a composition algebra with respect to $\langle\ ,\ \rangle=\langle e,e\rangle^{-1}\langle\ ,\ \rangle$,*

(iii) *X satisfies either* (5.1) *or* (5.2) *with $\alpha=\langle e,e\rangle$.*

Proof. We define multiplication on V by

$$ac=\langle e,e\rangle^{-1}(-X(a,e,c)+\langle a,e\rangle c+\langle e,c\rangle a-\langle c,a\rangle e). \tag{5.3}$$

Substituting e for a or c shows that e is a two-sided identity. To show that V is a composition algebra with respect to this multiplication and the bilinear form $\langle\ ,\ \rangle_1$, it is only necessary to check that $\langle ac,ac\rangle_1=\langle a,a\rangle_1\langle c,c\rangle_1$. We have

$$\begin{aligned}\langle ac,ac\rangle_1 &= \langle e,e\rangle^{-3}\{\langle X(a,e,c),X(a,e,c)\rangle-2\langle X(a,e,c),\langle a,e\rangle c\rangle\\ &\quad-2\langle X(a,e,c),\langle e,c\rangle a\rangle+2\langle X(a,e,c),\langle c,a\rangle e\rangle+\langle a,e\rangle^2\langle c,c\rangle\\ &\quad+2\langle a,e\rangle\langle e,c\rangle\langle c,a\rangle-2\langle a,e\rangle\langle c,a\rangle\langle c,e\rangle+\langle e,c\rangle^2\langle a,a\rangle\\ &\quad-2\langle e,c\rangle\langle c,a\rangle\langle a,e\rangle+\langle c,a\rangle^2\langle e,e\rangle\}\\ &=\langle e,e\rangle^{-3}\begin{vmatrix}\langle a,a\rangle & \langle a,e\rangle & \langle a,c\rangle\\ \langle e,a\rangle & \langle e,e\rangle & \langle e,c\rangle\\ \langle c,a\rangle & \langle c,e\rangle & \langle c,c\rangle\end{vmatrix}\\ &\quad+\langle e,e\rangle^{-3}\{\langle a,e\rangle^2\langle c,c\rangle+\langle e,c\rangle^2\langle a,a\rangle\\ &\quad-2\langle e,c\rangle\langle c,a\rangle\langle a,e\rangle+\langle c,a\rangle^2\langle e,e\rangle\}\\ &=\langle e,e\rangle^{-3}\langle a,a\rangle\langle e,e\rangle\langle c,c\rangle=\langle a,a\rangle_1\langle c,c\rangle_1.\end{aligned}$$

Next we show that either (5.1) or (5.2) is true. By the definition (5.3), we know that (5.1) and (5.2) are both true for $b=e$. Then by (5.3) and the skew-symmetry of X, (5.1) and (5.2) are true if a, b, or $c=e$. Thus, by linearity we may next assume that a, b, and c are orthogonal to e and divide the investigation into the two cases which follow.

Case I. a, b, c are orthogonal to e and lie in a four-dimensional subalgebra W. By linearity we may assume that e, a, b, c are mutually orthogonal, and then we may as well assume that $c=-X(a,e,b)=\langle e,e\rangle ab$. If $n=4$, by (1.1), $X(a,b,c)=\beta e$ for some β. Then

$$\begin{aligned}\beta=\langle\beta e,e\rangle_1 &=\langle X(a,b,c),e\rangle_1=\langle X(a,e,b),c\rangle_1\\ &=\langle-\langle e,e\rangle ab,\langle e,e\rangle ab\rangle_1=-\langle e,e\rangle^2\langle a,a\rangle_1\langle b,b\rangle_1.\end{aligned}$$

On the other hand

$$-\alpha a(\bar{b}c) = \langle e, e\rangle^2 a(b(ab)) = -\langle e, e\rangle^2 a^2 b^2 = -\langle e, e\rangle^2 \langle a, a\rangle_1 \langle b, b\rangle_1 e,$$

which finishes this case if $n=4$. Of course, if $n=4$, the proof of theorem (5.2) ends here, and both (5.1) and (5.2) hold because V is associative.

If $n=8$ the foregoing argument will still be valid provided that $X(a, b, c)=\beta e$ for some β. By (1.1) we know only that $X(a, b, c)=\beta e+d$, where $(d, W)=0$. Suppose g is any element such that $(g, W)=0$. Linearizing (1.2) with respect to a_1 and then a_2 yields

$$\langle X(a, b, c), X(gc, e, c)\rangle + \langle X(gc, b, c), X(a, e, c)\rangle = 0.$$

Since $X(a, e, c)=-\langle e, e\rangle ac=\gamma b$ for some γ, the second term on the left is zero. Hence $X(a, b, c)$ is orthogonal to $(gc)\,c$, which is a multiple of g; that is, if $(g, W)=0$, then $\langle \beta e+d, g\rangle=0$. Hence $d=0$.

In turning to the other case, the following definition will be convenient.

DEFINITION. Let V be an eight-dimensional composition algebra. Let $a\in e^{\perp}$ with $\langle a, a\rangle\neq 0$; $b\in(e, a)^{\perp}$ with $\langle b, b\rangle\neq 0$; $c\in(e, a, b, ab)^{\perp}$ with $\langle c, c\rangle\neq 0$. Then it is known [4] that a, b, c generate V; that $a(bc)=-(ab)\,c$; and that an orthogonal basis for V is given by $e, a, b, ab, c, ac, bc, a(bc)$. We shall call a, b, c so chosen a *normal set of generators* of V.

Case II. a, b, c are orthogonal to e but do not lie in a four-dimensional subalgebra. By linearity we may then assume that a, b, c is a normal set of generators of V. Then by case I $\langle X(a, b, c), ab\rangle=-\langle X(a, b, ab), c\rangle=0$, and similarly $\langle X(a, b, c), ac\rangle=\langle X(a, b, c), bc\rangle=0$. Also $\langle X(a, b, c), e\rangle=\langle X(a, e, b), c\rangle=0$. Hence by (1.1), $X(a, b, c)$ is a multiple of $a(bc)$. By (1.2),

$$X(a, b, c) = \varepsilon\langle e, e\rangle\, a(bc), \tag{5.4}$$

where $\varepsilon=\pm 1$. If $\varepsilon=1$, (5.1) is satisfied for this particular a, b, c; if $\varepsilon=-1$, (5.2). By linearity, to show that the same equation, either (5.1) or (5.2), is satisfied for all a, b, c, it is sufficient to rule out the possibility that ε is different for a different normal set of generators chosen from among $a, b, ab, c, ac, bc, a(bc)$. That is, we need to verify that with ε determined by our initial choice of a, b, c, (5.4) holds even when these other normal sets of generators are substituted for a, b, c.

The different normal sets of generators among $a, b, ab, c, ac, bc, a(bc)$ can be obtained from the following six sets by suitably permuting a, b, c: $\{ab, b, c\}$; $\{bc, c, ab\}$; $\{cb, b, ab\}$; $\{(ab)\,c, c, b\}$; $\{bc, c, (ab)\,c\}$; $\{ab, (ab)\,c, bc\}$. (It does not matter for purposes of this argument that some of these generating elements differ by sign from the corresponding basis elements.) Since both sides of (5.4) are skew-symmetric in the arguments, (5.4) holds after any permutation of a, b, c. Hence it is sufficient to verify (5.4) with a, b, c replaced by the six normal sets of generators just listed.

To verify (5.4) for the first set, we again use the linearizations of (1.2) with respect to a_1 and a_2:

$$\begin{aligned} 0 &= \langle X(a, b, c), X(a\,b, e, c)\rangle + \langle X(a\,b, b, c), X(a, e, c)\rangle \\ &= \langle \varepsilon\langle e, e\rangle\, a(b\,c), -\langle e, e\rangle(a\,b)\,c\rangle + \langle \varepsilon'\langle e, e\rangle(a\,b)(b\,c), -\langle e, e\rangle\, a\,c\rangle \\ &= \varepsilon\langle e, e\rangle^3 \langle a(b\,c), a(b\,c)\rangle_1 - \varepsilon'\langle e, e\rangle^3 \langle\langle b, b\rangle_1\, a\,c, a\,c\rangle_1 \\ &= (\varepsilon - \varepsilon')\langle e, e\rangle^3 \langle a, a\rangle_1 \langle b, b\rangle_1 \langle c, c\rangle_1 . \end{aligned}$$

Hence $\varepsilon = \varepsilon'$.

To verify (5.4) for the other five sets we successively replace the set a, b, c in the preceding paragraph by the sets $\{b, c, ab\}$; $\{c, b, ab\}$; $\{ab, c, b\}$; $\{b, c, (ab)\,c\}$; $\{c, (ab)\,c, bc\}$. This concludes the proof of theorem (5.2).

We now turn to deciding when two three-fold vector cross products X_1 and X_2 with respect to $\langle\ ,\ \rangle$ on V are isomorphic. Suppose that $e, f \in V$ and $\langle e, e\rangle \neq 0$, $\langle f, f\rangle \neq 0$. By theorem (5.2) there is a multiplication ab on V with identity e such that the multiplication makes V a composition algebra with respect to $\langle\ ,\ \rangle_1 = \langle e, e\rangle^{-1}\langle\ ,\ \rangle$, and X_1 is given by (5.1) or (5.2) with $\alpha = \langle e, e\rangle$. Similarly, we obtain a multiplication $a \cdot b$ with identity f and the bilinear form $\langle\ ,\ \rangle_2 = \langle f, f\rangle^{-1}\langle\ ,\ \rangle$ permitting composition, with X_2 given by (5.1) or (5.2) with respect to $a \cdot b$, $\langle\ ,\ \rangle_2$, and $\alpha = \langle f, f\rangle$. Our isomorphism conditions can be stated as follows: X_1 and X_2 are isomorphic if and only if they are both given by (5.1) or both by (5.2). The main step in the proof of this assertion is the following lemma.

LEMMA (5.3): *Let $n=8$ and suppose that X_1 is given by* (5.1) *and X_2 by* (5.2). *Then X_1 and X_2 are not isomorphic.*

Proof. The proof is by contradiction. Suppose that φ is an isomorphism: $X_1(\varphi a, \varphi b, \varphi c) = \varphi X_2(a, b, c)$. By definition of isomorphism, φ is an isometry of $\langle\ ,\ \rangle$ so that for all a, b, c

$$\langle f, f\rangle \langle e, e\rangle^{-1}\, \varphi a\,(\overline{\varphi b}\, \varphi c) = \varphi((a \cdot \hat{b}) \cdot c), \tag{5.5}$$

where $b \to \hat{b}$ is the involution with respect to the multiplication $a \cdot b$. If $\varphi g = e$, let $a = c = g$ in (5.5). We obtain

$$\langle f, f\rangle \langle e, e\rangle^{-1}\, \overline{\varphi b} = \varphi((g \cdot \hat{b}) \cdot g). \tag{5.6}$$

We substitute (5.6) into (5.5) and replace $\hat{b}$ by b to get

$$\varphi a\,(\varphi((g \cdot b) \cdot g)\, \varphi c) = \varphi((a \cdot b) \cdot c). \tag{5.7}$$

By (4.2), $g \cdot g = 2\langle g, f\rangle_2 g - \langle g, g\rangle_2 f$. This can be substituted into (5.7) after setting $b = f$, the identity for $a \cdot b$. We obtain

$$2\langle g, f\rangle_2\, \varphi a\,(\varphi g\, \varphi c) - \langle g, g\rangle_2\, \varphi a\,(\varphi f\, \varphi c) = \varphi(a \cdot c).$$

Using the facts that $\varphi g=e$ and that φ is an isometry of $\langle\ ,\ \rangle$, hence of $\langle\ ,\ \rangle_2$, this becomes

$$2\langle e,\varphi f\rangle_2\,\varphi a\,\varphi c-\langle e,e\rangle_2\,\varphi a(\varphi f\varphi c)=\varphi(a\cdot c). \tag{5.8}$$

According to (5.8),

$$\begin{aligned}\varphi((a\cdot b)\cdot g)&=2\langle e,\varphi f\rangle_2\,\varphi(a\cdot b)-\langle e,e\rangle_2\,\varphi(a\cdot b)\,\varphi f\\&=4\langle e,\varphi f\rangle_2^2\,\varphi a\,\varphi b-2\langle e,\varphi f\rangle_2\langle e,e\rangle_2\,\varphi a(\varphi f\varphi b)\\&\quad-2\langle e,\varphi f\rangle_2\langle e,e\rangle_2(\varphi a\,\varphi b)\,\varphi f+\langle e,e\rangle_2^2(\varphi a(\varphi f\varphi b))\,\varphi f.\end{aligned} \tag{5.9}$$

We now calculate $\varphi a(\varphi((g\cdot b)\cdot g))$ in two ways. First, in (5.9) we set $a=g$ and multiply the obtained equation on the left by φa. This yields

$$\begin{aligned}\varphi a(\varphi((g\cdot b)\cdot g))&=4\langle e,\varphi f\rangle_2^2\,\varphi a\,\varphi b-2\langle e,\varphi f\rangle_2\langle e,e\rangle_2\,\varphi a(\varphi f\varphi b)\\&\quad-2\langle e,\varphi f\rangle_2\langle e,e\rangle_2\,\varphi a(\varphi b\varphi f)+\langle e,e\rangle_2^2\,\varphi a\left[(\varphi f\varphi b)\varphi f\right].\end{aligned} \tag{5.10}$$

Secondly, in (5.7) we set $c=g$ and expand the right-hand side of the obtained equation using (5.9), obtaining

$$\begin{aligned}\varphi a(\varphi((g\cdot b)\cdot g))&=4\langle e,\varphi f\rangle_2^2\,\varphi a\,\varphi b-2\langle e,\varphi f\rangle_2\langle e,e\rangle_2\,\varphi a(\varphi f\varphi b)\\&\quad-2\langle e,\varphi f\rangle_2\langle e,e\rangle_2(\varphi a\,\varphi b)\,\varphi f+\langle e,e\rangle_2^2\left[\varphi a(\varphi f\varphi b)\right]\varphi f.\end{aligned}$$

Subtracting the preceding equation from (5.10) yields

$$2\langle e,\varphi f\rangle_2\langle e,e\rangle_2(\varphi a,\varphi b,\varphi f)=\langle e,e\rangle_2^2(\varphi a,\varphi f\varphi b,\varphi f), \tag{5.11}$$

where $(x,y,z)=(xy)z-x(yz)$. We rewrite (5.11) using the facts (1) in an alternative algebra $(x,zy,z)=(x,y,z)z$ [2, page 210]; (2) $\langle\ ,\ \rangle_1$ is a multiple of $\langle\ ,\ \rangle_2$; (3) $\langle e,e\rangle_1=1$.

$$2\langle e,\varphi f\rangle_1(\varphi a,\varphi b,\varphi f)=(\varphi a,\varphi b,\varphi f)\,\varphi f. \tag{5.12}$$

Then since $2\langle e,\varphi f\rangle_1=\varphi f+\overline{\varphi f}$, we obtain

$$(\varphi a,\varphi b,\varphi f)\,\overline{\varphi f}=0,$$

or multiplying on the right by φf,

$$\langle\varphi f,\varphi f\rangle_1(\varphi a,\varphi b,\varphi f)=0.$$

Since $\langle\varphi f,\varphi f\rangle_1\neq 0$, $(x,y,\varphi f)=0$ for all $x,y\in V$. But the only elements φf with this property in an eight-dimensional composition algebra are multiples of the identity element. Therefore, setting $b=f$ in (5.5), we obtain $\gamma\varphi a\varphi c=\varphi(a\cdot c)$ for some $\gamma\neq 0$; and setting $a=c=f$ in (5.5), $\delta\varphi b=\varphi\hat{b}$ for some $\delta\neq 0$. Then

$$\begin{aligned}\langle f,f\rangle\langle e,e\rangle^{-1}\,\varphi a(\overline{\varphi b}\,\varphi c)&=\varphi((a\cdot\hat{b})\cdot c)=\gamma\,\varphi(a\cdot\hat{b})\,\varphi c\\&=\gamma^2(\varphi a\,\varphi\hat{b})\,\varphi c=\gamma^2\delta(\varphi a\,\overline{\varphi b})\,\varphi c.\end{aligned}$$

Letting φa, $\overline{\varphi b}$, φc be a normal set of generators of V, we find that $\langle f, f\rangle\langle e, e\rangle^{-1} = -\gamma^2\delta$. Letting $\varphi a = \varphi b = \varphi c = e$, we find that $\langle f, f\rangle\langle e, e\rangle^{-1} = +\gamma^2\delta$, a contradiction.

COROLLARY 5.4: *Suppose that X is a three-fold vector cross product with respect to $\langle\,,\rangle$ on V. Let $e_1, e_2 \in V$ such that $\langle e_1, e_1\rangle \neq 0$ and $\langle e_2, e_2\rangle \neq 0$. Then either X is given by* (5.1) *with respect to both the multiplication* (5.3) *defined by e_1 and the multiplication* (5.3) *defined by e_2, or with respect to both multiplications X is given by* (5.2). (Note that in case $n=4$, this corollary is vacuous because four-dimensional composition algebras are associative.)

Proof. If X is given by (5.1) with respect to one multiplication and (5.2) with respect to the other, let $X_1 = X_2 = X$ in lemma (5.3). Then the identity map on V is an isomorphism between X_1 and X_2, a contradiction.

The following lemma provides the remaining fact about isomorphisms. Notice that since four-dimensional composition algebras are associative, the lemma says that for $n=4$ any two three-fold vector cross products are isomorphic.

LEMMA 5.5: *Suppose that X_1 and X_2 are three-fold vector cross products. Using the multiplications and notation from the paragraph before lemma* (5.3), *suppose that X_1 and X_2 are both given by* (5.1) *or both by* (5.2). *Then X_1* and *X_2 are isomorphic.*

Proof. Suppose X_1 and X_2 are both given by (5.1). The proof for (5.2) is identical. By theorem (5.2) there is a new multiplication a^*b on V with identity e such that X_2 is given by (5.1) or (5.2) with respect to the multiplication a^*b. By corollary (5.4), X_2 must be given by (5.1) with respect to a^*b. Since the involution for any multiplication with identity e leaves e fixed and for $\langle a, e\rangle = 0$ maps $a \to -a$, the involutions for ab and a^*b coincide. Then

$$X_1(a, b, c) = -\langle e, e\rangle^{-1} a(\bar{b}c) + \langle a, b\rangle c + \langle b, c\rangle a - \langle a, c\rangle b,$$
$$X_2(a, b, c) = -\langle e, e\rangle^{-1} a^*(\bar{b}^* c) + \langle a, b\rangle c + \langle b, c\rangle a - \langle a, c\rangle b.$$

The two multiplications on V form composition algebras with respect to the same bilinear form $\langle e, e\rangle^{-1}\langle\,,\rangle$, and so they are isomorphic. If φ is an isomorphism between them, then φ is also an isometry of $\langle e, e\rangle^{-1}\langle\,,\rangle$, hence an isometry of $\langle\,,\rangle$. Then $\varphi a(\overline{\varphi b}\varphi c) = \varphi(a^*(\bar{b}^*c))$ and $\langle\varphi a, \varphi b\rangle\varphi c = \langle a, b\rangle\varphi c$, so that $X_1(\varphi a, \varphi b, \varphi c) = \varphi X_2(a, b, c)$. Hence X_1 and X_2 are isomorphic.

We summarize the results of this section in the following theorem.

THEOREM 5.6: (i) *A necessary and sufficient condition that the non-degenerate bilinear form $\langle\,,\rangle$ possess a non-trivial* three-*fold vector cross product is that $n=4$ or 8 and that $\langle\,,\rangle$ be a multiple of the bilinear form of a composition algebra.*

(ii) *If $\langle\,,\rangle$ satisfies the conditions in* (i), *then any three-fold vector cross product on V is given as follows. Let $e \in V$ with $\langle e, e\rangle \neq 0$ and let $\langle\,,\rangle = \langle e, e\rangle^{-1}\langle\,,\rangle$. Then on V there is a multiplication ab (depending on X and e) with identity e such that $\langle\,,\rangle_1$ permits composition and either X is of type* (5.1):

$$X(a, b, c) = \langle e, e\rangle\big(- a(\bar{b} c) + \langle a, b\rangle_1 c + \langle b, c\rangle_1 a - \langle c, a\rangle_1 b\big),$$

or X is of type (5.2):

$$X(a, b, c) = \langle e, e\rangle\big(- (a \bar{b})\, c + \langle a, b\rangle_1 c + \langle b, c\rangle_1 a - \langle c, a\rangle_1 b\big).$$

(iii) *If $n=4$, any two three-fold vector cross products on V are isomorphic. If $n=8$ two three-fold vector cross products on V are isomorphic if and only if they are both of type* (5.1) *or both of type* (5.2).

6. The Nonexistence of Further Cases

PROPOSITION (6.1): *Let X be an r-fold vector cross product with respect to a bilinear form $\langle\ ,\ \rangle$ on a space V of dimension n. Suppose there exist $u_1, \ldots, u_k \in V$ with $\langle u_i, u_j\rangle = \delta_{ij}$ $(1 \leqslant i, j \leqslant k)$ where $k < r$. Let V_1 be the orthogonal complement of the subspace spanned by $u_1, \ldots, u_k$ and define $X_1 : (V_1)^{r-k} \to V_1$ by $X_1(a_1, \ldots, a_{r-k}) = X(a_1, \ldots, a_{r-k}, u_1, \ldots, u_k)$. Then X_1 is an $(r-k)$-fold vector cross product on V_1 with respect to the restriction of $\langle\ ,\ \rangle$ to V_1.*

Proof: It is easily verified that X_1 satisfies (1.1) and (1.2).

THEOREM (6.2): *There are no other vector cross products besides those described in theorems* (2.2), (3.3). (4.1), *and* (5.6).

Proof: It suffices to show that vector cross products do not exist for n and r not listed in theorem A. To prove this nonexistence we may assume the ground field F is algebraically closed; this assures the existence of $u_1, \ldots, u_k \in V$ with $\langle u_i, u_j\rangle = \delta_{ij}$. Suppose an r-fold vector cross product with $r \geqslant 4$ exists; we may use proposition (6.1) to obtain a three-fold product on a space of dimension $n-r+3$. Then $n-r=1$ or 5 by theorem (5.6). If $n-r=1$, we are in the case treated by theorem (3.3). If $n-r=5$, $r \geqslant 4$, we use proposition (6.1) to obtain a four-fold vector cross product on a space V of dimension 9. To complete the proof we show that no such vector cross product exists.

Assume X is a four-fold vector cross product on V. We first note that if $a_1, \ldots, a_5$ are pairwise orthogonal, linearization of (1.2) yields

$$\langle X(a_1, a_2, a_3, a_4), X(a_1, a_2, a_3, a_5)\rangle = 0. \tag{6.1}$$

Choose $v \in V$ with $\langle v, v\rangle = 1$ and let W be the orthogonal complement of V. By proposition (6.1) and theorem (5.6) there is a product on W with identity e such that for all $a, b, c \in W$ with $\{e, a, b, c\}$ pairwise orthogonal,

$$X(a, b, c, v) = \alpha\, a(b\, c) \tag{6.2}$$

for some non-zero $\alpha \in F$ depending on a, b, and c. This follows because for a, b, c orthogonal to e in W, $\bar{b} = -b$ and $a(bc) = \pm(ab)\, c$. Now let $\{a, b, c\}$ be a normal set of generators for W. From equations (6.1) and (6.2) it follows that

$$\langle X(a, b, c, a b), a(b c)\rangle = \alpha^{-1}\langle X(a, b, c, a b), X(a, b, c, v)\rangle = 0. \tag{6.3}$$

Since $\{a, ab, c\}$ and $\{b, ab, c\}$ are also normal systems of generators it follows from (6.3) and the skew-symmetry of X that

$$\langle X(a, b, c, a b), b c\rangle = \langle X(a, b, c, a b), a c\rangle = 0. \tag{6.4}$$

Finally from (6.1) and (6.2) it follows that

$$\begin{aligned}\langle X(a, b, c, a b), e\rangle &= \beta\langle X(a, b, c, a b), (a b)^2\rangle \\ &= \gamma\langle X(a, b, a b, c), X(a, b, a b, v)\rangle = 0.\end{aligned}$$

Therefore we conclude that $X(a, b, c, ab)=\varrho v$ for some non-zero $\varrho\in F$. Similarly $X(a, b, c, ac)=\sigma v$ for some non-zero $\sigma\in F$. Thus by (6.1)

$$0 = \langle X(a, b, c, a b), X(a, b, c, a c)\rangle = \varrho\sigma \neq 0,$$

a contradiction.

7. The Classification over Special Fields

Over certain fields we may determine all the vector cross products more explicitly because the associated bilinear forms are completely known. We first note one consequence of the preceding sections.

THEOREM (7.1): *A necessary condition for the existence of a vector cross product is that the discriminant of* $\langle\ ,\ \rangle$ *be* 1. *Except for* $r=n-1$, *this condition is not in general sufficient.*

Over an algebraically closed field all non-degenerate quadratic forms of the same dimension are equivalent. Hence for given n and r all vector cross products are isomorphic except in the case $n=8$, $r=3$; then there are two isomorphism classes, which are given by (5.1) and (5.2).

For the real numbers, except for the case $n=8$, $r=3$, the isomorphism class of a vector cross product is determined by the signature of the bilinear form associated with it. We obtain the following table:

n and r	number of isomorphism classes	possible signatures
n even, $r=1$	$n/2+1$	$n, n-4, \ldots, -n+4, -n$
$r=n-1$	$[n/2]+1$	$n, n-4, \ldots, n-2[n]$
$n=7, r=2$	2	$7, -1$
$n=8, r=3$	6	$8, 0, -8$

The first two rows of this table are consequences of sections 2 and 3. The case $n=7$, $r=2$, and $n=8$, $r=3$ are as stated because the quadratic form associated with a composition algebra over the real numbers must have signature 8 or 0. Each such quadratic form determines an isomorphism class of vector cross products in the case $n=7$, $r=2$. For $n=8$, $r=3$ there are two vector cross products for each quadratic form and they are given by (5.1) and (5.2). Furthermore, two additional vector cross products corresponding to the quadratic form with signature -8 are obtained by taking $\alpha=-1$ in theorem (5.1).

For finite fields the number of isomorphism classes of vector cross products for a given n and r is the same as for algebraically closed fields: one class except for $n=8$, $r=3$, when there are two. To verify this we use the classification of quadratic forms (for characteristic not 2), given for example by ARTIN [1, p. 144]. Let g be any fixed nonsquare in F. Then for odd dimensions any quadratic form is equivalent to either (7.1) or (7.2) below; for even dimensions (7.3) or (7.4):

$$x_1^2 - x_2^2 + x_3^2 - x_4^2 + \cdots + x_{n-2}^2 - x_{n-1}^2 + x_n^2 \tag{7.1}$$

$$x_1^2 - x_2^2 + x_3^2 - x_4^2 + \cdots + x_{n-2}^2 - x_{n-1}^2 + g x_n^2 \tag{7.2}$$

$$x_1^2 - x_2^2 + x_3^2 - x_4^2 + \cdots + x_{n-1}^2 - x_n^2 \tag{7.3}$$

$$x_1^2 - x_2^2 + x_3^2 - x_4^2 + \cdots + x_{n-3}^2 - x_{n-2}^2 + x_{n-1}^2 - g x_n^2 \tag{7.4}$$

The discriminants are $(-1)^{(n-1)/2}$, $(-1)^{(n-1)/2}g$, $(-1)^{n/2}$, $(-1)^{n/2}g$, respectively. We get the following table:

n and r	quadratic form giving the isomorphism class
$n\equiv 0 \pmod 4$, $r=1$	(7.3)
$n\equiv 2 \pmod 4$, $r=1$, -1 is a square	(7.3)
$n\equiv 2 \pmod 4$, $r=1$, -1 is not a square	(7.4)
$n\equiv 0 \pmod 4$, $r=n-1$	(7.3)
$n\equiv 1 \pmod 4$, $r=n-1$	(7.1)
$n\equiv 2 \pmod 4$, $r=n-1$, -1 is a square	(7.3)
$n\equiv 2 \pmod 4$, $r=n-1$, -1 is not a square	(7.4)
$n\equiv 3 \pmod 4$, $r=n-1$, -1 is a square	(7.1)
$n\equiv 3 \pmod 4$, $r=n-1$, -1 is not a square	(7.2)
$n=7$, $r=2$, -1 is a square	(7.1)
$n=7$, $r=2$, -1 is not a square	(7.2)
$n=8$, $r=3$	(7.3) with two isomorphism classes

This table is established as follows. For $r=1$ or $n-1$ the cases listed are precisely those whose quadratic form has discriminant 1, because if -1 is not a square, $-g$ is a square. Each of the quadratic forms for $r=1$ is also consistent with (2.2). The vector cross products for $n=7$, $r=2$ and $n=8$, $r=3$ arise from the unique 8-dimensional composition algebra which is split, as in theorems (4.1) and (5.1). To see that no additional products for $n=8$, $r=3$ are obtained from (5.1) or (5.2) by letting α vary, we note that in varying α the discriminant of the quadratic form remains a square, so that the resulting quadratic forms are still equivalent to (7.3).

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