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Autor(en): **Gross, Herbert / Miller, Vinnie H.**

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Continuous Forms in Infinite Dimensional Spaces (Quadratic Forms and Linear Topologies IV)

By HERBERT GROSS and VINNIE H. MILLER[1]) in Bozeman, Montana

Professor Dr. GOTTFRIED KÖTHE dedicated to his 60th birthday

Introduction

Let E be a k-vectorspace supplied with a symmetric, non-degenerate, bilinear form $\Phi : E \times E \to k$. In [4] the class of topologies making Φ continuous was briefly considered. It was found that there is a "coarsest" such topology associated with each totally isotropic subspace H of E; this topology we denote by $\tau_\Phi H$. As we shall see in Chapter III, these are fairly canonical topologies for the form Φ. It is our intention to utilize these topologies in framing and answering questions of a strictly algebraic nature concerning infinite dimensional vectorspaces (E, Φ). In particular we shall be concerned with groups of orthogonal automorphisms of such vectorspaces (Chapter IV) and with the possibility of orthogonal decompositions (Chapter V).

The defining neighborhoods for the $\tau_\Phi H$ topologies are given in Chapter II below, and some of the elementary properties of such topologies are developed. Also in Chapter II the $\tau_\Phi H$-completions of spaces are discussed. The given form Φ extends uniquely to a form $\tilde{\Phi} : \tilde{E} \times \tilde{E} \to k$ on the completion $\tilde{E}$ of E. If Φ is non-degenerate then $\tilde{\Phi}$ is nondegenerate if and only if H is orthogonally closed ($H^{\perp\perp} = H$). $\tau_\Phi H$-completions coincide with the locally linearly compact spaces with continuous forms.

In Chapter III the Clifford algebra $C(\Phi)$ associated with a linearly topologized space (E, Φ) is discussed. The fruitfulness of Clifford algebras in the study of finite dimensional spaces and their orthogonal groups is well known. Starting from a linear topology τ on (E, Φ) it seems desirable to construct linear topologies τ' on the associated Clifford algebra $C(\Phi)$ in such a way that τ' will induce the initial topology τ on E if E is thought of as embedded in $C(\Phi)$. We first extend τ to suitable topologies on the tensor products $\overset{p}{\otimes} E$ and then, by the usual sum and quotient operations, to a topology τ' on $C(\Phi)$. The construction discussed here will make use of the projective tensor product topology $\overset{p}{\otimes}\tau$ (of τ) on $\overset{p}{\otimes} E$ introduced in [6]. It is shown that the resulting topology on $C(\Phi)$, denoted by $\otimes\tau$, induces the initial topology τ on E if and only if τ is finer than some $\tau_\Phi H$ topology. Surprisingly enough, this condition is also seen to be equivalent with the condition that $\otimes\tau$ on $C(\Phi)$ be Hausdorff. These natural requirements for topologies on $C(\Phi)$ thus lead us again to the topologies $\tau_\Phi H$. The

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main result of this chapter is the theorem saying that multiplication in $C(\Phi)$ is $\otimes\tau$-continuous when (E, Φ) is of denumerably infinite dimension and $\tau = \tau_\Phi H$, H orthogonally closed. Under these conditions $C(\Phi)$ with the $\otimes\tau$ topology is in fact a topological algebra. Topologies can be given which are finer than some $\tau_\Phi H$ for which multiplication fails to be continuous, both in the denumerable case and in higher dimensional cases. Whether multiplication in $C(\Phi)$ *can* be $\otimes\tau$-continuous for suitable τ is an open question in the nondenumerable case.

In Chapter IV of the paper we investigate in some detail the algebraic structure of groups of $\tau_\Phi H$-continuous (orthogonal) automorphisms of spaces $(E, \Phi, \tau_\Phi H)$. The spaces treated in this discussion are those which are either of denumerable dimension or else $\tau_\Phi H$-complete. For a large number of underlying fields it turns out that the full orthogonal group of a $\tau_\Phi H$-complete space coincides with the group of all $\tau_\Phi H$-continuous automorphisms; (H has to be a maximal totally isotropic subspace of E).

In the last part we discuss some examples of nondenumerable spaces (E, Φ), some of which were suggested by topological investigations. For example, it has long been known that spaces of nondenumerable infinite dimension do not in general have orthogonal bases. Topological considerations point to classes of spaces (E, Φ) which do not even contain an orthogonal summand of infinite dimension less than that of the whole space. Clearly such spaces admit no infinite orthogonal decompositions whatever.

Some of our theorems have obvious extensions to the more general case of ε-Hermitean forms over arbitrary fields (of any characteristic.) In many cases examples illustrating the more general context were not at hand; we have therefore not considered these possible generalizations here.

I. Notations and Definitions

I.1. In the following E will always be a vectorspace over some commutative field k and $\Phi: E \times E \to k$ will be a symmetric, bilinear form. We assume throughout that char $k \neq 2$. If Φ is nondegenerate we say that E is semisimple. $\|x\|$ is $\Phi(x, x)$. Subspaces H of (E, Φ) are usually endowed with the induced form $\Phi|_{H \times H}$. $H \cap H^\perp$ is called the radical of H (rad H). H is totally isotropic if $H \subset H^\perp$ and anisotropic when $\|x\| = 0$ only if $x = 0$. The subspace H is called orthogonally closed ($\perp$-closed) if $H^{\perp\perp} = H$ and orthogonally dense ($\perp$-dense) if $H^{\perp\perp} = E$. If H is $\perp$-closed and F finite dimensional then $H + F$ is $\perp$-closed. A semisimple space (E, Φ) which possesses no proper $\perp$-dense subspace is of finite dimension. (This is proved by showing that Φ induces an epimorphism of E onto E^*, the algebraic dual of E.) In particular, if every subspace of a semisimple space is $\perp$-closed then the space is finite dimensional. Bases of a vectorspace are algebraic bases throughout; $k(e_\alpha)_{\alpha \in I}$ denotes a k-space with basis vectors e_α. If $A \oplus B$ is an orthogonal decomposition (for some Φ) we write $A \overset{\perp}{\oplus} B$.

I.2. A Witt decomposition of (E, Φ) is a decomposition $E=(R\oplus R')\overset{\perp}{\oplus}G$, R and R' totally isotropic subspaces spanned by the two halves of a symplectic basis $\{r_\alpha, r_\alpha'\}_{\alpha\in I}$ of E: $R=k(r_\alpha)_{\alpha\in I}$, $R'=k(r_\alpha')_{\alpha\in I}$, $\Phi(r_\alpha, r_\beta')=\delta_{\alpha\beta}$ (Kronecker), and G with an orthogonal basis. The following theorem is often used ([10], Theorem 7): *Let (E, Φ) be a semi-simple space of denumerably infinite dimension, and let R be an orthogonally closed totally isotropic subspace of E. Then E admits a Witt decomposition $E=(R\oplus R')\overset{\perp}{\oplus}G$.* We shall frequently find ourselves in the following situation: *Let $E=R\oplus R'$ be a space of denumerably infinite dimension with R and R' totally isotropic; further let $\{r_i, r_i'\}_{1\leqslant i\leqslant n}$ be a set of vectors with $r_i\in R$, $r_i'\in R'$* and $\Phi(r_i, r_j')=\delta_{ij}$. *Then we can extend $\{r_i, r_j'\}_{i\leqslant n}$ to a symplectic basis $\{r_i, r_i'\}_{i\geqslant 1}$ of E whose two halves span R and R' respectively.*

I.3. Terminology and conventions concerning linear topologies are consistent with [4] and [6]. We should like to recall that the "orthogonal" $F^\perp$ for $F\subset(E, \Phi)$ is a subspace $\{x\in E;\ x\perp F\}$ of E; whereas the "orthogonal" F^0 is a subspace of some dual G of E, viz. $\{x\in G;\ \langle F, x\rangle=0\}$ where $\langle,\rangle$ defines the duality between E and G. $\sigma(E, G)$ denotes the weak topology on E induced by G.

I.4. We conclude this chapter with a few words about the underlying fields. We shall usually assume later that the spaces (E, Φ) admit infinite dimensional totally isotropic subspaces, a requirement on the form Φ. However, there is an impressive list of fields k such that every infinite dimensional k-space (E, Φ) admits infinite dimensional totally isotropic subspaces. Choosing Φ diagonal, it is clear that such fields are necessarily non-formally real. All fields in the following list have the property that there is an integer m, depending only on k, such that every form Φ in $m+1$ variables over k has a non-trivial zero. Finite algebraic extensions K of fields with this property are again of this kind, m^n in lieu of m will do, m the appropriate number for k and $n=[K:k]$. (m^n is not, in general, the most economic choice; see for example the end of item 2 in the list below.) Fields for which there is such a number m are called *ortho-normal* in [14]. (The name is derived from the fact that every semisimple space (E, Φ) of denumerable dimension over such a field possesses an orthonormal basis [10].) In particular, the following fields are orthonormal.

(1) A Kneser field k is a non-formally real field of characteristic unequal 2 for which $g_k=k^*/k^{*2}$, the multiplicative group of nonzero elements modulo squaie factors, has finite order $o(g_k)$. All Kneser fields are orthonormal; they have $m=o(g_k)$ (see [5]). When $o(g_k)$ is finite it is necessarily a power of 2 as all of the group elements are of this order. For every $n\geqslant 0$ there are Kneser fields k with $o(g_k)=2^n$; special examples are the algebraically closed fields, the finite fields, and the local fields.

(2) A further class of orthonormal fields which does not fall into the previous category is the class of all transcendental extensions of finite transcendence degree r of finite fields and algebraically closed fields. Here m is 2^{1+r} and 2^r respectively (see the second lemma in [13]). Hence also orthonormal are the function fields in r variables

over a constant field k which is finite or algebraically closed; it follows from [13] that 2^{1+r} and 2^r respectively will do for m in this case.

(3) Finite or infinite non-formally real algebraic extensions of the rationals are orthonormal; here $m=4$ by Hasse-Minkowski theory.

II. The $\tau_\Phi H$ Topologies

Throughout Chapter II, (E, Φ) will be a semisimple space over an arbitrary (commutative) field k with characteristic unequal 2; k invariably carries the discrete topology.

II.1 *Elementary Properties*

DEFINITION 1: Let H be a totally isotropic subspace of (E, Φ). The linear topology defined by the neighborhood filter $\{H \cap F^\perp\}$ of $0 \in E$, F running through the finite dimensional subspaces of E, will be denoted by $\tau_\Phi H$ (cf. [4], 2.3).

As the notation suggests the topology depends in a fundamental way on the form as the following corollary shows, (cf. Theorem 17 below).

COROLLARY: *Let τ be a linear topology on (E, Φ). The form $\Phi: E \times E \to k$ is τ-continuous if and only if τ is finer than some $\tau_\Phi H$ ($\tau \geqslant \tau_\Phi H$ for a suitable H).*

Indeed if Φ is continuous then it is continuous at $(0, 0) \in E \times E$, i.e. there is a τ-neighborhood H with $\Phi(H, H) = (0)$, k being discrete. Furthermore, for arbitrary fixed x, $\Phi(x, y)$ is continuous in y; thus there is a τ-neighborhood V_x with $\Phi(x, V_x) = \{0\}$, i.e. $V_x \subset k(x)^\perp$. Taking finite intersections we see that all the $H \cap F^\perp$ are τ-neighborhoods of $0 \in E$, and therefore $\tau \geqslant \tau_\Phi H$. The converse follows by the same argument, noticing that Φ is continuous if and only if it is continuous at $(0, 0)$ and separately continuous.

It is easy to see that the topology $\tau_\Phi H$ is discrete if and only if H is finite dimensional. The only spaces of interest in this connection are therefore those (E, Φ) which admit infinite dimensional, totally isotropic subspaces H (see I.4). We finally notice that the semisimplicity of (E, Φ) implies that $\tau_\Phi H$ is always Hausdorff.

Since every totally isotropic H is contained in a maximal totally isotropic subspace V, the $\tau_\Phi V$ with maximal V are precisely the coarsest linear topologies making Φ continuous.

We remark that in contrast to the locally convex case, continuity of Φ at $(0, 0)$ does not imply continuity of Φ on $E \times E$. We shall illustrate this by an example. In Chapter IV we shall describe semisimple spaces $(V \oplus W, \Phi)$ of the following kind: V will be a totally isotropic space spanned by an infinite basis $\{v_\iota\}_{\iota \in I}$, and for $w \in W$, if $\Phi(w, v_\iota) = 0$ for infinitely many $\iota \in I$ then $w = 0$. On such a space $V \oplus W$ we now define a linear topology τ as follows: For every finite subset $G \subset I$ let $I - G$ be its complement in I and let $F_G = k(v_\iota)_{\iota \in I - G}$. τ is defined by taking the sets $\{F_G\}$ as a zero-neighborhood basis, G running through the finite subsets of I. τ is Hausdorff as $\cap F_G = (0)$; further Φ is continuous at (0, 0) as all F_G are totally isotropic. However, for $0 \neq w \in W$, $\Phi(w, y)$ is not continuous in y. Otherwise $\Phi(w, F_{G_0}) = (0)$ for suitable F_{G_0}, i.e. $\Phi(w, v_\iota) = 0$ for all $\iota \in I - G_0$, and thus $w = 0$ contrary to our assumption.

We now turn to the comparison of certain $\tau_\Phi H$ topologies.

LEMMA: *Let V and H be totally isotropic, $\perp$-closed subspaces of (E, Φ). We have $\tau_\Phi V = \tau_\Phi(V \cap H) \geqslant \tau_\Phi H$ if and only if* $\dim V/V \cap H$ *is finite.*

Proof: If $\tau_\Phi V \geqslant \tau_\Phi V \cap H$ then there is a finite dimensional F such that $V \supset V \cap H \supset V \cap F^\perp$. And therefore $\dim V/V \cap H \leqslant \dim V/V \cap F^\perp \leqslant \dim F$.

Conversely since $H \supset V \cap H$, $\tau_\Phi H \leqslant \tau_\Phi V \cap H$ and $\tau_\Phi V \cap H \geqslant \tau_\Phi V$. It remains to show that $\tau_\Phi V \cap H \leqslant \tau_\Phi V$. By hypothesis $V = (V \cap H) \oplus G$ with G finite dimensional and $V \cap H$ $\perp$-closed. So $\dim (V \cap H)^\perp / (V \cap H)^\perp \cap G^\perp \leqslant \dim E/G^\perp = \dim G$ is finite; i.e. $(V \cap H)^\perp = ((V \cap H)^\perp \cap G^\perp) \oplus K$ for finite dimensional K. Therefore $V \cap H = (V \cap H)^{\perp\perp}$ $= ((V \cap H) + G) \cap K^\perp = V \cap K^\perp$. From which we conclude that $\tau_\phi(V \cap H) \leqslant \tau_\phi V$.

THEOREM 1: *Let V and H be maximal totally isotropic subspaces of (E, Φ). The following are equivalent*:

(i) $\tau_\Phi V = \tau_\Phi H$

(ii) *$\tau_\Phi V$ and $\tau_\Phi H$ are comparable*

(iii) *$V/V \cap H$ and $H/V \cap H$ are of the same finite dimension*

(iv) *$V/V \cap H$ and $H/V \cap H$ are finite dimensional.*

Proof: (i)→(ii) is trivial. If (ii) is the case we have for instance $\tau_\Phi V \geqslant \tau_\Phi H$. Hence $V = (V \cap H) \oplus G$ for finite dimensional G by the preceding lemma. (If V is totally isotropic then so is $V^{\perp\perp}$, so $V = V^{\perp\perp}$ for maximal V.) We set $H = (V \cap H) \oplus G'$ and claim that $G \oplus G'$ is semisimple. Indeed if $x \in \operatorname{rad} G \oplus G'$ then x is isotropic and $x \perp V$ and $x \perp H$ as $G \oplus G' \perp V \cap H$. Hence $x \in V$ and $x \in H$ as both V and H are maximal. Therefore, $x = 0$ since $(G \oplus G') \cap V \cap H = (0)$. Since G is finite dimensional, $G \oplus G'$ semisimple, G and G' both totally isotropic, we have $\dim G = \dim G'$. This proves (ii)→(iii). (iii)→(iv) is trivial. (iv)→(i) is a direct consequence of the previous lemma.

We consider an example which shows that the assumptions on V and H can not be weakened in the previous theorems. Let $E = V \oplus V'$ be of denumerable dimension, V and V' totally isotropic for Φ, $\Phi(v_i, v'_j) = \delta_{ij}$ for $\{v_i\}_{i \geqslant 1}$ and $\{v'_i\}_{i \geqslant 1}$ bases of V and V' respectively. Let $H = k(v_1 + v_i)_{i > 1}$. V is maximal, in particular $\perp$-closed; further $H \subset V$ and $\dim V/V \cap H = \dim V/H = 1$. Nevertheless we do not have $\tau_\Phi V = \tau_\Phi H$. For if $\tau_\Phi H \leqslant \tau_\Phi V$, we should have $H \supset V \cap F^\perp$ for some finite dimensional F. Since $V \supset H \supset V \cap F^\perp$ we see that $\dim H/V \cap F^\perp$ is finite; further $V \cap F^\perp$ is $\perp$-closed as both V and $F^\perp$ are $\perp$-closed. Hence H is $\perp$-closed (I.1). This is a contradiction as it is easily verified that $H^{\perp\perp} = V \neq H$. We see that $\tau_\Phi H$ is strictly finer than $\tau_\Phi V$ in spite of the fact that H falls short of V by only one dimension.

Let $T \in \mathfrak{O}(E, \Phi)$, the orthogonal group of (E, Φ). T is $\tau_\Phi V$ continuous if and only if $\tau_\Phi V \leqslant \tau_\Phi T(V)$.

THEOREM 2: *Let H be a totally isotropic $\perp$-closed subspace of (E, Φ). Let $T \in \mathfrak{O}(E, \Phi)$. T is $\tau_\Phi H$-continuous if and only if $T(H)/H \cap T(H)$ is finite dimensional. T^{-1} is $\tau_\Phi H$-continuous if and only if $H/H \cap T(H)$ is finite dimensional.*

It seems natural to call a subspace $L \subset E$ *almost invariant* for T, $T \in \mathfrak{O}(E, \Phi)$, if $L \cap T(L)$ is of finite codimension in both L and $T(L)$. It is straightforward to verify that all T which leave L almost invariant form a subgroup of $\mathfrak{O}(E, \Phi)$. In particular, if L is totally isotropic and $\perp$-closed, then the previous theorem shows that the group of all T leaving L almost invariant is the largest subgroup of $\mathfrak{O}(E, \Phi)$ consisting of $\tau_\Phi L$-continuous automorphisms. A $\tau_\Phi H$-continuous T does not, in general, have a continuous inverse. Let $E = V \oplus V'$ be the space of denumerable dimension of the previous example, $V = k(v_i)_{i \geqslant 1}$, $V' = k(v')_{i \geqslant 1}$. The index map $2i \to 4i$, $6i + 2j - 1 \to 4i + j$, $j = 1, 2, 3$ and $i \geqslant 1$ defines an orthogonal automorphism leaving V and V' invariant. If we set $H = k(v_{4i})_{i \geqslant 1}$ then T is $\tau_\Phi H$-continuous but not open. This cannot happen in the case of maximal totally isotropic spaces:

THEOREM 3: *Let V be a maximal totally isotropic subspace of (E, Φ), and let $T \in \mathfrak{O}(E, \Phi)$. The following are equivalent.*

(i) *T is $\tau_\Phi V$-continuous*

(ii) *T^{-1} is $\tau_\Phi V$-continuous*

(iii) *$V/V \cap T(V)$ and $T(V)/V \cap T(V)$ are of (the same) finite dimension.*

Proof: This is an immediate consequence of the lemma above and Theorems 2 and 3.

We conclude this introductory section with an example which will be of importance later on.

LEMMA: *Let H^* be the algebraic dual of the k vectorspace H, and let Φ be defined on $E = H \oplus H^*$ by $\Phi(h^*, h) = h^*(h)$, H and H^* both totally isotropic for Φ. Then H^* is almost invariant under any $T \in \mathfrak{O}(E, \Phi)$. In other words, every $T \in \mathfrak{O}(E, \Phi)$ is $\tau_\Phi H^*$-continuous.* (We note that E is semisimple, and both H and H^* are maximal totally isotropic subspaces.)

Proof: For fixed $T \in \mathfrak{O}(E, \Phi)$ we set $D = H^* \cap T(H^*)$, $H^* = D \oplus K$, $T(H^*) = D \oplus S$. In particular $S \cap H^* = (0)$ and $D \perp K \oplus S$. Let $\{s_\iota\}_{\iota \in I}$ be a basis of S. We decompose s_ι into $s_\iota = h_\iota + h_\iota^*$, $h_\iota \in H$, $h_\iota^* \in H^*$. Since $S \cap H^* = (0)$, the h_ι, $\iota \in I$, must be linearly independent. We put $\bar{S} = k(h_\iota)_{\iota \in I}$ and have $\dim \bar{S} = \dim S$. We have $D \perp \bar{S}$ since $D \perp S$ and $D \subset H^* \subset H^{*\perp}$. For arbitrary $h^* \in H^*$, we decompose h^* into $d + k$, $d \in D$, $k \in K$. For every $h \in \bar{S}$, $\Phi(h^* - k, h) = \Phi(d, h) = 0$ so $h^*(h) = \Phi(h^*, h) = \Phi(k, h) = k(h)$. Thus K possesses a subspace isomorphic to $\bar{S}^*$ and $\dim K \geqslant \dim \bar{S}^* = \dim S^*$.

Applying T^{-1} to the decompositions given above for H^* and $T(H^*)$ yields

$T^{-1}(H^*)=(T^{-1}(H^*)\cap H^*)+T^{-1}K$ and $H^*=(T^{-1}(H^*)\cap H^*)+T^{-1}S$. By reasoning exactly as above we obtain $\dim T^{-1}S\geqslant\dim(T^{-1}K)^*$, so $\dim S\geqslant\dim K^*$. But this combined with the inequality of the previous paragraph shows that K and S are of the same finite dimension.

As we shall see later (IV, Theorem 21), the subspace H is by no means left almost invariant by $\mathfrak{O}(E, \Phi)$.

II.2 *Completions*

Let V be a totally isotropic subspace of E and equip E with the topology $\tau=\tau_\Phi V$. In several of the theorems which follow it will be convenient to consider the following decomposition: $E=V\oplus H_1\oplus H_2$ with $V^\perp=H_1\oplus V$, $V=k(v_\alpha)_{\alpha\in I}$, $H_1=k(h_{1\alpha})_{\alpha\in J}$, $H_2=k(h_{2\alpha})_{\alpha\in K}$, $H=H_1\oplus H_2$. Such a decomposition is of course always possible.

The symbol $\sim$ will denote completion. The topology τ under consideration is always $\tau_\Phi V$. Thus $\tilde{\tau}$ denotes the completion of the $\tau_\Phi V$ topology.

In this chapter, V' will denote the topological dual of V.

The first theorem shows that the problem of completing E reduces to that of completing V.

THEOREM 4: $\tilde{E}=\tilde{V}\oplus H$. *$\tilde{\tau}|_H$ is the discrete topology.*

Indeed, every algebraic complement H of a linear zero-neighborhood is a discrete topological supplement.

The completion is only of interest if Φ induces a continuous bilinear form on $\tilde{E}$. The next theorem guarantees that this will be the case.

THEOREM 5: *The quadratic form $Q:E\to k$ extends to a unique continuous function $\tilde{Q}:\tilde{E}\to k$. $\tilde{Q}$ is quadratic, $\tilde{V}$ is totally isotropic and, with respect to the associated bilinear form $\tilde{\Phi}$, $H_1\perp\tilde{V}$. $\tilde{\tau}\geqslant\tau_{\tilde{\Phi}}W$ for some totally isotropic subspace W of $(\tilde{E}, \tilde{\Phi})$.*

Proof: Although Q is not a uniformly continuous function it can still be extended to $\tilde{E}$ provided that for all $\tilde{\tau}$-Cauchy systems $\langle x_\beta\rangle$ of elements of E which converge to $\tilde{x}$ in $\tilde{E}$ the directed systems $\langle Q(x_\beta)\rangle$ have one and the same limit in k (see [12], page 17).

Let $\langle v_\alpha+h_\alpha\rangle$ and $\langle v'_\alpha+h'_\alpha\rangle$ be two directed systems in E both converging to $\tilde{x}\in\tilde{E}$. Since both are Cauchy and $\tilde{V}$ is a $\tilde{\tau}$ zero neighborhood, h_α equals some fixed h for α sufficiently large and $h'_\alpha=h$ for sufficiently large α. And since both directed systems converge to $\tilde{x}$, $\langle v_\alpha+h_\alpha-v'_\alpha-h'_\alpha\rangle$ is also Cauchy so $h=h'$; in particular $\langle v_\alpha-v'_\alpha\rangle$ is Cauchy.

Now consider $\langle Q(v_\alpha+h_\alpha)\rangle$. For α and β sufficiently large, $Q(v_\alpha+h_\alpha)-Q(v_\beta+h_\beta)=Q(v_\alpha+h)-Q(v_\beta+h)=2\Phi(v_\alpha, h)-2\Phi(v_\beta, h)=2\Phi(v_\alpha-v_\beta, h)=0$ since we may assume $v_\alpha-v_\beta\in k(h)^\perp\cap V$. Therefore $\langle Q(v_\alpha+h_\alpha)\rangle$ is a Cauchy system in the complete Hausdorff space k so has unique limit λ. Similarly $\lim Q(v'_\alpha+h'_\alpha)=\lambda'$.

By computations similar to those above, $Q(v_\alpha+h_\alpha)-Q(v'_\alpha+h'_\alpha)=0$ for α sufficiently large. So $\lambda=\lambda'$. Therefore Q extends uniquely to a continuous function $\tilde{Q}:\tilde{E}\to k$.

If two continuous functions mapping the topological space X into the Hausdorff space Y agree on a dense subset D of X then they are identical. Applying this well known result gives immediately that $\tilde{Q}$ is quadratic, $\tilde{V}$ is totally isotropic and $\tilde{\Phi}(\tilde{v}, h_1) = 0$, $\tilde{v} \in \tilde{V}$, $h_1 \in H_1$.

Since $\tilde{Q}$ is continuous, $\tilde{\tau} \geqslant \tau_{\tilde{\Phi}} W$ for some totally isotropic W by II.1.

The results to this point are of an existential nature. In the next three theorems the form of the completion is made moie precise and a computing formula is given for $\tilde{\Phi}$.

THEOREM 6: *$(\tilde{V}, \tilde{\tau}|_{\tilde{V}}) = (H_2^*, \sigma(H_2^*, H_2))$ so $\tilde{E} = H_2^* \oplus H$ with the $\sigma(H_2^*, H_2)$ topology on H_2^* and the discrete topology on H.*

Proof: V is $\tau_\Phi V$ linearly bounded, since for arbitrary $V \cap F^\perp$, F finite dimensional, $\dim V + (V \cap F^\perp)/V \cap F^\perp = \dim V/V \cap F^\perp = \dim V + F^\perp/F^\perp \leqslant \dim E/F^\perp = \dim F$. Therefore $\tilde{V}$ is $\tilde{\tau}$ linearly compact, $\tilde{V}$ is topologically isomorphic to $\tilde{V}'^*$ and $\tilde{\tau}|_{\tilde{V}} = \sigma(\tilde{V}, \tilde{V}')$ (see KÖTHE [12], page 101).

To complete the proof we show that $\tilde{V}'$ is in fact H_2, but first we prove a useful

LEMMA: *If $E = V^\perp \oplus H_2$ with $V \subset V^\perp$ and $\langle V, H_2 \rangle$ a dual pair for Φ then $\tau_\Phi V|_V = \sigma(V, H_2)$. It is not necessary to assume that E is semisimple.*

Proof of the lemma: $\sigma(V, H_2)$ has a zero neighborhood basis of sets $G^0 = \{v \in V;\ \Phi(h_2, v) = 0 \text{ for all } h_2 \in G\} = V \cap G^\perp$, G a finite dimensional subspace of H_2. The sets $V \cap G^\perp$ are in the $\tau_\Phi V|_V$ zero neighborhood basis. In fact for an arbitrary set $V \cap F^\perp$ in the $\tau_\Phi V|_V$ zero neighborhood basis, F being finite dimensional is contained in some $V^\perp + G$ so $V \cap F^\perp \supset V \cap V^{\perp\perp} \cap G^\perp = V \cap G^\perp$. So the sets $V \cap G^\perp$ are even a zero neighborhood basis for $\tau_\Phi V|_V$.

Returning to the proof of the theorem we show the lemma applies. $\langle V, H_2 \rangle$ is a dual pair foi Φ, for if $\Phi(v, h_2) = 0$ for all $h_2 \in H_2$ then $h_2 \in V^\perp$ so $h_2 = 0$, and if $\Phi(v, h_2) = 0$ for all $v \in V$ then since v is also orthogonal to $V^\perp$, $v \perp E$ which implies $v = 0$ by the semisimplicity of E. By the lemma, $\tau_\Phi V|_V = \sigma(V, H_2)$ and under these conditions $V' = H_2$.

Each element h_2 of H_2 corresponds to a function on V namely Φ_{h_2} with $\Phi_{h_2}(v) = \Phi(h_2, v)$. The Φ_{h_2} are linear, and they are even continuous since Φ is separately continuous. Φ_{h_2} extends uniquely to a linear continuous function $\Psi_{h_2}: \tilde{V} \to k$. $\tilde{V}' = \{\Psi_{h_2};\ h_2 \in H_2\} = H_2$ so $\tilde{V}'^* = H_2^*$. We also have $\tilde{v}(h_2) = \tilde{v}(\Psi_{h_2}) = \Psi_{h_2}(\tilde{v})$, (see [12], page 101).

Combining with the earlier result, $\tilde{V} \cong \tilde{V}'^* = H_2^*$ with topology $\tilde{\tau} = \sigma(\tilde{V}, \tilde{V}') = \sigma(H_2^*, H_2)$. This completes the proof of Theorem 6.

Applying the same proof technique to an arbitrary dual pair gives the

COROLLARY: *If $\langle V_1, V_2 \rangle$ is a dual pair, the completion of $(V_1, \sigma(V_1, V_2))$ is $(V_2^*, \sigma(V_2^*, V_2))$.*

THEOREM 7: *The unique bilinear form $\tilde{\Phi}$ of Theorem 5 has $\tilde{\Phi}(\tilde{v}, h_2) = \tilde{v}(h_2)$ for*

$\tilde{v} \in \tilde{V} = H_2^*$ *and* $h_2 \in H_2$. $\tilde{\tau} = \tau_{\tilde{\Phi}} \tilde{V}$. $(\tilde{E}, \tilde{\Phi})$ *is semisimple if and only if V is* $\perp$*-closed. If V is a maximal totally isotropic (resp.* $\perp$*-closed) subspace of E then* $\tilde{V}$ *is a maximal totally isotropic (resp.* $\perp$*-closed) subspace of* $\tilde{E}$.

Proof: To define the extension $\tilde{\Phi}$ of our bilinear form Φ it suffices to specify $\tilde{\Phi}(\tilde{v}, h_2)$ since other values are known from Theorem 5. We take as definition $\tilde{\Phi}(\tilde{v}, h_2) = \Psi_{h_2}(\tilde{v}) = \tilde{v}(h_2)$ and define $\tilde{\Phi}$ symmetrically and on sums bilinearly. This results in the general formula $\tilde{\Phi}(\tilde{u}+h_1+h_2, \tilde{u}'+h_1'+h_2') = \tilde{u}(h_2') + \tilde{u}'(h_2) + \Phi(h_1+h_2, h_1'+h_2')$. The associated $\tilde{Q}$ has $\tilde{Q}(\tilde{u}+h_1+h_2) = 2\tilde{u}(h_2) + Q(h_1+h_2)$.

To show $\tilde{Q}$ is continuous, let $\tilde{x} = \tilde{u}+h_1+h_2$ be arbitrary in $\tilde{E}$. $h_2 \in k(h_{2\beta})_{\beta \in B}$ for some finite set B. If $\tilde{v} \in V_B = \tilde{V} \cap k(h_{2\beta})_{\beta \in B}$ then $\tilde{Q}(\tilde{u}+h_1+h_2+\tilde{v}) = 2(\tilde{u}+\tilde{v})(h_2) + Q(h_1+h_2) = 2\tilde{u}(h_2) + Q(h_1+h_2) = Q(\tilde{u}+h_1+h_2)$. And by Theorem 6, V_B is a space in the $\tilde{\tau}$ zero neighborhood basis. So $\tilde{Q}$ is continuous.

Finally, $\tilde{Q}$ agrees with Q on E, for if $v \in V$ then $\tilde{Q}(v+h_1+h_2) = 2v(h_2) + Q(h_1+h_2) = 2\Phi(v, h_2) + Q(h_1+h_2) = Q(v+h_1+h_2)$ since $\Phi(v, h_2) = \Phi_{h_2}(v) = \Psi_{h_2}(v)$.

Thus $\tilde{Q}$ is the unique function of Theorem 5, hence in particular quadratic.

We turn our attention now to the completion topology. The conditions of the lemma apply to $(\tilde{E}, \tilde{\Phi})$. For $(H_2^*)^{\perp_{\tilde{\Phi}}} = H_2^* \oplus H_1$ since if $\Phi(\tilde{v}+h_1+h_2, \tilde{u}) = \tilde{u}(h_2) = 0$ identically in $\tilde{u}$ then $h_2 = 0$, while from Theorem 5, $\tilde{\Phi}(v+h_1, \tilde{u}) = 0$ for all $\tilde{u} \in H_2^*$. And $\langle H_2^*, H_2 \rangle$ is a dual pair for $\tilde{\Phi}$ since $\tilde{\Phi}(\tilde{u}, h_2) = \tilde{u}(h_2)$. Applying the lemma, $\sigma(H_2^*, H_2) = \tau_{\tilde{\Phi}} \tilde{V}|_{\tilde{V}}$ is the completion topology on $\tilde{V}$. Since the topology on H is discrete and the sum $\tilde{V} \oplus H$ is topological, the completion topology is $\tau_{\tilde{\Phi}} \tilde{V}$.

We now determine the conditions under which $(\tilde{E}, \tilde{\Phi})$ will be semisimple. First we prove that $\tilde{\Phi}$ is nondegenerate iff H_1 is semisimple.

If H_1 is semisimple we must show that $\tilde{x} = \tilde{u}+h_1+h_2 \perp \tilde{E}$ implies $\tilde{x} = 0$. If h_2 were not zero there would be a v in V with $1 = \Phi(v, h_2) = \tilde{\Phi}(v, h_2) = \tilde{\Phi}(v, \tilde{u}+h_1+h_2)$, so $h_2 = 0$. If $\tilde{x} = \tilde{u}+h_1$, with $h_1 \neq 0$ then by the semisimplicity of H_1, there exists an $h_1' \in H_1$, with $1 = \Phi(h_1, h_1') = \tilde{\Phi}(h_1', \tilde{u}+h_1)$ so $h_1 = 0$. Finally if $\tilde{u} \neq 0$ then there is an $h_2 \in H_2$ with $\tilde{\Phi}(\tilde{u}, h_2) = \tilde{u}(h_2) = 1$ so $\tilde{u} = 0$. Conversely if H_1 is not semisimple then there is an $h_1' \in H_1$ with $h_1' \perp H_1$. Let $\Phi_{h'_1} \in H_2^*$ with $\Phi_{h'_1}(h_2) = \Phi(h_1', h_2)$. For arbitrary $\tilde{u}+h_1+h_2$ in E, $\tilde{\Phi}(-\Phi_{h'_1}+h_1', \tilde{u}+h_1+h_2) = -\Phi_{h'_1}(h_2) + \Phi(h_1', h_1+h_2) = 0$, so $\tilde{\Phi}$ is degenerate, and $\dim(\operatorname{rad} H_1) \leqslant \dim(\operatorname{rad} \tilde{E})$. Hence $\dim(\operatorname{rad} H_1) = \dim(\operatorname{rad} \tilde{E})$.

The proof of the following lemma now shows that $\tilde{\Phi}$ is nondegenerate if and only if V is $\perp$-closed.

LEMMA: H_1 *is semisimple iff V is* $\perp$*-closed; in fact* $\dim(\operatorname{rad} H_1) = \dim(\operatorname{rad} \tilde{E}) = \dim V^{\perp\perp}/V$.

Proof: $V^\perp = V \oplus H_1$ so $V^{\perp\perp} = (V+H_1)^\perp = V^\perp \cap H_1^\perp = (V+H_1) \cap H_1^\perp = V + (H_1 \cap H_1^\perp)$. Since $V^{\perp\perp} = V \oplus (\operatorname{rad} H_1)$, $V = V^{\perp\perp}$ iff $\operatorname{rad} H_1 = (0)$.

To complete the proof of the theorem, sufficient conditions will be given for $\tilde{V}$ to be an orthogonally closed or a maximal totally isotropic subspace. As observed above,

$\tilde{V}^{\perp_{\tilde{\Phi}}} = V^{\perp} \oplus H_1$. If V is $\perp$-closed then H_1 is Φ-semisimple, so H_1 is $\tilde{\Phi}$-semisimple, therefore $\tilde{V}$ is $\perp$-closed.

For V a maximal totally isotropic subspace of E, H_1 must be anisotropic. So if $\tilde{u}+h_1 \in \tilde{V}^{\perp}$ then $\tilde{Q}(\tilde{u}+h_1)=\tilde{Q}(h_1) \neq 0$ unless $h_1=0$. Therefore $\tilde{V}$ is a maximal totally isotropic subspace of $(\tilde{E}, \tilde{\Phi})$.

A normal form for the decomposition of $(\tilde{E}, \tilde{\Phi})$ is given by

THEOREM 8: *$\tilde{E}=(G_2^* \oplus G_2) \overset{\perp}{\oplus} G_1$ with G_2, G_2^* totally isotropic,* $\dim G_i = \dim H_i$ *and $\Phi(g_2^*, g_2) = g_2^*(g_2)$ for all $g_2^* \in G_2^*$ and $g_2 \in G_2$.*

Proof: From Theorem 6, $\tilde{E}=H_2^* \oplus H_2 \oplus H_1$ with $H_2 = k(h_{2\alpha})_{\alpha \in K}$ and $H_1 = k(h_{1\alpha})_{\alpha \in J}$. As usual let Φ_h denote the function $H_2 \to k$ with $\Phi_h(h_2)=\Phi(h, h_2)$. Put $G_1 = k(h_{1\alpha} - \Phi_{h_{1\alpha}})$ and $G_2 = k(h_{2\alpha} - \frac{1}{2}\Phi_{h_{2\alpha}})_{\alpha \in K}$. The dimensions of G_1 and G_2 are clearly as specified. Every element $\tilde{u}+h_1+h_2$ of E can be written in the form $(\tilde{u}+\Phi_{h_1}+\frac{1}{2}\Phi_{h_2})+(h_1-\Phi_{h_1})+(h_2-\frac{1}{2}\Phi_{h_2})$ and the spaces H_2^*, G and G_2 have (0) intersection so $\tilde{E}=H_2^* \oplus G_2 \oplus G_1$. The remaining relationships are verified by routine calculation. Extending each $h_2^* \in H_2^*$ by zero to all of E we obtain h^{**}. $h^* \to h^{**}|_G$ is a topological isomorphism $H^* \to G^*$.

In the next theorem we show that the completion of the $(E, \tau_\Phi V)$ spaces coincide with the locally linearly compact spaces on which the form $\tilde{\Phi}$ is continuous.

THEOREM 9: *If $(E, \bar{\tau})$ is a locally linearly compact space and if the nondegenerate, bilinear form Φ is continuous (*i.e., *$\bar{\tau} \geqslant \tau_\Phi V$ for some totally isotropic V) then E is $\bar{\tau}$-complete and $\bar{\tau}=\tau_\Phi D$ for some linearly $\bar{\tau}$-compact $D \subset V$. Further $\tau_\Phi D = \tau_\Phi V$ if and only if* $\dim V/D$ *is finite. Conversely if $(E, \Phi, \tau_\Phi W)$ is complete then E is locally linearly $\tau_\Phi W$-compact.*

Proof: Since E is locally linearly $\bar{\tau}$-compact, there is a linearly $\bar{\tau}$-compact zero neighborhood U. V is $\bar{\tau}$-closed so $D=V \cap U$ is linearly $\bar{\tau}$-compact. And for finite dimensional F, $D \cap F^{\perp} = V \cap U \cap F^{\perp}$ is a $\bar{\tau}$-zero neighborhood. Therefore $\bar{\tau} \geqslant \tau_\Phi D$. But $(D, \tau_\Phi D|_D)$ is a linearly topologized space and D with the finer $\bar{\tau}|_D$ topology is a linearly compact space, so $\tau_\Phi D|_D = \bar{\tau}|_D$ (KÖTHE [12], page 98). Since D is both a $\bar{\tau}$ and a $\tau_\Phi D$ zero neighborhood, $E = D \oplus D_0$ is a topological sum and D_0 is discrete for both topologies (KÖTHE [12], page 96). Therefore $\bar{\tau}=\tau_\Phi D$.

To demonstrate that $\tau_\Phi D = \tau_\Phi V$ we apply the lemma to Theorem 1.

Conversely, if $(E, \tau_\Phi W)$ is complete then since W is $\tau_\Phi W$-closed, $W=\tilde{W}$. But W is also linearly $\tau_\Phi W$-bounded as we have already shown in Theorem 3. Hence $W=\tilde{W}$ is linearly $\tau_\Phi W$-compact, which shows that $(E, \tau_\Phi W)$ is locally linearly $\tau_\Phi W$-compact.

The proof gives the following interesting corollaries (cf. [9]). The first is an immediate consequence of Theorems 8 and 9.

COROLLARY 1: *If $(E, \bar{\tau})$ is locally linearly compact and Φ is continuous then E has a* *decomposition* $(D_2^* \oplus D_2) \overset{\perp}{\oplus} D_1$ *of the form given in Theorem* 8 *with* D_2^* *the locally linearly compact* D *of Theorem* 9.

COROLLARY 2: *If* $(E, \tau_\Phi W)$ *is complete and* A *is a semisimple* $\tau_\Phi W$*-closed subspace of* E *then either* A *is discrete or* $\dim A \geqslant \|k\|^{\aleph_0}$.

Proof: By the previous theorem, $(E, \tau_\Phi W)$ is locally linearly $\tau_\Phi W$-compact. Since A is closed, A is locally linearly compact with respect to the induced topology. Also Φ is continuous when restricted to A, so the previous corollary applies and $A = (D_2^* \oplus (D_2 \oplus D_1))$ with the $\tau_\Phi D_2^*$ topology on A. If $\dim D_2 < \aleph_0$ then $\dim D_2^* < \aleph_0$, and the topology on D_2^* would be discrete. Since D_2^* is a linear zero neighborhood, $D_2 + D_1$ is a topological complement of D_2^*. And the topology on $D_2 + D_1$ is discrete as well. In this case $\tau_\Phi D_2^*$ is the discrete topology.

If on the other hand $\dim D_2 \geqslant \aleph_0$ then $\|D_2^*\| \geqslant \|k\|^{\aleph_0}$, so $\dim A \geqslant \dim D_2^* \geqslant \|k\|^{\aleph_0}$.

III. Clifford Algebras

III.1 *Canonical Topologies on the Clifford Algebra*

With the tensor algebra $T(E)$ defined as usual over the vector space (E, Φ), let Q be the quadratic form associated with Φ and let I be the two-sided ideal generated by the elements $x \otimes x - Q(x)$ in $T(E)$. Then the Clifford algebra $C(E)$ is by definition $T(E)/I$. The equivalence class of $x_1 \otimes \cdots \otimes x_n$ will be denoted by $x_1 \circ \cdots \circ x_n$. If $E = k(e_\alpha)_{\alpha \in J}$ with J asymmetrically ordered by $<$, then for $S = \{\alpha_1, \ldots, \alpha_n\}$, $\alpha_1 < \cdots < \alpha_n$ let $e_S = e_{\alpha_1} \circ \cdots \circ e_{\alpha_n}$. The e_S together with the scalar 1 are a basis for $C(E)$ (for a proof see [1]). In particular if $x_1, \ldots, x_n$ are linearly independent elements of E, then $x_1 \circ \cdots \circ x_n \neq 0$.

If $f: E \to E$ is an isometry of (E, Φ) onto itself then f extends to an algebra homomorphism $g: T(E) \to T(E)$ which is the identity on k and has $g(x_1 \otimes \cdots \otimes x_n) = f(x_1) \otimes \cdots \otimes f(x_n)$. Since $f(x) \circ f(x) = Q(f(x)) = Q(x)$, f similarly induces an algebra homomorphism $h: C(E) \to C(E)$ with $h|_k = 1|_k$ and $h(x_1 \circ \cdots \circ x_n) = f(x_1) \circ \cdots \circ f(x_n)$. Further let E have basis $(e_\alpha)_{\alpha \in I}$ then $(f(e_\alpha))_{\alpha \in I}$ is also a basis for E. Basis elements $e_{i_1} \circ \cdots \circ e_{i_n}$ of $C(E)$ are mapped by h onto a complete set of basis elements $f(e_{i_1}) \circ \cdots \circ f(e_{i_n})$ of $C(E)$, so h is bijective. Thus every isometry f of E induces an algebra isomorphism h of $C(E)$. Conversely, if h is an algebra isomorphism of $C(E)$ which maps E onto E and is the identity on k then the restriction of h to E is an isometry, for $\Phi(h(x), h(x)) = h(x) \circ h(x) = h(x \circ x) = h(\Phi(x, x)) = \Phi(x, x)$. Because of this canonical relation between the isometries of E and algebra isomorphisms of $C(E)$, we shift our attention to the problem of topologizing the Clifford Algebra.

Starting with a linearly topologized space (E, τ), there are many ways of constructing linear topologies on the tensor products $\overset{p}{\underset{1}{\otimes}} E$. Here we shall consider two tensor product topologies, the τ_ε topology, corresponding to the ε-product of

SCHWARTZ and the "projective" topological tensor product topology corresponding to that of GROTHENDIECK. These topologies have been studied in [6].

A linear topology on the tensor products extends canonically by taking the direct sum topology on the tensor algebra and then the quotient topology to a linear topology on the Clifford algebra. If this extension is to be useful it must induce the initial topology when restricted to E. We now investigate whether this is the case for either the ε-product or the projective tensor product extensions.

Since it will quickly become apparent that the ε-product topology is not suitable in the sense just mentioned, we shall describe it only briefly. For further detail the reader is referred to [6]. The τ_e topology is the finest linear topology on $\overset{p}{\underset{1}{\otimes}} E$ for which the canonical multilinear map $\prod_1^p E_i \to \overset{p}{\underset{1}{\otimes}} E_i$ is uniformly continuous. For each p, τ_e has a neighborhood basis at zero of sets

$$\hat{U}_p = U_p \otimes E \otimes E \otimes \cdots \otimes E + E \otimes U_p \otimes E \otimes \cdots \otimes E + \cdots + E \otimes E \otimes E \otimes \cdots \otimes U_p$$

each summand containing p factors and the U_p running through a zero neighborhood basis for the topology τ on E. A zero neighborhood basis for the tensor algebra consists of sets $\hat{U} = \bigoplus_{p=1}^{\infty} \hat{U}_p$ and a zero neighborhood basis for the Clifford algebra of the sets $\sigma \hat{U}$ where σ is the canonical map $T(E) \to C(E)$. These extensions as well as the ε-product topologies on the tensor products will be denoted by τ_e.

THEOREM 10: *If (E, τ) is discrete then $(C(E), \tau_e)$ is discrete. If (E, τ) is not discrete then $(C(E), \tau_e)$ is trivial.*

Proof: If (E, τ) is discrete then (0) is in the zero neighborhood basis for τ. In the expression for $\hat{U}_p$ in the preceding paragraph taking $U_p = (0)$ for every p gives $\hat{U} = (0)$ and $\sigma(\hat{U}) = (0)$. Since (0) is thus in the zero neighborhood bases for $(C(E), \tau_e)$, the latter is discrete in this case.

On the other hand if (E, τ) is not discrete and $\sigma(\hat{U}) = \sigma(U_1 + U_2 \otimes E + E \otimes U_2 + U_3 \otimes E \otimes E + \cdots)$ is an arbitrary set in the zero neighborhood bases of $(C(E), \tau_e)$ then every element of the form $x_1 \circ x_2 \circ \cdots \circ x_n$ is in $\sigma \hat{U}$. For since τ is not discrete, there is an element $y \neq 0$ in U_{n+2}, and since E is semisimple, there is a $z \in E$ with $\Phi(y, z) = \frac{1}{2}$. $y \circ z + z \circ y = 1$, and so $x_1 \circ \cdots \circ x_n = y \circ z \circ x_1 \circ \cdots \circ x_n + z \circ y \circ x_1 \circ \cdots \circ x_n \in \sigma \hat{U}$. $\sigma(\hat{U})$ is thus seen to be a subspace of $C(E)$ containing a set of generators of $C(E)$, hence $\sigma(\hat{U}) = C(E)$. In this case τ_e is the trivial topology on $C(E)$.

So requiring that the τ_e topology on $C(E)$ induce the initial topology τ on E would leave for consideration only the uninteresting cases where τ is discrete or trivial. For this reason the τ_e topology will not be discussed further.

We now turn our attention to the projective tensor product topology $\overset{p}{\underset{1}{\otimes}}\tau$ on the tensor product $\overset{p}{\underset{1}{\otimes}}E$. In [6] it is shown that there is a unique linear topology on $E\otimes E$ with the following properties: (1) the canonical bilinear map $\omega_2: E\times E\to E\otimes E$ is continuous and (2) if f is a bilinear continuous map of $E\times E$ into a linearly topologized k-vector space G then the induced linear map $E\otimes E\to G$ is continuous. The proof extends to $\overset{p}{\underset{1}{\otimes}}E$. $\overset{p}{\underset{1}{\otimes}}\tau$ is by definition this unique topology. Clearly $\overset{p}{\underset{1}{\otimes}}\tau$ is the finest linear topology on $\overset{p}{\underset{1}{\otimes}}E$ for which $\omega_p: \prod_1^p E\to \overset{p}{\underset{1}{\otimes}}E$ is continuous. If τ is Hausdorff so is $\overset{p}{\underset{1}{\otimes}}\tau$ (for details see [6]).

A neighborhood basis at zero for the $\overset{2}{\underset{1}{\otimes}}\tau$ topology is given by the subspaces $\hat{U}_2=U_2\otimes U_2+\sum_{x\in E}[x]\otimes U_{2x}+\sum_{x\in E}U_{2x}\otimes[x]$ with U_{2x} and U_2 running through a zero neighborhood basis of τ. This is so since ω_2 is continuous if and only if it is continuous at (0, 0) and is separately continuous at $(x, 0)$ and $(0, x)$ for every $x\in E$. If ω_2 is to be continuous for a topology $\bar{\tau}$ on $E\otimes E$ then every set in the $\bar{\tau}$ zero neighborhood basis must contain a space of the form $\hat{U}_2$. Conversely, the spaces $\hat{U}_2$ define a linear topology on E for which ω_2 is continuous. The same reasoning applies for any p, so a zero neighborhood basis for $\overset{p}{\underset{1}{\otimes}}\tau$ consists of the sets

$$\begin{aligned}\hat{U}_p = U_p\otimes\cdots\otimes U_p &+ \sum_{x\in E}\sum_{\text{perms}}[x]\otimes U_{px}\otimes\cdots\otimes U_{px}\\ &+ \sum_{x,y\in E}\sum_{\text{perms}}[x]\otimes[y]\otimes U_{pxy}\otimes\cdots\otimes U_{pxy}\\ &+\cdots+\sum_{x_1,\ldots,x_{p-1}\in E}\sum_{\text{perms}}[x_1]\otimes[x_2]\otimes\cdots\otimes[x_{p-1}]\otimes U_{px_1x_2\ldots x_{p-1}},\end{aligned}$$

with the subscripted U's running through a zero neighborhood basis for τ and

$$\begin{aligned}\sum_{\text{perms}}[x]\otimes U_{px}\otimes\cdots\otimes U_{px} &= [x]\otimes U_{px}\otimes\cdots\otimes U_{px}+\\ &+U_{px}\otimes[x]\otimes U_{px}\otimes\cdots\otimes U_{px}+\cdots+U_{px}\otimes U_{px}\otimes\cdots\otimes[x]\end{aligned}$$

and with similar meanings for the other $\sum_{\text{perms}}$ symbols. Henceforth in this chapter $\sum_{x_1,\ldots,x_q\in E}\sum_{\text{perms}}$ will be abbreviated by $\sum$ and $\hat{U}_p$ will be written simply

$$\begin{aligned}U_p\otimes\cdots\otimes U_p+\sum[x]\otimes U_{px}\otimes\cdots\otimes U_{px}&+\\ \sum[x]\otimes[y]\otimes U_{pxy}\otimes\cdots\otimes U_{pxy}&+\cdots.\end{aligned}$$

Taking sums and quotients, the projective tensor product topologies induce linear opologies on $T(E)$ and $C(E)$, both denoted by $\otimes\tau$.

We should like to remark that it is the requirement of separate continuity for ω_p which is responsible for the complexity of the $\hat{U}_p$'s and the subsequent difficulties in the proof of Theorem 18 below.

We now determine for which topologies τ on E the induced topology $\otimes\tau|_E$ is equal to τ.

THEOREM 11: *If E has a zero neighborhood basis of subspaces no one of which is totally isotropic then $\otimes\tau|_E$ is trivial.*

Proof: Let $\sigma(\hat{U})=\sigma(\overset{\infty}{\underset{1}{\oplus}}\hat{U}_p)=\sigma(U_1+U_2\otimes U_2+\sum[x]\otimes U_{2x}+U_3\otimes U_3\otimes U_3+\cdots)$ be an arbitrary space from the zero neighborhood basis for $\otimes\tau$ on $C(E)$. We claim that an arbitrary element x of E is an element of $\sigma(\hat{U})$. For U_x is not totally isotropic so there exists a $y\in U_x$ with $Q(y)\neq 0$. $x=1/Q(y)\, y\circ y\circ x=\sigma((1/Q(y))\, y\otimes y\otimes x)\in\hat{U}_3\subset\sigma(\hat{U})$. But this implies $\sigma(\hat{U})\cap E=E$ for arbitrary $\sigma(\hat{U})$, hence the assertion of the theorem.

On the other hand, if the conditions of Theorem 11 are not met then some linear neighborhood V of zero is totally isotropic. Intersecting V with the spaces of the zero neighborhood basis gives a zero neighborhood basis for totally isotropic subspaces. In this case we have

THEOREM 12: *Let (E,τ) have a zero neighborhood basis $\{U_\alpha\}$ of totally isotropic subspaces. Then $\otimes\tau|_E=\tau$ if and only if $E=\bigcup_\alpha U_\alpha^\perp$.*

Proof of necessity: Let $x\in E$. We shall show $x\in\bigcup U_\alpha^\perp$. For $x=0$ this conclusion is immediate so suppose $x\neq 0$. Since τ is Hausdorff there is a U_α with $x\notin U_\alpha$; $\otimes\tau|_E=\tau$ so there is a $\hat{U}$ with $\sigma(\hat{U})\cap E\subset U_\alpha$; $\hat{U}=U_1+U_2\otimes U_2+\sum[x]\otimes U_{2x}+U_3\otimes U_3\otimes U_3+\sum[x]\otimes U_{3x}\otimes U_{3x}+\sum[x]\otimes[y]\otimes U_{3xy}+\cdots$. Suppose by way of contradiction that $x\notin U_{3xx}^\perp$. Then there is a $y\in U_{3xx}$ with $\phi(x,y)=1$; therefore $x=\Phi(x,y)\,x=x\circ y\circ x+y\circ x\circ x\in\sigma(\hat{U})$ which, since $x\in E$, implies $x\in U_\alpha$ a contradiction. We conclude $x\in U_{3xx}^\perp$.

Proof of sufficiency: Since each $\sigma(\hat{U})\cap E=\sigma(U_1+\cdots)\cap E\supset U_1$, $\otimes\tau|_E\leqslant\tau$. Now suppose U_1 is an arbitrary space in the τ zero neighborhood basis. We shall construct $\hat{U}$ such that $\sigma(\hat{U})\cap E\subset U_1$. First take $U_n\subset U_1$ for all n. By hypothesis for every x there is a U_x such that $x\perp U_x$. Take $U_{nx_1\ldots x_m}=U_n\cap\bigcap_{i=1}^{m}U_{x_i}$ and $\hat{U}=U_1+U_2\otimes U_2+\sum[x]\otimes U_{2x}+\cdots$. We first note that $\sigma(\sum_{\text{perms}}[x_1]\otimes\cdots\otimes[x_m]\otimes U_{nx_1\ldots x_m}\otimes\cdots\otimes U_{nx_1\ldots x_m})=\sigma([x_1]\otimes\ldots[x_m]\otimes U_{nx_1\ldots x_m}\otimes\cdots\otimes U_{nx_1\ldots x_m})$, because if $u\in U_{nx_1\ldots x_m}$ then $u\circ x_i=-x_i\circ u$, $1\leqslant i\leqslant m$. So every element of $\sigma(\hat{U})$ is of the form $t=\sum_{\text{finite}}t_i\circ u_i$, $u_i\in U_1$. Suppose by way of contradiction $x\in\sigma(\hat{U})\cap E$ but $x\notin U_1$. Let $(e_\alpha)_{\alpha\in A}$ be a basis for the vectorspace U_1. The set $(x,e_\alpha)_{\alpha\in A}$ being linearly independent can be extended to a basis for E. Since $x\in\sigma(\hat{U})$, by the remarks above x is of the form $\sum_{i=1}^{n}t_i\circ u_i=\sum_{i=1}^{n}t_i\circ(\sum_{j=1}^{m}\lambda_{ij}e_{\alpha_j})=$

$\sum_{i=1}^{n} \sum_{j=1}^{m} \lambda_{ij} t_i \circ e_{\alpha_j}$. Multiplying through by $e_{\alpha_1} \circ e_{\alpha_2} \circ \cdots \circ e_{\alpha_m}$ gives $x \circ e_{\alpha_1} \circ \cdots \circ e_{\alpha_m} = 0$ since U_1 is totally isotropic. But this is not possible since $x \circ e_{\alpha_1} \circ \cdots \circ e_{\alpha_m}$ is an element of a basis of the Clifford algebra. Hence $\sigma(\hat{U}) \cap E \subset U_1$. $\therefore \otimes\tau|_E = \tau$.

As an immediate consequency of the construction in the proof of Theorem 12 we have for future reference the

COROLLARY: *If (E, τ) has a zero neighborhood basis of totally isotropic subspaces $\{U_\alpha\}$ then $(T(E), \otimes\tau)$ has a zero neighborhood basis of sets $\hat{U}$ such that t in $\sigma(\hat{U})$ implies $t = \sum_{\text{finite}} t_j \circ e_j$ with the e_j linearly independent elements from a single totally isotropic U_α.*

Next we shall describe the topologies for which the conditions of Theorem 12 are realized.

THEOREM 13: *If (E, τ) is a linearly topologized space with a zero neighborhood basis of totally isotropic subspaces U_α and $E = \bigcup_\alpha U_\alpha^\perp$ then $\tau \geqslant \tau_\Phi U_\alpha$ for every α. Conversely, if (E, τ) is a linearly topologized space with $\tau \geqslant \tau_\Phi V$ for some totally isotropic subspace V of E then τ has a zero neighborhood basis of totally isotropic subspaces U_α and $E = \bigcup_\alpha U_\alpha^\perp$.*

Proof: To show $\tau \geqslant \tau_\Phi U_{\alpha_0}$, let $U_{\alpha_0} \cap F^\perp$ be a set in the $\tau_\Phi U_{\alpha_0}$ zero neighborhood basis, $F = k(x_i)_{1 \leqslant i \leqslant n}$. Since $E = \bigcup_\alpha U_\alpha^\perp$ there exist zero neighborhoods U_{α_i} with $x_i \perp U_{\alpha_i}$. $U_{\alpha_0} \cap \bigcap_{i=1}^{n} U_{\alpha_i} \subset U_{\alpha_0} \cap F^\perp$ proving the contention.

Conversely suppose $\tau \geqslant \tau_\Phi V$. The spaces $V \cap F_\beta^\perp$ with F_β a finite dimensional subspace of E are by hypothesis part of a zero neighborhood basis $\{U_\alpha\}$ for τ. Since $V \cap F_\beta^\perp$ is totally isotropic the U_α may be chosen totally isotropic. But the $(V \cap F_\beta^\perp)^\perp$ already cover E since $E = \bigcup_\beta F_\beta$ and $F_\beta \subset F_\beta^{\perp\perp} \subset (V \cap F_\beta^\perp)^\perp$. Therefore $E = \bigcup_\alpha U_\alpha^\perp$.

It is interesting to note that when $\tau = \tau_\Phi V$, V of infinite dimension and codimension, the $\tau \otimes \tau$ topology on $E \otimes E$ is strictly finer than the τ_e topology. This will be proved at the end of this chapter at which time certain lemmas and theorems will be available to make the proof easy.

The topology on $C(E)$ can only be considered admissible if continuous orthogonal automorphisms of E induce continuous algebra isomorphisms of $C(E)$ and conversely. The projective tensor product topology has this essential property as the following theorems shows.

THEOREM 14: *Let f be an orthogonal automorphism of (E, Φ), g the corresponding algebra isomorphism of $T(E)$ (with $g(x_1 \otimes \cdots \otimes x_n) = f(x_1) \otimes \cdots \otimes f(x_n)$), and let h be the corresponding algebra isomorphism of $C(E)$ (with $h(x_1 \circ \cdots \circ x_n) = f(x_1) \circ \cdots \circ f(x_n)$). If $\tau \geqslant \tau_\Phi V$, V totally isotropic, then f is τ-continuous if and only if h is $\otimes\tau$-continuous.*

Proof: Suppose f is τ-continuous. Let $\hat{U}=U_1+U_2\otimes U_2+\sum[x]\otimes U_{2x}+\cdots$ be a set in the $T(E)$ zero neighborhood basis. For every U_n (resp. $U_{nf(x)f(y)\ldots}$) there is a V_n (resp. $V_{nxy\ldots}$) such that $f(V_n)\subset U_n$ (resp. $f(V_{nxy\ldots})\subset U_{nf(x)f(y)\ldots}$). $g(\hat{V})=g(V_1+V_2\otimes V_2+\sum[x]\otimes V_{2x}+\cdots\subset U_1+U_2\otimes U_2+\sum[f(x)]\otimes U_{2f(x)}+\cdots=\hat{U}$ establishing the continuity of g.

Clearly $h\sigma=\sigma g$ so $h\sigma(\hat{V})=\sigma g(\hat{V})\subset\sigma(\hat{U})$, and h is likewise $\otimes\tau$-continuous.

Conversely if h is an algebra isomorphism of $C(E)$ with $h|_E=f$ and $h|_k=1|_k$ then we already know $h|_E=f$ is an orthogonal automorphism of E. Since $\tau\geqslant\tau_\Phi V$, $\tau=\otimes\tau|_E$, so the continuity of h implies the continuity of $h|_E$.

Applying the theorem to f^{-1} and h^{-1} gives the

COROLLARY: *With the hypothesis of Theorem* 14, *f is open if and only if h is.*

Although not essential, it would be desirable to have a Hausdorff topology on $C(E)$. We first consider separate continuity of multiplication in $T(E)$ since this result will be used in the proof of Hausdorff. Later in the chapter the subject of continuity of multiplication will be discussed in more detail.

THEOREM 15: *Multiplication is separately continuous in* $(T(E), \otimes\tau)$.

Proof: First consider multiplication on the left by $S=x_1\otimes\cdots\otimes x_p\in\overset{p}{\underset{1}{\otimes}}E$. For every q the map $\prod_1^q E\to\prod_1^{p+q}E\to\overset{p+q}{\underset{1}{\otimes}}E$ with $(y_1,\ldots,y_q)\to(x_1,\ldots,x_p,y_1,\ldots,y_q)\to x_1\otimes\cdots\otimes x_p\otimes y_1\otimes\cdots\otimes y_q$ is continuous and so induces a continuous map $\overset{q}{\underset{1}{\otimes}}E\to\overset{p+q}{\underset{1}{\otimes}}E$ with $y_1\otimes\cdots\otimes y_q\to x_1\otimes\cdots\otimes x_p\otimes y_1\otimes\cdots\otimes y_q$ (by the definition of $\overset{p+q}{\underset{1}{\otimes}}E$). Addition gives a $\otimes\tau$ continuous map $T(E)\to T(E)$ with $t\to s\otimes t$. The argument readily extends to multiplication by a sum of such s's.

Using Theorem 15 we can prove the

COROLLARY: *If A is a two sided ideal in $(T(E), \otimes\tau)$ then $\bar{A}$, the topological closure of A is also a two sided ideal.*

This follows from the separate continuity of multiplication.

With the corollary above we are in a position to prove $(C(E), \otimes\tau)$ is Hausdorff for $\tau\geqslant\tau_\Phi V$:

THEOREM 16: *$\tau\geqslant\tau_\Phi V$ for V some totally isotropic subspace of E if and only if $(C(E), \otimes\tau)$ is Hausdorff.*

Proof: Using Theorem 13 it suffices to show that $(C(E), \otimes\tau)$ is Hausdorff iff τ has a zero neighborhood basis of totally isotropic subspaces U_α and $E=\bigcup_\alpha U_\alpha^\perp$. The topology $\otimes\tau$ on $C(E)$ was obtained by quotients from the $\otimes\tau$ topology on $T(E)$. Under these circumstances it is well known (see for example [12]) that $(C(E), \otimes\tau)$ is

Hausdorff iff $I=\bar{I}$ (I the two sided ideal in $T(E)$ generated by the elements $x\otimes x-Q(x)$, or equally well by the elements $x\otimes y+y\otimes x-2\Phi(x, y)$.)

Suppose $\otimes\tau$ is Hausdorff. $I=\bar{I}$ is a proper ideal in $T(E)$ so in particular $-1\notin\bar{I}$. Therefore there exists $\hat{U}=U_1+U_2\otimes U_2+\sum[x]\otimes U_{2x}+\cdots$ in the usual zero neighborhood basis for $T(E)$ with $-1+\hat{U}$ disjoint from I. We claim U_2 is totally isotropic. For if this were not so there would be an $x\in U_2$ with $\|x\|\neq 0$. Put $y=x(2\|x\|)^{-1}$ then y is also in U_2 and $\Phi(x, y)=\frac{1}{2}$. So $x\otimes y+y\otimes x-1\in(-1+\hat{U})\cap I$, contradiction. We may therefore assume that all the U_α are totally isotropic.

We claim in addition that for each $x\in E$, $x\perp U_{2x}$. If not there would be a $y\in U_{2x}$ with $\Phi(x, y)=\frac{1}{2}$, and then $x\otimes y+y\otimes x-1\in(-1+\hat{U})\cap I$ as before; contradiction. $\therefore E=\bigcup_\alpha U_\alpha^\perp$.

To prove the converse we assume $E=\bigcup_\alpha U_\alpha^\perp$ for some totally isotropic zero neighborhood basis $\{U_\alpha\}$. By the corollary to Theorem 12 proved earlier, $T(E)$ has a zero neighborhood basis of sets $\hat{U}$ such that if $t\in\sigma(\hat{U})$ then $t=\sum_{j=1}^{n} t_j\circ e_j$ with the e_j linearly independent elements from a single totally isotropic zero neighborhood U_{α_0}. We claim this implies $1\notin\bar{I}$. For if $1\in\bar{I}$ then $1+\hat{U}$ meets I, and so $1+\sigma(\hat{U})$ meets (0) say in $1+t$. We have $0=1+t=1+\sum_{j=1}^{n} t_j\circ e_j$. Multiplying by $e_1\circ\cdots\circ e_n$ gives $0=e_1\circ\cdots\circ e_n$ which is impossible since the e_j are linearly independent.

Thus it is clear that $1\notin\bar{I}$; in particular $\bar{I}\neq T(E)$. But $C(E)=T(E)/I$ is a simple algebra and $I\subset\bar{I}$, so $I=\bar{I}$. Thus $\otimes\tau$ is Hausdorff.

Summarizing some of the properties of the $\tau_\Phi V$ topologies we have the following

THEOREM 17: *If (E, τ) is a linearly topologized space and τ is not the trivial topology then the following are equivalent:*

(i) $\tau\geqslant\tau_\Phi V$ *for some totally isotropic* V

(ii) $\phi: E\times E\to k$ *is continuous*

(iii) τ *has a zero neighborhood basis of sets* U_α *with* $E=\bigcup_\alpha U_\alpha^\perp$

(iv) $\otimes\,\tau|_E=\tau$

(v) $(C(E), \otimes\tau)$ *is Hausdorff.*

III.2 *Continuity of Multiplication*

We turn our attention to the question of continuity of multiplication in $(C(E), \otimes\tau_\Phi V)$. For denumerable (E, Φ) we shall establish the remarkable fact that $(C(E), \otimes\tau_\Phi V)$ is a topological algebra for closed V, (Theorem 18). It is not clear whether a similar result holds in the nondenumerable case. For τ strictly finer than $\tau_\Phi V$, we shall give an example of a denumerable (E, Φ) for which multiplication fails to be continuous in $(C(E), \otimes\tau)$. First we prove

LEMMA: *If* $E = k(e_\alpha)_{\alpha\in I}$ *then the sets*

$$\hat{U} = U_1 + U_2 \otimes U_2 + \sum_{e_\alpha} \sum_{\text{perms}} [e_\alpha] \otimes U_{2\,e_\alpha} + U_3 \otimes U_3 \otimes U_3 + \sum_{e_\alpha} \sum_{\text{perms}} [e_\alpha] \otimes U_{3\,e_\alpha} \otimes U_{3\,e_\alpha} + \sum_{e_\alpha e_\beta} \sum_{\text{perms}} [e_\alpha] \otimes [e_\beta] \otimes U_{3\,e_\alpha e_\beta} + \cdots$$

form a zero neighborhood basis for $\otimes\tau$ *on* $T(E)$ *when the subscripted* U*'s run through a zero neighborhood basis for* τ.

We shall again write $\sum$ to mean $\sum_{e_\alpha e_\beta \cdots} \sum_{\text{perms}}$.

Proof: Clearly each $\otimes\tau$ zero neighborhood contains such a $\hat{U}$. Conversely, put $V_n = U_n$ and for $x_1, x_2, \ldots, x_m \in k(e_\alpha)_{\alpha\in A}$, A finite, put $V_{m\,x_1\ldots x_m} = \bigcap_{\alpha_i \in A} U_{n\,e_{\alpha_1}\ldots e_{\alpha_m}}$. Then

$$\sum_{\alpha_i \in A} [e_{\alpha_1}] \otimes \cdots \otimes [e_{\alpha_m}] \otimes U_{n\,e_{\alpha_1}\ldots e_{\alpha_m}} \otimes \cdots \otimes U_{n\,e_{\alpha_1}\ldots e_{\alpha_m}} \supset [x_1] \otimes \cdots \otimes [x_m] \otimes V_{n\,x_1\ldots x_m} \otimes \cdots \otimes V_{n\,x_1\ldots x_m}$$

so $\hat{U}$ contains a $\otimes\tau$ zero neighborhood.

THEOREM 18: *If* $\dim E = \aleph_0$ *and* V *is a closed totally isotropic subspace of* E *then* $(T(E), \otimes\tau_\Phi V)$ *and* $(C(E), \otimes\tau_\Phi V)$ *are topological algebras.*

Proof: Since $\dim E = \aleph_0$ and V is closed and totally isotropic there is a decomposition of E into $(V \oplus V') \overset{\perp}{\oplus} G$ with $V = k(v_i)_{i\geqslant 1}$ and $V' = k(v'_i)_{i\geqslant 1}$ both totally isotropic and $\Phi(v_i, v'_j) = \delta_{ij}$. The $\tau_\Phi V$ topology has a zero neighborhood basis of sets $k(v_i)_{i>n}$ since for F finite dimensional, $F \subset V + k(v'_i)_{i\leqslant n} + G$ so $V \cap F^\perp \supset V \cap V^\perp \cap k(v'_i)^\perp_{i\leqslant n} \cap G^\perp = V \cap k(v'_i)^\perp_{i\leqslant n} = k(v_i)_{i>n}$. We shall need an enumerated basis for E, so let $E = k(e_i)_{i\geqslant 1}$ with $v_i = e_{2i}$ for $i \geqslant 1$. Then the sets $U^\star_n = k(e_i)_{i>n,\ i\ \text{even}}$ are a $\tau_\Phi V$ zero neighborhood basis. (They are not distinct. In fact $U^\star_1 = k(v_i)_{i>0}$, $U^\star_2 = U^\star_3 = k(v_i)_{i>1}$, etc.). The advantage of this numbering is that it yields the following simple criterion: $e_i \in U^\star_n$ iff $e_i \in V$ and $i > n$. The $U^\star_n$ will be referred to as $\star$-sets in the rest of the proof.

To show multiplication in $T(E)$ is continuous at $(0, 0)$ let $\hat{U}' = U'_1 + U'_2 \otimes U'_2 + \sum [e_i] \otimes U'_{2\,e_i} + \cdots$ be a set in the $(T(E), \otimes\tau)$ zero neighborhood basis. We must find $\hat{V} \otimes \hat{V} \subset \hat{U}'$. Clearly it suffices to find $\hat{V} \otimes \hat{V} \subset \hat{U} \subset \hat{U}'$. With this in mind we shrink $\hat{U}'$ somewhat, in order to make it more manageable, as follows. Choose inductively sets U_n which are $\star$-sets and such that $U_1 \subset U'_1 \cap U^\star_1$ and $U_n \subset U_1 \cap U_2 \cap \cdots \cap U_{n-1} \cap U'_n \cap U^\star_n$. Denote by $U''_{n\,e_m}$ the set $\bigcap_{t\leqslant n-1} \bigcap_{j_1\ldots j_t \leqslant m} U'_{n\,e_{j_1}\ldots e_{j_t} e_m}$ (i.e., the intersection of all sets $U'_{n\,e_{j_1}\ldots e_{j_t} e_m}$ for which m is the largest e-subscript). Define the sets $U_{n\,e_m}$ by induction on m to be $\star$-sets contained in $U_n \cap U_{n\,e_1} \cap \cdots \cap U_{n\,e_{m-1}} \cap U''_{n\,e_m} \cap U^\star_m$ with $U_{n\,e_1}$ a $\star$-set contained in $U_n \cap U''_{n\,e_1} \cap U^\star_1$. Since the U_n and $U_{n\,e_m}$ are $\star$-sets there are functions g_n with $U_n = U^\star_{g_n(0)}$ and $U_{n\,e_m} = U^\star_{g_n(m)}$. As a consequence of the construction we have $U^\star_{g_n(0)} = U_n \subset U^\star_n$ so

$$0 < n \leqslant g_n(0)$$

and $U^{\star}_{g_n(m)}=U_{n\,e_m}\subset U^{\star}_m$ so

$$m\leqslant g_n(m).$$

Also if $0<i\leqslant j$ then $U_{n\,e_j}\subset U_{n\,e_i}\subset U_n$ so $U^{\star}_{g_n(j)}\subset U^{\star}_{g_n(i)}\subset U^{\star}_{g_n(0)}$ therefore

$$g_n(0)\leqslant g_n(i)\leqslant g_n(j)\quad \text{for}\quad 0<i\leqslant j.$$

Take

$$\hat{U}=U_1+U_2\otimes U_2+\sum[e_i]\otimes U_{2\,e_i}+U_3\otimes U_3\otimes U_3+\sum[e_i]\otimes U_{3\,e_i}\otimes U_{3\,e_i}+\sum[e_i]\otimes[e_j]\otimes U_{3\,e_i\,e_j}+\cdots$$

with $U_{n\,e_{i_1}\ldots e_{i_m}}=U_{n\,e_{i_k}}$ where $i_k=\max(i_1,\ldots,i_m)$.

To define $\hat{V}$ we shall make use of a function used in enumerating $N\times N$, N the nonnegative integers. For n, $m\in N$ put $f(n,m)=\frac{1}{2}(n+m)(n+m+1)+n+1$. Then $f(n_1,m_1)\leqslant f(n_2,m_2)$ iff either $n_1+m_1<n_2+m_2$ or $n_1+m_1=n_2+m_2$ and $n_1\leqslant n_2$. For our purposes it suffices that f have the property that for any two pairs (n_1,m_1) and (n_2,m_2), $f(n_1,m_1)$ and $f(n_2,m_2)$ are comparable, and for only finitely many (n_1,m_1) is $f(n_1,m_1)\leqslant f(n_2,m_2)$. We now define the $\hat{V}_q$ for our $\hat{V}$. For prescribed q, $e_{j_1'},\ldots,e_{j_m'}$ let

$$V_{q\,e_{j_1'}\ldots e_{j_m'}}=\bigcap_{\substack{\text{all } p,\, i_n\geqslant 0 \text{ with}\\ f(p,i_n)\leqslant f(q,j_m)}}\Big[U^{\star}_{g_{p+q}(j_m)}\cap U^{\star}_{g_{p+q}(i_n)}\cap\bigcap_{i=1}^{g_{p+q}{}^{p}(j_m)}U^{\star}_{g_{p+q}(i)}\cap\bigcap_{i=1}^{g_{p+q}{}^{p}(i_n)}U^{\star}_{g_{p+q}(i)}\Big]\cap U^{\star}_{j_m}$$

where $j_m=\max(j_1',\ldots,j_m')$. Finally take the expression for V_q to be the same as that for $V_{q\,e_{j_1'}\ldots e_{j_m'}}$ with j_m replaced by 0 throughout. g^p_{p+q} is defined iteratively by $g^1_{p+q}(j_m)=g_{p+q}(j_m)$, and $g^p_{p+q}(j_m)=g_{p+q}(g^{p-1}_{p+q}(j_m))$. Put $\hat{V}=\oplus\hat{V}_p=V_1+V_2\otimes V_2+\sum[e_i]\otimes V_{2\,e_i}+\cdots$ as usual. The reason for the choice of each part of $V_{q\,e_{j_1'}\ldots e_{j_m'}}$ will become apparent in the cases we consider in showing that $\hat{V}\otimes\hat{V}\subset\hat{U}$.

Let $s'=e_{i_1'}\otimes\cdots\otimes e_{i_n'}\otimes e_{i_{n+1}'}\otimes\cdots\otimes e_{i_p'}\in\hat{V}_p$ with $e_{i_k'}\in V_{p\,e_{i_1'}\ldots e_{i_n'}}$, $n+1\leqslant k\leqslant p$. Let $t=e_{j_1'}\otimes\cdots\otimes e_{j_q'}$ with $e_{j_k'}\in V_{q\,e_{j_1'}\ldots e_{j_m'}}$, $m+1\leqslant k\leqslant q$. Let $i_1\leqslant i_2\leqslant\cdots\leqslant i_n$ be the subscripts $i_1',\ldots,i_n'$ in their natural order and $i_{n+1}\leqslant\cdots\leqslant i_p$ the subscripts $i_{n+1}',\ldots,i_p'$ in their natural order. Since $e_{i_{n+1}}\in V_{p\,e_{i_1'}\ldots e_{i_n'}}\subset U^{\star}_{i_n}$, $i_{n+1}>i_n$ giving the combined ordering $i_1\leqslant\cdots\leqslant i_n<i_{n+1}\leqslant\cdots\leqslant i_p$. Similarly let $j_1\leqslant\cdots\leqslant j_m<j_{m+1}\leqslant\cdots\leqslant j_q$ be the natural order of the j_k'. Note that $e_{i_{n+1}},\ldots,e_{i_p},e_{j_{m+1}},\ldots,e_{j_q}$ are all in V.

We now show $s'\otimes t'\in\hat{U}$. The general nature of the next steps in the proof is this. Let $l_1\leqslant l_2\leqslant\cdots\leqslant l_s\leqslant l_{s+1}\leqslant\cdots\leqslant l_{p+q}$ be the subscripts $i_1,\ldots,i_p$, $j_1,\ldots,j_q$ arranged in order. Let $l_{s+1}>i_n,j_m$. Then $e_{l_{s+1}},e_{l_{s+2}},\ldots$ are all in V. We show that we can always choose s so that $e_{l_{s+1}}\in U^{\star}_{g_{p+q}(l_s)}=U_{p+q,e_{l_1},\ldots,e_{l_s}}$. Then $e_{l_{s+2}},e_{l_{s+3}},\ldots,e_{l_{p+q}}$ are also in $U_{p+q,e_{l_1},\ldots,e_{l_s}}$ so $s'\otimes t'\in\sum_{\text{perms}}[e_{l_1}]\otimes\cdots\otimes[e_{l_s}]\otimes U_{p+q\,e_{l_1}\ldots e_{l_s}}\otimes\cdots\otimes U_{p+q\,e_{l_1}\ldots e_{l_s}}\subset\hat{U}$.

We assume without loss of generality that $f(p,i_n)\leqslant f(q,j_m)$ hence i_n and j_m will not play symmetric roles in the sequel. Since $j_m=\max(j_1',\ldots,j_m')$, by the definition of

$V_{q e_{j_1}' \dots e_{j_m}'}$ we have $e_{j_{m+1}} \in U^{\star}_{g_{p+q}(j_m)}$, $e_{j_{m+1}} \in U^{\star}_{g_{p+q}(i_n)}$, $e_{j_{m+1}} \in U^{\star}_{g_{p+q}(i)}$ for $i \leqslant g^{p}_{p+q}(j_m)$ and $e_{j_{m+1}} \in U^{\star}_{g_{p+q}(i)}$ for $i \leqslant g^{p}_{p+q}(i_n)$. In particular from the second of these conditions we have $j_{m+1} > g_{p+q}(i_n) \geqslant i_n$.

Case A: i_n or j_m is the immediate predecessor of j_{m+1} in the ordered list of subscripts. Since as noted above $e_{j_{m+1}} \in U^{\star}_{g_{p+q}(i_n)}$ and $U^{\star}_{g_{p+q}(j_m)}$ in these cases $s' \otimes t' \in \hat{U}$.

Since $i_n < j_{m+1}$ the only other possibility is that j_{m+1} is the immediate successor of some i_s, $s > n$.

Case B: $i_n \leqslant \cdots \leqslant j_m \leqslant i_{s-k} \leqslant \cdots \leqslant i_s \leqslant j_{m+1} \leqslant \cdots$ Note that only i-subscripts occur between i_{s-k} and i_s. If $i_{s-k} > g_{p+q}(j_m)$ then $e_{i_{s-k}} \in U^{\star}_{g_{p+q}(j_m)}$ (by the basic definition of the $\star$-sets), and we're done. Similarly if $i_t > g_{p+q}(i_{t-1})$ for any t with $s-k < t \leqslant s$ then $e_{i_t} \in U^{\star}_{g_{p+q}(i_{t-1})}$ as desired. If on the other hand none of these alternatives occurs then $i_s \leqslant g_{p+q}(i_{s-1})$ and $i_{s-1} \leqslant g_{p+q}(i_{s-2})$ etc. So $i_s \leqslant g_{p+q}(i_{s-1}) \leqslant g^{2}_{p+q}(i_{s-2}) \leqslant \cdots \leqslant g^{k}_{p+q}(i_{s-k}) \leqslant g^{k+1}_{p+q}(j_m) \leqslant g^{p}_{p+q}(j_m)$; these inequalities follow since g_{p+q} is nondecreasing. But then $i_s \leqslant g^{p}_{p+q}(j_m)$, so as noted earlier $e_{j_{m+1}} \in U^{\star}_{g_{p+q}(i_s)}$.

Case C: $j_m \leqslant \cdots \leqslant i_n \leqslant i_{n+1} \leqslant \cdots \leqslant i_s \leqslant j_{m+1} \leqslant \cdots$. The proof is the same as for Case B but with i_n replacing j_m throughout.

In the case where $i_{n+1} = i_1$ (resp. $j_{m+1} = j_1$) take $i_n = 0$ (resp. $j_m = 0$), and the proof goes through as above. This would be the case when $e_{i_1'} \otimes \cdots \otimes e_{i_p'} \in V_p \otimes \cdots \otimes V_p$ (resp. $e_{j_1'} \otimes \cdots \otimes e_{j_p'} \in V_q \otimes \cdots \otimes V_q$).

In every instance $s' \otimes t' \in \hat{U}$. Now a product of two arbitrary elements of $\hat{V}$ is a sum of terms of the form $s' \otimes t'$ hence also in $\hat{U}$, completing the proof that multiplication in $(T(E), \otimes\tau)$ is continuous at (0, 0).

In Theorem 15 it was shown that multiplication in $(T(E), \otimes\tau)$ is separately continuous. Thus $(T(E), \otimes\tau)$ is a topological vector space with continuous multiplication, hence a topological algebra.

We now prove that continuity of multiplication in $(T(E), \otimes\tau)$ implies continuity of multiplication in $(C(E), \otimes\tau)$. Let $m:(s, t) \to s \otimes t$ be the multiplication in $T(E)$ and σ the canonical map: $T(E) \to C(E) = T(E)/I$. Then $\sigma \circ m: T(E) \times T(E) \to C(E)$ is continuous and constant on equivalence classes modulo I, so it induces a well defined map $\bar{m}:(\sigma(s), \sigma(t)) \to \sigma(s) \circ \sigma(t)$ which is in fact multiplication in $C(E)$. Given $\sigma(s) \circ \sigma(t) \in \mathcal{O}$, $\mathcal{O}$ open in $C(E)$, there exist $\mathcal{O}(s)$ and $\mathcal{O}(t)$ containing s and t respectively with $\sigma \circ m(\mathcal{O}(s) \times \mathcal{O}(t)) \subset \mathcal{O}$. Since $\bar{m} \circ (\sigma \times \sigma) = \sigma \circ m$, $\bar{m}(\sigma(\mathcal{O}(s)) \times \sigma(\mathcal{O}(t))) \subset \mathcal{O}$ and so $\bar{m}$ is continuous.

Multiplication need not be continuous in $(T(E), \otimes\tau)$ for $\tau > \tau_{\Phi} V$ even when $\dim E = \aleph_0$ and V is a maximal (hence $\perp$-closed) totally isotropic subspace as the example below will show. The next lemma will be used in the example and in the next theorem.

LEMMA: *Let $E = V \oplus W$ with $V = k(e_\alpha)_{\alpha \in I}$ and $W = k(e_\alpha)_{\alpha \in J}$ have a topology for which there is a neighborhood basis at zero composed of sets of the form $U_L = k(v_\alpha)_{\alpha \in L}$,* *L running through some of the subsets of I. Let*

$$\hat{U}_n = U_n \otimes \cdots \otimes U_n + \sum_{\alpha \in I \cup J} [e_\alpha] \otimes U_{n\,e_\alpha} \otimes \cdots \otimes U_{n\,e_\alpha} + \sum [e_\alpha] \otimes [e_\beta] \otimes U_{n\,e_\alpha e_\beta} \otimes \cdots \otimes U_{n\,e_\alpha e_\beta} + \cdots$$

with $U_{n\,e_\alpha} \subset U_n$, $U_{n\,e_\alpha e_\beta} \subset U_{n\,e_\alpha} \cap U_{n\,e_\beta} \cap \cdots$ *and all subscripted U's from the zero neighborhood basis. If* $e_{\alpha_0} \notin U_{n\,e_{\alpha_1}}$ *and* $e_{\alpha_1} \notin U_n$ *then* $e_{\alpha_1} \otimes (\bigotimes_1^{n-1} e_{\alpha_0}) \notin \hat{U}_n$, e_{α_0} *and* e_{α_1} *elements of the basis* $\{e_\alpha\}_{\alpha \in I \cup J}$.

Proof: The summands of $\hat{U}_n$ are of these types: either of the form $[e_{\alpha_1}] \otimes A$ with A containing a factor $U_{n\,e_{\alpha_1}-}$ or of the form $U_{n,-} \otimes B$, or of the form $[e_\alpha] \otimes C$ with $\alpha \neq \alpha_1$. Since by hypothesis $[e_{\alpha_0}] \notin U_{n\,e_{\alpha_1}-}$, $[e_{\alpha_1}] \otimes A \subset F = [e_{\alpha_1} \otimes e_{\beta_1} \otimes \cdots \otimes e_{\beta_{n-1}};$ some $\beta_i \neq \alpha_0]$. While $U_{n-} \otimes B$ and $[e_\alpha] \otimes C \subset G = [e_{\gamma_1} \otimes e_{\gamma_2} \otimes \cdots \otimes e_{\gamma_n}; \gamma_1 \neq \alpha_1]$. $e_{\alpha_1} \otimes e_{\alpha_0} \otimes \cdots \otimes e_{\alpha_0} \notin F \oplus G$ and $\hat{U}_n \subset F \oplus G$ concluding the proof.

Example: Let $E = V \oplus W$ with $V = k(v_i)_{i \geq 1}$ and $W = k(w_i)_{i \geq 1}$ both totally isotropic and $\Phi(v_i, w_j) = \delta_{ij}$. Take for τ the topology with neighborhood basis at zero of sets $U_n^{\star\star} = k(v_{2^n i})_{i \geq 1}$. As proved in Theorem 18, the $\tau_\Phi V$ topology has a zero neighborhood basis of sets $U_n^{\star} = k(v_i)_{i > n}$. Each $U_n^{\star\star}$ contains some $U_m^{\star}$ (for example $U_n^{\star\star} \supset U_n^{\star}$) but not conversely, so τ is strictly finer than $\tau_\Phi V$.

In the zero neighborhood basis for $(T(E), \otimes\tau)$ consider any set $\hat{U} = U_1 + U_2 \otimes U_2 + \sum [v_i] \otimes U_{2\,v_i} + \sum [w_i] \otimes U_{2w_i} + \ldots$ of the general form given in the preceding lemma and in particular with $U_n = U_n^{\star\star}$ and $U_{n\,v_i} = U_n^{\star\star} \cap U_i^{\star\star}$. Let $\hat{V} = V_1 + V_2 \otimes V_2 + \ldots$ with the subscripted V's from the τ zero neighborhood basis, and suppose by way of contradiction that $\hat{V} \otimes \hat{V} \subset \hat{U}$. $V_1 = U_q^{\star\star} = k(v_{2^q i})_{i \geq 1}$ for some q. For i odd, $v_{2^q i} \in V_1$ but $v_{2^q i} \notin U_{q+1}$. $\bigcap_{i\,\text{odd}} U_{q+1, v_{2^q i}} \subset \bigcap_{i\,\text{odd}} U_{2^q i}^{\star\star} = (0)$, so $V_q \not\subset \bigcap_{i\,\text{odd}} U_{q+1, v_{2^q i}}$. There is an odd i_1, and a $v_{j_0} \in V_q$ such that $v_{j_0} \notin U_{q+1, v_{2^q i_1}}$. And since i_1, is odd, $v_{2^q i_1} \notin U_{q+1}$. Therefore by the lemma $v_{2^q i_1} \otimes v_{j_0} \otimes \cdots \otimes v_{j_0} \notin \hat{U}_{q+1}$. On the other hand $v_{2^q i_1} \otimes v_{j_0} \otimes \cdots \otimes v_{j_0} \in V_1 \otimes V_q \otimes \cdots \otimes V_q \subset \hat{U}_{q+1}$, a contradiction.

Examples can be given with V orthogonally closed and totally isotropic, $\tau > \tau_\Phi V$ and $\dim V > \aleph_0$ for which multiplication is not continuous. The state of affairs when $\tau = \tau_\Phi V$ and $\dim V > \aleph_0$ is an open question.

In this chapter two topologies were considered on the tensor product $E \otimes E$. It is apparent from a comparison of the neighborhood basis at zero that $\tau_e \leq \tau \otimes \tau$. In [6] it is shown that $\tau_e = \tau \otimes \tau$ when τ is the weak topology. On the other hand using several of our earlier results it is now easy to show that τ_e is strictly coaser than $\tau \otimes \tau$ for τ a $\tau_\Phi V$ topology, V of infinite dimension and codimension.

THEOREM 19: *Let* $E = V \oplus H$ *have topology* $\tau_\Phi V$, *V totally isotropic and of infinite dimension,* $H = k(h_\alpha)_{\alpha \in I}$ *also of infinite dimension. Then* $\tau_e < \tau \otimes \tau$.

Proof: Since $\operatorname{card} I \geq \aleph_0$ there is a bijective function f mapping I onto its finite subsets. There is a neighborhood basis at zero for the $\tau_\Phi V$ topology of sets $U_\alpha = V \cap k(h_\beta)^\perp_{\beta\in f(\alpha)}$. For if F is finite dimensional then $V\cap F^\perp \supset V\cap(V+k(h_\beta)_{\beta\in f(\alpha)})^\perp = V\cap k(h_\beta)^\perp_{\beta\in f(\alpha)}$. Let $V=k(v_\gamma)_{\gamma\in C}$. $\hat{U}_2 = V\otimes V + \sum [v_\gamma]\otimes V + \sum [h_\alpha]\otimes U_\alpha$ is a space in the $\tau\otimes\tau$ zero neighborhood basis (see the lemma to Theorem 18). Suppose by way of contradiction that $\hat{U}_2 \supset E\otimes U + U\otimes E$, U in the $\tau_\Phi V$ neighborhood basis at 0. $\tau_\Phi V$ is Hausdorff but not discrete since $\dim V \geqslant \aleph_0$, so there is a U_{α_0} with $U \not\subset U_{\alpha_0}$ i.e., some $v_{\alpha_1}\in U$, $v_{\alpha_1}\notin U_{\alpha_0}$. Then $h_{\alpha_0}\otimes v_{\alpha_1}$, h_{α_0} in the basis for H is clearly in $E\otimes U+U\otimes E$ but by the preceding lemma it is not in $\hat{U}_2$. Thus $\hat{U}_2$ contians no τ_e zero neighborhood so $\tau_e < \tau\otimes\tau$.

IV. Groups of $\tau_\Phi H$-continuous Automorphisms

In this chapter (E, Φ) is a semisimple k-space possessing infinite dimensional totally isotropic subspaces (cf. I.4). In particular every maximal totally isotropic subspace is then of infinite dimension. $\mathfrak{O}(E, \Phi)$ is the (full) orthogonal group of (E, Φ); k_d is the additive group of a d-dimensional linear space over k. We write (x, y) for $\Phi(x, y)$.

We shall discuss groups of $\tau_\Phi H$-continuous automorphisms of (E, Φ) under the following special assumptions: (A) H is a maximal totally isotropic subspace of (E, Φ); (B) the space (E, Φ) is either of denumerable dimension or else (E, Φ) is $\tau_\Phi H$-complete.

Under these conditions (E, Φ) admits a decomposition:

$$E = (H \oplus \bar{H}) \overset{\perp}{\oplus} G, \quad H \text{ and } \bar{H} \text{ totally isotropic}, \quad G \text{ anisotropic}. \tag{1}$$

Corresponding to the two cases in (B) we have either $\dim\bar{H} = \dim H = \aleph_0$ or else $H = \bar{H}^*$ the algebraic dual of $\bar{H}$, and $\Phi(h^*, h') = h^*(h')$, $h^*\in\bar{H}^*$, $h'\in\bar{H}$, (I.2 and Theorem 8).

Since H is assumed maximal, the collection of all $\tau_\Phi H$-continuous T in $\mathfrak{O}(E, \Phi)$. form a subgroup (see Theorem 3) denoted by $\mathfrak{T}(H, \Phi)$. The discussion of this group will proceed by describing in a geometrical fashion the groups and factors of a normal series. We shall therefore start out with the investigation of various special subgroups of $\mathfrak{T}(H, \Phi)$.

IV.1 We start by assuming that $\dim E = \aleph_0$. Let $\mathfrak{K}$ be the subgroup of all $T\in\mathfrak{O}(E, \Phi)$ with the property that H and G are left pointwise fixed under T and $T(H\oplus\bar{H}) \subset H\oplus\bar{H}$ (hence $T(H\oplus\bar{H}) = H\oplus\bar{H}$). The restriction of these T to the space $H\oplus\bar{H}$ form a group which we identify with $\mathfrak{K}$. In other words in the study of $\mathfrak{K}$ we may assume that $G=(0)$. We are going to describe $\mathfrak{K}$ in some detail.

Let $T\in\mathfrak{K}$. We set $T = 1 + L$. For $x\in\bar{H}$, $Lx\perp H$, so $Lx\in H$. We have L a linear map $H\oplus\bar{H}\to H$ with $L(H)=(0)$. For every $z\in H\oplus\bar{H}$, we have $\|z\| = \|Tz\| = \|z+Lz\|$, hence $(z, Lz)=0$. Since $\operatorname{char} k \neq 2$, the last condition is equivalent to

$$(Lx, y) = -(x, Ly) \quad \text{for all} \quad x, y\in H\oplus\bar{H}. \tag{2}$$

(L is "antiselfadjoint" or "skew".) Since $\operatorname{Im} L \subset H$ and H is totally isotropic we see that the map $T=1+L \to L$ is a group isomorphism of $\mathfrak{K}$ onto the additive group of linear maps $L: H \oplus \bar{H} \to H$ satisfying (2) and $L(H)=(0)$. The spaces H and $\bar{H}$ are spanned by the two halves of some symplectic basis, $H=k(h_i)_{i \geqslant 1}$, $\bar{H}=k(h_i')_{i \geqslant 1}$, $\Phi(h_i, h_j')=\delta_{ij}$. With respect to this basis the matrix of L is of the form $\begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}$ where A is a denumerable column- and row-finite skew matrix, $A=-{}^tA$, ${}^tA$ the transpose of A. Hence we have the group isomorphism

$$\mathfrak{K} \cong k_{\aleph_0}, \qquad \dim E = \aleph_0 . \tag{3}$$

We now turn to the discussion of the transformations $T \in \mathfrak{K}$, $T=1+L$. By (2) we see that $\operatorname{Ker} L=(\operatorname{Im} L)^{\perp}$, and in particular $\operatorname{Ker} L$ is $\perp$-closed. We have the following sequence of subspaces in $H \oplus \bar{H}$:

$$\begin{aligned}(0) \subset \operatorname{Im} L \subset (\operatorname{Ker} L)^{\perp} = (\operatorname{Im} L)^{\perp\perp} \subset H^{\perp\perp} = H \subset \operatorname{Ker} L = \\ (\operatorname{Ker} L)^{\perp\perp} = (\operatorname{Im} L)^{\perp} \subset H \oplus \bar{H} .\end{aligned} \tag{4}$$

We shall prove that $T=1+L$ in uniquely determined up to orthogonal similarity by the dimensions of the three spaces $\operatorname{Im} L$, $(\operatorname{Im} L)^{\perp\perp}/\operatorname{Im} L$ and $(\operatorname{Im} L)^{\perp}/(\operatorname{Im} L)^{\perp\perp}$ (cf. [11] for a similar theorem on selfadjoint L). Further we shall see that $H \oplus \bar{H}$ is the orthogonal sum of finite dimensional subspaces that are invariant under $T=1+L$ if and only if $(\operatorname{Im} L)^{\perp\perp}/\operatorname{Im} L \cong (0)$.

It will be convenient to have the following two examples at our disposal. (I) Let (E_0, Φ) be an orthogonal sum $\underset{I}{\oplus} P_i$ of hyperbolic planes $P_i=k(h_i, h_i')$, $i \in I$, I either denumerable or finite and even. We set $H=k(h_i)_I$, $\bar{H}=k(h_i')_I$, and define an automorphism $T=1+L$ as follows: $L(H)=(0)$, $Lh_{2i-1}'=h_{2i}$ $Lh_{2i}'=-h_{2i-1}$. L (and consequently T) leaves the pairs $P_{2i-1} \overset{\perp}{\oplus} P_{2i}$ invariant. It is also easy to see that $(Lz, z)=0$ for all $z \in E_0$. We have $(\operatorname{Im} L)^{\perp\perp}/\operatorname{Im} L \cong (0)$ in this case. (This of course automatically takes place when $H \oplus \bar{H}$ is an orthogonal sum of finite dimensional invariant subspaces, whatever $\dim H \oplus \bar{H}$ may be.) (II) In order to obtain an example with $(\operatorname{Im} L) \neq (\operatorname{Im} L)^{\perp\perp}$ we consider a space (E_1, Ψ), $E_1=E_0 \oplus k(h_0)$, E_0 as before (i.e. $\Psi|_{E_0}=\Phi$) with denumerable I; further $\Psi(h_0, h_i)=0$ $(i \geqslant 0)$, $\Psi(h_0, h_i')=1$, $(i>0)$. Since I is not finite, (E_1, Ψ) is easily seen to be semisimple. Defining L on E_0 as before and setting $Lh_0=0$ we have again $(Lz, z)=0$ for all $z \in E_1$. However, this time we find $(\operatorname{Im} L)^{\perp\perp}=k(h_i)_{i \geqslant 1}^{\perp\perp}=k(h_i)_{i \geqslant 0}=\operatorname{Im} L \oplus k(h_0)$, i.e. $\dim (\operatorname{Im} L)^{\perp\perp}/\operatorname{Im} L=1$.

THEOREM 20: *Let $E=H \oplus \bar{H}$ be the sum of the totally isotropic spaces H and $\bar{H}$, E semisimple and of denumerable dimension. An automorphism $T \in \mathfrak{K}$ (the automorphisms leaving H pointwise fixed) is uniquely determined up to orthogonal similarity by the dimensions d_1, d_2, d_3 of the three spaces:*

$$\operatorname{Im}(T-1), \operatorname{Im}(T-1)^{\perp\perp}/\operatorname{Im}(T-1), \operatorname{Im}(T-1)^{\perp}/\operatorname{Im}(T-1)^{\perp\perp}$$

(i.e. if T and $\bar{T}$ have the same invariants d_i, $i=1, 2, 3$, then $\bar{T}=ATA^{-1}$ for some orthogonal automorphism.[2]*) Further, if $d_2=0$ then E is an orthogonal sum $E=E_{00}\overset{\perp}{\oplus}E_0$ where E_{00} is left pointwise fixed under T, $\dim E_{00}=d_3$, E_0 is (and is transformed) as E_0 in the example above, and $\dim E_0=2d_1$. Conversely, if E is of the particular form $E_{00}\overset{\perp}{\oplus}E_0$ then $d_2=0$. On the other hand, if $d_2=\mathfrak{c}\neq 0$ then E is an orthogonal sum $E_{00}\oplus E_2$ where E_{00} is left pointwise fixed under T, $\dim E_{00}=d_3$ and E_2 is the orthogonal sum of $\mathfrak{c}$ replicas of a space E_1 which is (and is transformed) as E_1 in the example above;* $\dim E_2=d_1=\mathfrak{c}\cdot\aleph_0=\aleph_0$, $d_2=\mathfrak{c}$.

Proof: Again we write $T=1+L$. We first reduce the general case to the case with $\operatorname{Ker}L=(\operatorname{Im}L)^{\perp\perp}$ by splitting off an orthogonal summand E_{00} of E with $\dim E_{00}=d_3$ (cf. (4) above). The $\perp$-closed subspace $(\operatorname{Im}L)^{\perp\perp}\subset H$ induces a decomposition by I.2 as follows: $E=((\operatorname{Im}L)^{\perp\perp}\oplus S)\overset{\perp}{\oplus}(U_1\oplus\bar{U}_1)$ where $S+\bar{U}_1=\bar{H}$, $(\operatorname{Im}L)^{\perp\perp}\oplus U_1=H$. Hence $\operatorname{Ker}L=(\operatorname{Im}L)^{\perp\perp\perp}=(\operatorname{Im}L)^{\perp\perp}\overset{\perp}{\oplus}(U_1\oplus\bar{U}_1)$. So putting $E_{00}=U_1\oplus\bar{U}_1$, E_{00} is left pointwise fixed and $\dim E_{00}=d_3$. We may therefore concentrate on the semisimple space $(\operatorname{Im}L)^{\perp\perp}\oplus S$. (We note for later that its orthogonal supplement E_{00} is uniquely determined up to orthogonal isomorphism by d_3 since E_{00} is a sum of hyperbolic planes). Denoting the restrictions of T and L to the subspace $(\operatorname{Im}L)^{\perp\perp}\oplus S$ again by "T" and "L" we are now in the situation where, in addition to (4), $(\operatorname{Im}L)^{\perp\perp}=\operatorname{Ker}L(=\operatorname{Im}L^{\perp})$.

The subspace $(\operatorname{Im}L)\oplus S$ of the semisimple space $(\operatorname{Im}L)^{\perp\perp}\oplus S$ is itself semisimple. (Quite generally $\operatorname{rad}(A+B)\subset\operatorname{rad}(A^{\perp\perp}+B)$.) We set $(\operatorname{Im}L)^{\perp\perp}=\operatorname{Im}L\oplus R$ and distinguish two cases $R=(0)$ and $R\neq(0)$. Note that $\dim R=d_2$.

Case A: $R=(0)$. We shall decompose $\operatorname{Im}L\oplus S$ into an orthogonal sum of four dimensional subspaces which are invariant under L. Let $\{e_i\}_{i\in M}$ be a basis of S and assume that we have already constructed 4-dimensional semisimple subspaces F_1, $F_2, \ldots, F_{n-1}$ which are pairwise orthogonal, invariant under L and of the following shape:

$$F_i = k(y_i, x_i, L y_i, L x_i) = k(y_i, L x_i) \oplus k(x_i, -L y_i), \quad x_i \text{ and } y_i\in S,$$

the two dimensional summands being hyperbolic planes with bases as indicated. Let e_m be the first basis vector of S not contained in $K=\bigoplus_1^{n-1} F_i$. We shall construct a four dimensional semisimple subspace $F_n\subset K^{\perp}$ which is invariant under L (and again of the same shape as the F_i's with $i<n$) such that $e_m\in\bigoplus_1^n F_i$. In this fashion we construct an

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[2]) As the proof will show, we can always find such an A with $A(H)=H$; thus $A\in\mathfrak{T}(H, \Phi)$.

orthogonal sum $\oplus F_i$ of invariant subspaces F_i such that $S \subset \oplus F_i$. Since $L(S) = \operatorname{Im} L$ we then also have $\operatorname{Im} L \subset \oplus F_i$, hence $\oplus F_i = (\operatorname{Im} L) \oplus S$. It thus remains to show how to construct F_n. Since K is finite dimensional and semisimple we have $(\operatorname{Im} L) \oplus S = K \oplus K^\perp$, and we decompose e_m accordingly, $e_m = e'_m + e''_m$, $e'_m \in K$, $e''_m \in K^\perp$. By I.2 the space $K^\perp$ is of the form $K^\perp = R_1 \oplus \bar{R}_1$ with $R_1 \subset \operatorname{Im} L$, $\bar{R}_1 \subset S$. Since $e_m \in S$, S totally isotropic, we have $e_m - \sum_1^{n-1} (\lambda_i x_i + \mu_i y_i) \perp K$, $x_i, y_i \in S$ by assumption about the spaces $F_i (i < n)$. In other words $e''_m \in S$ and thus $e''_m \in \bar{R}_1$. Since $\operatorname{Ker} L \cap S = (0)$ we have $0 \neq Le''_m$ and $e''_m \in K^\perp$ implies $Le''_m \in K^\perp \cap (\operatorname{Im} L) = R_1$. Since furthermore $K^\perp = R_1 \oplus \bar{R}_1$ is semisimple (both R_1 and $\bar{R}_1$ are totally isotropic), there exists $y \in \bar{R}_1$ with $(Le''_m, y) = 1$. It is readily checked that $F_n = k(y, e''_m, Ly, Le''_m) = k(y, Le''_m) \overset{\perp}{\oplus} k(e''_m, -Ly)$ satisfies all the induction assumptions.

Case B: $R \neq (0)$, $(\operatorname{Im} L)^{\perp\perp} = (\operatorname{Im} L) \oplus R$. We shall prove the assertion of the theorem for the special case $d_2 = \dim R = 1$. Example I, discussed earlier in this section, may then be taken as representative. It follows from Witt's theorem in the denumerable case that any semisimple space $E = V^{\perp\perp} \oplus S$ of denumerable dimension, V and S both totally isotropic with $\dim V^{\perp\perp}/V = \mathfrak{c} \neq 0$ is an orthogonal sum of $\mathfrak{c}$ replicas of a semisimple space $V_0^{\perp\perp} \oplus S_0$ of denumerable dimension, V_0 and S_0 both totally isotropic such that $\dim V_0^{\perp\perp}/V_0 = 1$ and $V = \oplus_{\mathfrak{c}} V_0$ (see [8], the application following Theorem 4). It is therefore sufficient to discuss the case where $\dim R = 1$, $(\operatorname{Im} L)^{\perp\perp} \oplus S = (\operatorname{Im} L) \oplus S \oplus k(r)$. In contrast to the former case where we have set up directly a canonical form for $\operatorname{Im} L \oplus S$, this time we shall prove uniqueness up to orthogonal similarity by considering a second map $\bar{T} = 1 + \bar{L}$ with the same invariants d_i as for T and then proceed to give orthogonal decompositions as follows:

$$\operatorname{Im} L \oplus S = \overset{\infty}{\oplus} F_i = \overset{\infty}{\oplus} [k(y, Lx) + k(x, -Ly)]$$

$$\operatorname{Im} \bar{L} \oplus \bar{S} = \overset{\infty}{\oplus} \bar{F}_i = \overset{\infty}{\oplus} [k(\bar{y}, \bar{L}\bar{x}) + k(\bar{x}, -\bar{L}\bar{y})]$$

as in case A but in addition we have

$$(r, y) = (\bar{r}, \bar{y}) \quad \text{and} \quad (r, x) = (\bar{r}, \bar{x}) \tag{5}$$

for all summands in the decompositions above. Note that we shall automatically have $(r, Ly) = (\bar{r}, \bar{L}\bar{y})$ and $(r, Lx) = (\bar{r}, \bar{L}\bar{x})$ as all these numbers are zero, the vectors belonging to the totally isotropic spaces $(\operatorname{Im} L)^{\perp\perp}$ and $(\operatorname{Im} \bar{L})^{\perp\perp}$ respectively. If A is the linear extension of the map sending r into $\bar{r}$ and y, x, Ly, Lx into $\bar{y}, \bar{x}, \bar{L}\bar{y}, \bar{L}\bar{x}$ respectively, then A is an orthogonal isomorphism, and we have $\bar{L} \circ A = A \circ L$ (and $\bar{T} \circ A = A \circ T$). A can be extended to all of E by extending it to the orthogonal supplements of $(\operatorname{Im} L)^{\perp\perp} \oplus S$ and $(\operatorname{Im} \bar{L})^{\perp\perp} \oplus \bar{S}$ respectively (cf. the beginning of our proof).

Assume then that we have already constructed the spaces $F_1, \ldots, F_{n-1}$ and

$F_1, \ldots, F_{n-1}$ such that (5) holds for these summands. As in case A, we find an F_n such that $\overset{n}{\underset{1}{\oplus}} F_i$ contains one more prescribed basis vector of S. We have to construct a suitable match $\bar{F}_n$ for F_n such that (5) holds again for the spaces F_n and $\bar{F}_n$. We have $F_n = k(y_n, Lx_n) \overset{\perp}{\oplus} k(x_n, -Ly_n)$, y_n and x_n in S. Let $\bar{K} = \overset{n-1}{\underset{1}{\oplus}} \bar{F}_i$. Furthermore let $\bar{K}^{\perp o}$ be the orthogonal of $\bar{K}$ in $(\mathrm{Im}\,\bar{L}) \oplus \bar{S}$, $(\mathrm{Im}\,\bar{L} \oplus \bar{S}) = \bar{K} \oplus \bar{K}^{\perp o}$. Since the orthogonal of $(\mathrm{Im}\,\bar{L}) \oplus \bar{S}$ in $(\mathrm{Im}\,\bar{L}) \oplus \bar{S} \oplus k(\bar{r})$ is (0), we cannot have $\bar{r} \perp \bar{K}^{\perp o}$ or else, for suitable x in $\mathrm{Im}\,\bar{L}$, $\bar{r} - x$ would be orthogonal to all of $(\mathrm{Im}\,\bar{L}) \oplus \bar{S}$ (since $\bar{K}$ is finite dimensional and semisimple). Hence there exists $\bar{y}_n \in \bar{K}^{\perp o}$ with $(\bar{y}_n, \bar{r}) \neq 0$. Further, by I.2, the space $\bar{K}^{\perp o}$ is of the form $\bar{K}^{\perp o} = R_1 \oplus \bar{R}_1$ with $R_1 \subset \mathrm{Im}\,\bar{L}$ and $\bar{R}_1 \subset \bar{S}$. Since $\bar{r} \perp \mathrm{Im}\,\bar{L}$, hence $\bar{r} \perp R_1$, we may even pick $\bar{y}_n$ in $\bar{R}_1 \subset \bar{S}$ with $(\bar{y}_n, \bar{r}) \neq 0$.

Thus if we should have $(y_n, r) \neq 0$ in F_n then we have found $\bar{y}_n \in \bar{S}$ with $(\bar{y}_n, \bar{r}) \neq 0$. Replacing $\bar{y}_n$ by a suitable multiple we may assume $(\bar{y}_n, \bar{r}) = (y_n, r)$. On the other hand, if it should be the case that $(y_n, r) = 0$ then we simply pick some $\bar{y}_n$ in the infinite dimensional space $(\bar{K} \oplus k(\bar{r}))^{\perp} \cap \bar{S}$. In either case we have found $\bar{y}_n \in \bar{S}$ with $(\bar{y}_n, \bar{r}) = (y_n, r)$.

Since $\|\bar{L}\bar{y}_n\| = 0$ and $\bar{K}^{\perp o} = R_1 + \bar{R}_1$ is semisimple, there is an $\bar{x}_0 \in \bar{R}_1 \subset \bar{S}$ with $(\bar{x}_0, \bar{L}\bar{y}_n) = 1$. The space $G = k(\bar{y}_n, \bar{x}_0, \bar{L}\bar{y}_n, \bar{L}\bar{x}_0)$ is semisimple. It remains to adjust the value $(\bar{x}_0, \bar{r})$. Again by I.2 the orthogonal of $(\bar{K} \overset{\perp}{\oplus} G)$ in $(\mathrm{Im}\,\bar{L}) \oplus \bar{S}$ is of the form $R_2 \oplus \bar{R}_2$ with $R_2 \subset \mathrm{Im}\,\bar{L}$, $\bar{R}_2 \subset \bar{S}$. As before we cannot have $\bar{r} \in (R_2 \oplus \bar{R}_2)^{\perp}$. Hence we find $z \in \bar{R}_2 \subset \bar{S}$ with $(\bar{r}, z) \neq 0$. $\bar{x}_0 + \lambda z$ is isotropic for every λ and $(\bar{x}_0 + \lambda z, \bar{L}\bar{y}_n) = (\bar{x}_0, \bar{L}\bar{y}_n) = 1$. Since $(\bar{r}, z) \neq 0$ we may pick λ such that $(\bar{x}_0 + \lambda z, \bar{r}) = (x_n, r)$. We then put $\bar{x}_n = \bar{x}_0 + \lambda z$ for this choice of λ and $\bar{F}_n = k(\bar{y}_n, \bar{x}_n, \bar{L}\bar{y}_n, \bar{L}\bar{x}_n)$ enjoys all the required properties.

It is to be observed that in the construction of the spaces F_i and $\bar{F}_i$ we have to alternate between the roles of the spaces $(\mathrm{Im}\,L) \oplus S$ and $(\mathrm{Im}\,\bar{L}) \oplus \bar{S}$ so as to make sure that $\oplus F_i$ and $\oplus \bar{F}_i$ exhaust the spaces $(\mathrm{Im}\,L) \oplus S$ and $(\mathrm{Im}\,\bar{L}) \oplus \bar{S}$ respectively. In other words, in the next step we first construct $\bar{F}_{n+1}$ according to case A so that $\overset{n+1}{\underset{1}{\oplus}} \bar{F}_i$ contains one more prescribed basis vector; after that we find a suitable match F_{n+1} such that (5) holds by the immediately preceding construction. This completes the proof of Theorem 20.

For any $T = 1 + L \in \mathfrak{K}$ we have $T^2 = 1 + 2L$. Hence T and T^2 have the same invariants d_i. Therefore $T^2 = A \circ T \circ A^{-1}$ for a suitable orthogonal automorphism A by Theorem 20 with $A(H) = H$. Thus $T = A \circ T \circ A^{-1} \circ T^{-1}$ and we have the

Corollary: *$\mathfrak{K}$ is contained in the commutator subgroup of $\mathfrak{T}(H, \Phi)$.*

IV.2 There is at least one nondenumerable case for which Theorem 20 can be salvaged, namely for the spaces $H \oplus \bar{H}$ where $\bar{H}$ is of denumerable dimension and $H = \bar{H}^*$:

THEOREM 21: *Let H^* be the algebraic dual of H, H of denumerable dimension, both H^* and H totally isotropic for Φ and $\Phi(h^*, h) = h^*(h)$. Let $\mathfrak{K}$ be the group of those orthogonal automorphisms of $H^* \oplus H$ which leave H^* pointwise fixed.*

(i) *Every $T \in \mathfrak{K}$ is uniquely determined up to orthogonal similarity by the invariants d_1, d_2 and d_3 of Theorem 20;*

(ii) *$\mathfrak{K}$ is contained in the commutator subgroup of $\mathfrak{T}(H^*, \Phi)$.*

Proof: For fixed $T = 1 + L \in \mathfrak{K}$ we set $V = (\operatorname{Ker} L) \cap H$, $H = V \oplus S$. We have a canonical decomposition $H^* \oplus H = (V^* \oplus V) \overset{\perp}{\oplus} (S^* \oplus S)$. Furthermore $\operatorname{Ker} L = H^* \oplus V = (V^* \oplus S^*) \oplus V$. Hence $(\operatorname{Im} L)^{\perp\perp} = (\operatorname{Ker} L)^{\perp} = S^*$ and $(\operatorname{Im} L)^{\perp} = S^{*\perp} = S^* \overset{\perp}{\oplus} (V^* \oplus V)$. Thus $\dim(V^* + V) = \dim(\operatorname{Im} L)^{\perp}/(\operatorname{Im} L)^{\perp\perp} = d_3$. Furthermore $V^* + V$ is left pointwise fixed by T since $V^* + V \subset H^* + V = \operatorname{Ker} L$. As in the previous case we are left with the semisimple space $S^* \oplus S = (\operatorname{Im} L)^{\perp\perp} \oplus S$. If S is finite dimensional then $\dim S = \dim S^*$ and $S + S^*$ is an orthogonal sum of an even number of hyperbolic planes which are left pairwise invariant under L. Nothing remains to be proved in this case. If S is of denumerable dimension, then the semisimple subspace $(\operatorname{Im} L) \oplus S$ of $(\operatorname{Im} L)^{\perp\perp} \oplus S$ is of denumerable dimension ($\operatorname{Im} L = L(S)$) and admits, as we know by the proof of Theorem 20, an orthogonal decomposition

$$(\operatorname{Im} L) \oplus S = \bigoplus_i \left[k(x_i, L y_i) \overset{\perp}{\oplus} k(y_i, -L x_i)\right] \qquad x_i, y_i \in S$$

$\Phi(x_i, L y_i) = \Phi(y_i, -L x_i) = 1$. Consider a second automorphism $\bar{T} = 1 + \bar{L}$ with the same invariants d_i, $i = 1, 2, 3$. There is a similar decomposition $\operatorname{Im} \bar{L} \oplus \bar{S} = \bigoplus_i [k(\bar{x}_i, \bar{L}\bar{y}_i) \overset{\perp}{\oplus} k(\bar{y}_i, -\bar{L}\bar{x}_i)]$, $\Phi(\bar{x}_i, \bar{L}\bar{y}_i) = \Phi(\bar{y}_i, -\bar{L}\bar{x}_i) = 1$. Let $A_0 : S \to \bar{S}$ be the isomorphism which sends x_i and y_i into $\bar{x}_i$, $\bar{y}_i$ respectively; and let $A_0^* : \bar{S}^* \to S^*$ be its transpose: $(A_0^* \bar{s}^*)(s) = \bar{s}^*(A_0 s)$ for all $s^* \in S^*$, $s \in S$. The isomorphism $A : \bar{S}^* \oplus \bar{S} \to S^* \oplus S$ defined by $A|_{\bar{S}^*} = A_0^*$ and $A|_{\bar{S}} = A_0^{-1}$ is orthogonal, and we have $L \circ A = A \circ \bar{L}$. Since the invariant d_3 is the same for T and $\bar{T}$, A can be extended to all of $H^* + H$ in a trivial fashion. This concludes the proof of Theorem 21.

IV.3 In this section let (E, Φ) be a semisimple space of the following sort:

$$E = (H \oplus \bar{H}) \overset{\perp}{\oplus} G, \quad H \text{ and } \bar{H} \text{ totally isotropic}, \quad G \text{ anisotropic}. \tag{6}$$

Dimensions are arbitrary. The spaces H and $\bar{H}$ form a dual pairing under the form $\langle , \rangle$ induced by Φ. Let furthermore $\mathfrak{T}_H$ be the subgroup of $\mathfrak{O}(E, \Phi)$ of those automorphisms T which leave H invariant, $T(H) \subset H$. It is readily verified that this implies $T(H) = H$; the fact that G is anisotropic is however crucial. We shall investigate here the group $\mathfrak{T}_H$. For arbitrary fixed $T \in \mathfrak{T}_H$ and all $h' \in \bar{H}$ we set $Th' = T_1 h' + T_2 h' + T_3 h'$ with $T_1 h' \in \bar{H}$, $T_2 h' \in H$ and $T_3 h' \in G$. Since $T(H) = H$ we have $T(H^{\perp}) = H^{\perp}$, in particular $T(G) \subset H \oplus G$. For all $g \in G$ we set $T_g = T_4 g + T_5 g$ where $T_4 g \in G$ and $T_5 g \in H$.

The linear map $T_4\colon G\to G$ is orthogonal, injective and epijective, i.e. $T_4\in\mathfrak{O}(G,\Phi|_G)$. We now define a map $T^*\colon E\to E$ by linear extension of $T^*|_H=T|_H$, $T^*|_{\bar H}=T_1$ and $T^*|_G=T_4$. It is easy to check that T^* is orthogonal, injective and epijective. We put $\tilde T=T^{*-1}\circ T$. $\tilde T$ has the following properties

$$\begin{aligned}&\text{for all } h\in H, && \tilde T h=h\\ &\text{for all } h'\in\bar H, && \tilde T h'=h'+L_1h'+L_2h' \quad \text{where} \quad L_1h'\in H \quad \text{and} \quad L_2h'\in G\\ &\text{for all } g\in G, && \tilde T g=g+L_3 g \quad \text{where} \quad L_3 g\in H.\end{aligned} \tag{7}$$

Let $\mathfrak{T}_0$ be the subgroup of $\mathfrak{T}_H$ consisting of all $\tilde T$ of the form (7). The maps satisfying (7) satisfy the conditions

$$(T-1)\,H=(0)\,,\qquad (T-1)\,G\subset H\,. \tag{7'}$$

Conversely, if $(T-1)\,H=(0)$ then the orthogonality conditions give $(T-1)\,S\subset H^\perp$ for any subspace $S\subset E$. In particular, $(T-1)\,\bar H\subset H\oplus G$ which is equivalent to the second equation in (7). We thus see that (7) and (7′) are equivalent descriptions of the subgroup $\mathfrak{T}_0\subset\mathfrak{T}_H$. Since $T(H)=H$ and thus $T(H^\perp)=H^\perp=H+G$ for the elements of $\mathfrak{T}_H$, it is easily seen from (7′) that $\mathfrak{T}_0$ is an invariant subgroup of $\mathfrak{T}_H$.

We may summarize our reduction thus far as follows. Every $T\in\mathfrak{T}_H$ is of the form $T=T^*\circ\tilde T$, $\tilde T\in\mathfrak{T}_0$, and T^* has the properties $T^*|_H=T|_H$, $T^*|_{\bar H}=T_1$ and $T^*|_G=T_4$. Since $T^*\in\mathfrak{T}_0$ implies $T^*=1$, we have $\mathfrak{T}_H/\mathfrak{T}_0$ isomorphic to the group of all T^*. This group can conveniently be described as follows. First of all, every T^* can be identified with some element in $\mathfrak{O}(H\oplus\bar H,\Phi)\times\mathfrak{O}(G,\Phi|_G)$ since T^* leaves both $H\oplus\bar H$ and G invariant. As T runs through $\mathfrak{T}_H$, $T^*|_G=T_4$ runs through the whole group $\mathfrak{O}(G,\Phi|_G)$. On the other hand, since the restriction $T^*|_{H+\bar H}$ leaves both H and $\bar H$ invariant, the orthogonality conditions imply that $T^*|_H$ is the transpose of $(T^*|_{\bar H})^{-1}$ with respect to the dual pairing $\langle H,\bar H\rangle$. $T^*|_H$ and $T^*|_{\bar H}$ determine each other uniquely. In other words, if $\mathscr{L}(\bar H)$ is the group of all $\sigma(\bar H,H)$-continuous vectorspace automorphisms of $\bar H$ then $T^*|_{\bar H}=T_1\in\mathscr{L}(\bar H)$, (see [2], §4,1). Conversely, every $T_1\in\mathscr{L}(\bar H)$ gives rise to an orthogonal automorphism T^* of $H\oplus\bar H$ by letting $T^*|_H$ be the inverse of the transpose of T_1. Thus $T_1=T^*|_{\bar H}$ runs through the whole group $\mathscr{L}(\bar H)$ as T runs through $\mathfrak{T}_H$. Since $T^*\to T_1=T^*|_{\bar H}$ is a homomorphism, we have thus shown that the group of all T^* is isomorphic to $\mathscr{L}(\bar H)\times\mathfrak{O}(G,\Phi|_G)$:

$$\mathfrak{T}_H/\mathfrak{T}_0\cong\mathscr{L}(\bar H)\times\mathfrak{O}(G,\Phi|_G) \tag{8}$$

$\mathscr{L}(\bar H)$ the group of all $\sigma(\bar H,H)$-continuous vectorspace automorphisms of $\bar H$.

We now return to the group $\mathfrak{T}_0$ of all automorphisms T satisfying (7). The orthogonality conditions for these T give:

$$(L_1x,y)+(x,L_1y)=-(L_2x,L_2y)\qquad x,y\in\bar H \tag{9}$$

$$(L_2h',g)+(h',L_3g)=0\qquad h'\in\bar H, g\in G \tag{10}$$

Conversely, if linear maps $L_1:\bar{H}\to H$, $L_2:\bar{H}\to G$ and $L_3:G\to H$ satisfy (9) and (10) then (7) defines an orthogonal automorphism T in $\mathfrak{T}_0$. We now discuss these equations. Considering the dual pairings $\langle\bar{H}, H\rangle$ and $\langle G, G\rangle$ induced by Φ, (10) shows that $L_3:G\to H$ is the negative transpose of $L_2:\bar{H}\to G$. Hence L_2 and L_3 are continuous for the corresponding weak topologies. On the other hand, every L_2 in $\mathscr{L}(\bar{H}, G)$ (the additive group of continuous linear maps $\bar{H}\to G$) uniquely determines a map L_3 satisfying (10). Setting $L_1=\frac{1}{2}L_3\circ L_2$ we see that L_1 is a particular solution of (9). We have thus shown that the system (9)–(10) has solutions L_1, L_3 for prescribed $L_2\in\mathscr{L}(\bar{H}, G)$. In other words, as T runs through $\mathfrak{T}_0$, L_2 in (7) runs through the whole group $\mathscr{L}(\bar{H}, G)$. It is easily verified using (7) that the map $T\to L_2$ is a homomorphism. We therefore have an epimorphism $\eta:\mathfrak{T}_0\to\mathscr{L}(\bar{H}, G)$. Assume that T is in the kernel of η; $L_2=0$ for the corresponding L_2. Hence $L_3(G)\subset H\cap\bar{H}^{\perp}=(0)$ by (10), and $L_3=0$ also. Further L_1 is skew by (9). Conversely if L_1 is skew then $(L_2x, L_2y)=0$ for all $x, y\in\bar{H}$ by (9). In particular $\|L_2x\|=0$ for all $x\in\bar{H}$, and thus $L_2x=0$ as G is anisotropic. In other words $L_2=0$ and T belongs to $\operatorname{Ker}\eta$. Thus $\operatorname{Ker}\eta$ contains precisely the maps $T=1+L_1$, L_1 any linear map $\bar{H}\to H$ which is skew:

$$(L_1x, y)+(x, L_1y)=0 \qquad x, y\in\bar{H}. \tag{11}$$

If the conditions of Theorems 20 and 21 are satisfied then the group $\mathfrak{K}$ in these theorems is precisely $\operatorname{Ker}\eta$ restricted to $H\oplus\bar{H}$. We therefore put $\mathfrak{K}=\operatorname{Ker}\eta$ in general and have

$$\mathfrak{T}_0/\mathfrak{K}=\mathscr{L}(\bar{H}, G), \tag{12}$$

$\mathfrak{K}$ the additive group of linear maps $\bar{H}\to H$ satisfying (11). (13)

In a slightly different way we may account for our normal series as follows. $\mathfrak{T}_H$ can be described by the condition $(T-1)H\subset H$ for all T. We then select subgroups of $\mathfrak{T}_H$:

$$\begin{aligned}
&\mathfrak{T}_H;\,(T-1)\,H\subset H\\
&\mathfrak{T}_1\,;\,(T-1)\,H=(0)\\
&\mathfrak{T}_0\,;\,(T-1)\,H=(0), \quad (T-1)\,G\subset H.\\
&\mathfrak{K}\ \ ;\,(T-1)\,H^{\perp}=(0)
\end{aligned} \tag{14}$$

Since the elements of $\mathfrak{T}_H$ map H onto H (and consequently $H^{\perp}$ onto $H^{\perp}$), it is easily seen that these subgroups are invariant in $\mathfrak{T}_H$. The series $\mathfrak{T}_H\supset\mathfrak{T}_1\supset\mathfrak{T}_0\supset\mathfrak{K}$ is a fortiori normal. We have

$$\mathfrak{T}_H/\mathfrak{T}_1\cong\mathscr{L}(\bar{H}), \quad \mathfrak{T}_1/\mathfrak{T}_0\cong\mathfrak{O}(G, \Phi|_G), \quad \mathfrak{T}_0/\mathfrak{K}\cong\mathscr{L}(\bar{H}, G). \tag{15}$$

Remarks: 1. The elements of $\mathfrak{T}_0$ are algebraic; every $T\in\mathfrak{T}_0$ satisfies the polynomial equation $(\xi-1)^3=0$, i.e. $(T-1)^3x=0$ for all $x\in E$; the elements of the subgroup $\mathfrak{K}$ satisfy the equation $(\xi-1)^2=0$. The proof is straightforward using (7).

2. If $\dim E = \aleph_0$ then $\mathscr{L}(\bar{H})$ in (8) and (15) is isomorphic to the multiplicative group of all nonsingular row- and column-finite denumerable matrices. This is seen by introducing a symplectic basis $\{h_i, h_i'\}_{i\geqslant 0}$ for $H\oplus\bar{H}$, its two halves spanning H and $\bar{H}$ respectively. Furthermore this matrix group is generated by all those $T\in GL(\bar{H})$ for which $\bar{H}$ splits into a direct sum of finite dimensional subspaces $H_j = k(h_i')_{n_{j-1}\leqslant i\leqslant n_j}$ $(j=1, 2, \ldots)$, the decomposition depending on T. In fact every row- and column-finite denumerable matrix can be written as a product of two matrices each of which appears as diagonal under a suitable decomposition into finite blocks ([16]).

3. When $H=\bar{H}^*$ then $\mathscr{L}(\bar{H})$ is simply $GL(\bar{H})$.

IV.4 The connection between $\mathfrak{T}(H, \Phi)$ and $\mathfrak{T}_H$ is a simple one. For (E, Φ) a semisimple space we denote by $\mathfrak{J}(E, \Phi)$ (or simply $\mathfrak{J}$) the normal subgroup of $\mathfrak{O}(E, \Phi)$ generated by all reflections about hyperplanes of E. In other words, the elements of $\mathfrak{J}$ are precisely those orthogonal automorphisms of E which leave orthogonal summands of E of finite codimension pointwise fixed. (For a discussion of $\mathfrak{J}$ see [7].) We have the following

LEMMA: *For every $T\in\mathfrak{T}(H, \Phi)$, H maximal, there exists a $T_0\in\mathfrak{J}$ such that $T_0T(H) = H$.*

(E, Φ) is semisimple as usual and here is of arbitrary dimension. Indeed by II Theorem 3 we have $H=(H\cap T(H))\oplus F$, $T(H)=(H\cap T(H))\oplus G$ with $\dim F=\dim G= n<\aleph_0$. $F\oplus G$ is semisimple, and therefore an orthogonal sum of n hyperbolic planes $k(f_i, g_i)$, $1\leqslant i\leqslant n$. $E=(F\oplus G)\oplus(F\oplus G)^\perp$, $H\cap T(H)\subset(F\oplus G)^\perp$. We define T_0 to be the identity on $(F\oplus G)^\perp$. On $(F\oplus G)$ we define T_0 by $T_0 g_i = f_i$ and $T_0 f_i = g_i$, $1\leqslant i\leqslant n$. Thus $T_0T(H)=H$. In particular

$$\mathfrak{T}(H, \Phi)/\mathfrak{J}(E, \Phi) \cong \mathfrak{T}_H/\mathfrak{T}_H\cap\mathfrak{J}(E, \Phi). \tag{16}$$

We end this section with two theorems which apply whenever the ground field k belongs to the class described in I.4, independently of the form Φ. More generally they deal with spaces of the type (1) with G of finite dimension. (In the following, $X*Y$ denotes the semidirect product of the groups X and Y. See for example [15] page 212.)

THEOREM 22: *Let E be as in* (1) *with G of finite dimension.*

(i) *If E is of denumerable dimension we have $\mathfrak{T}(H, \Phi)/\mathfrak{J}(E, \Phi)\cong\mathscr{L}_0 * \mathfrak{K}_0$.*

(ii) *If $H=\bar{H}^*$ we have $\mathfrak{O}(E, \Phi)=\mathfrak{T}(\bar{H}^*, \Phi)$ and $\mathfrak{O}(E, \Phi)/\mathfrak{J}(E, \Phi)\cong\mathfrak{G}_0 * \mathfrak{K}_0$.*

$\mathfrak{K}_0$ is the quotient of the abelian group $\mathfrak{K}$ (of (1), (13) *and Theorem* 21) *modulo the transformations in $\mathfrak{K}$ of finite rank; $\mathscr{L}_0$ is the quotient of the multiplicative group of denumerable row- and column-finite matrices modulo its matrices of finite rank; $\mathfrak{G}_0$ is the quotient of $GL(\bar{H})$ modulo its transformations of finite rank.*

Proof: Combining the decompositions of the previous section with the above lemma we find for every $T\in\mathfrak{T}(H, \Phi)$ a decomposition $T=T_*\circ\tilde{T}\circ T_0$ *with* $T_0\in\mathfrak{J}(E, \Phi)$,

$\tilde{T}\in\mathfrak{K}$, $T_*|_G=1_G$, $T_*|_H$ and $-T_*|_{\bar{H}}$ transposes of each other ($T_*|_H\in\mathscr{L}(\bar{H})$). One verifies that the factors in this representation are unique modulo transformations T_*', $\tilde{T}'$, T_0' of finite rank. We obtain thus a bijective $j:\mathfrak{T}/\mathfrak{J}\to\mathscr{L}_0\times\mathfrak{K}_0$ (resp. $\mathfrak{T}/\mathfrak{J}\to\mathfrak{G}_0\times\mathfrak{K}_0$). The group structure of $\mathfrak{T}/\mathfrak{J}$ is readily transferred to the Cartesian product under j: For $S, T\in\mathfrak{T}$, $S=S_*\circ\tilde{S}\circ S_0$, $T=T_*\circ\tilde{T}\circ T_0$, the coset $(S\circ T)$ is mapped into the pair of cosets $(S_*\circ T_*)$, $(\tilde{S}^{T_*}, \tilde{T})$ under j where $\tilde{S}^{T_*}=T_*^{-1}\tilde{S}T_*\in\mathfrak{K}$. Further, by the last lemma in II.1, we have $\mathfrak{O}(E,\Phi)=\mathfrak{T}(\bar{H},\Phi)$ in case (ii). Q.E.D.

THEOREM 23: *Let* $\mathfrak{T}_\mathfrak{J}$ *and* $\mathfrak{O}_\mathfrak{J}$ *be the quotients* $\mathfrak{T}|\mathfrak{J}$ *and* $\mathfrak{O}|\mathfrak{J}$ *in* (i) *and* (ii) *of Theorem 22. For X a group let* K_X *be its commutator subgroup. Corresponding to* (i) *and* (ii) *respectively we have*

$$\text{(j)}\quad K_{\mathfrak{T}_\mathfrak{J}}\cong K_{\mathscr{L}_0}*\mathfrak{K}_0,\quad \mathfrak{T}_H/K_{\mathfrak{T}_\mathfrak{J}}\cong\mathscr{L}_0/K_{\mathscr{L}_0}$$

$$\text{(jj)}\quad K_{\mathfrak{O}_\mathfrak{J}}\cong K_{\mathfrak{G}_0}*\mathfrak{K}_0,\quad \mathfrak{O}_\mathfrak{J}/K_{\mathfrak{O}_\mathfrak{J}}\cong\mathfrak{G}_0/K_{\mathfrak{G}_0}$$

where in (jj) *we make the additional assumption that the field contains more than three elements when* $\dim\bar{H}>\aleph_0$.

Proof: We first show that every element $u\in\mathfrak{K}_0$ is a commutator of the form $u=[v,s]=vsv^{-1}s^{-1}=vsv^{-1}\circ s^{-1}$ where $v\in\mathscr{L}_0$ (resp. $\mathfrak{G}_0$) $s\in\mathfrak{K}_0$, and $vsv^{-1}\in\mathfrak{K}_0$. We choose a representative T in u, $T=1+L\in\mathfrak{K}$. For every $\gamma\in k$ we have $T_\gamma=1+\gamma L\in\mathfrak{K}$. Let further $T_\sigma^*\in\mathfrak{T}(E,\phi)$ be of the following sort: $T_\sigma^*|_G=1_G$, $T_\sigma^*|_{\bar{H}}=\sigma\cdot 1|_{\bar{H}}$, $T_\sigma^*|_H=\sigma^{-1}\cdot 1|_H$ for $0\neq\sigma\in k$. Since $T_\gamma|_G=1|_G$ also, we find $[T_\sigma^*,T_\gamma]=T_\sigma^*T_\gamma T_\sigma^{*-1}T_\gamma^{-1}=1+\gamma(\sigma^2-1)L$. Thus, if k contains more than three elements, there is a $0\neq\sigma\in k$ with $\sigma^2-1\neq 0$, and we may choose $\gamma=(\sigma^2-1)^{-1}$. For such a choice we have $[T_\sigma^*,T_\gamma]=1+L=T$. T_σ^* corresponds to an element in $\mathscr{L}_0$ (resp. $\mathfrak{G}_0$) under the isomorphism $j:\mathfrak{T}/\mathfrak{J}\to\mathscr{L}_0*\mathfrak{K}_0$ (resp. $\mathfrak{O}/\mathfrak{J}\cong\mathfrak{G}_0*\mathfrak{K}_0$), namely the coset (T_σ^*). (If $\dim\bar{H}=\aleph_0$ then the results follows from the proof of the corollary to Theorem 20 and the second part of Theorem 21 without assumptions on k). On the other hand, let us write (x,y) for an element in the semidirect product $X*Y$, $x\in X$, $y\in Y$. Multiplication goes as follows $(x,y)*(u,v)=(xu,y^uv)=(xu,u^{-1}yuv)$. It is straightforward to verify that the commutator $[(1,y)(x,1)]$ equals $(1,[y,x])=(1,yxy^{-1}x^{-1})$. Since we have shown that every $u\in\mathfrak{K}_0$ is of the form $yxy^{-1}x^{-1}$ with $y\in\mathscr{L}_0$ (resp. $\mathfrak{G}_0$) and $x\in\mathfrak{K}_0$ we see that every element $(1,u)\in\mathscr{L}_0*\mathfrak{K}_0$ (resp. $\mathfrak{G}_0*\mathfrak{K}_0$) is a commutator. Hence every element $([y_1, y_2],x)\in K_{\mathscr{L}_0}*\mathfrak{K}_0$ (resp. $K_{\mathfrak{G}_0}*\mathfrak{K}_0$) is a product of two commutators since $([y_1,y_2],x)=[(y_1,1),(y_2,1)]*(1,x)$. Thus $K_{\mathscr{L}_0}*\mathfrak{K}_0$ (resp. $K_{\mathfrak{G}_0}*\mathfrak{K}_0$) is contained in the commutator subgroup of $\mathscr{L}_0*\mathfrak{K}_0$ (resp. $\mathfrak{G}_0*\mathfrak{K}_0$). The converse is trivial. Q.E.D.

V. Non-denumerable Spaces

Among the infinite dimensional k-spaces (E,Φ), k arbitrary, essentially only the denumerable case has thus far been treated with success. There are at least two reasons for this: practically all of the techniques applied in the finite and denumerable case prove to be rather useless when $\dim E > \aleph_0$ (e.g. proofs by induction in the finite case and inductive constructions in the denumerable case); secondly, many of the vital theorems of the finite or denumerable case actually cease to be valid if $\dim E > \aleph_0$. For example later we shall give an example of a space $E = V_1 \oplus V_2$, V_1 and V_2 totally isotropic and of the same dimension, for which there is no symplectic basis whose two halves span V_1 and V_2 respectively (whereas for $\dim E \leqslant \aleph_0$ there is always such a basis). In the following we shall list some theorems and examples illustrating features of the nondenumerable case.

Generalizing an example given in [4], we start by constructing spaces useful for various examples. Let $\mathfrak{a}$ and $\mathfrak{b}$ be arbitrary infinite cardinals; let k be a field of any characteristic with $\operatorname{card} k \geqslant \max(\mathfrak{a}, \mathfrak{b})$, V and W k-spaces of dimension $\mathfrak{a}$ and $\mathfrak{b}$ respectively. We shall define symmetric forms Φ on $V \oplus W$ by specifying only the values $\Phi(V, W)$. This is done in such a way that (E, Φ) will be semisimple if in addition we merely require that at least one of V and W have an orthogonal basis for Φ, isotropic or not; otherwise Φ may be defined completely arbitrarily on the subspaces V and W. We proceed as follows: If char $k = 2$, let I and J be disjoint subsets of k with $\operatorname{card} I = \mathfrak{a}$ and $\operatorname{card} J = \mathfrak{b}$. If $\operatorname{char} k \neq 2$, decompose k into $k_1 \cup k_2$ such that $\alpha \in k_1$ if and only if $-\alpha \in k_2$, $k_1 \cap k_2 = \{o\}$. Since $\operatorname{card} k_1 \geqslant \max(\mathfrak{a}, \mathfrak{b})$ we may let I and J be disjoint subsets of k_1 with $\operatorname{card} I = \mathfrak{a}$ and $\operatorname{card} J = \mathfrak{b}$. In either case $\alpha + \beta \neq 0$ for $\alpha \in I$, $\beta \in J$.

In V and W we introduce bases $V = k(v_\alpha)_{\alpha \in I}$ and $W = k(w_\beta)_{\beta \in J}$. We set

$$\phi(v_\alpha, w_\beta) = \frac{1}{\alpha + \beta}.$$

Since $\det(1/(\alpha_i + \beta_j))_{1 \leqslant i; j \leqslant n} = \prod_{i<j} (\alpha_i - \alpha_j) \prod_{i<j} (\beta_i - \beta_j) \prod_{i,j} (\alpha_j + \beta_i)^{-1} \neq 0$ provided the α_i's are distinct and the β_j's are distinct, it is easily seen that $V^\perp \cap W = (0)$ and $W^\perp \cap V = (0)$. More precisely, if in $v = \sum \lambda_i v_{\alpha_i}$ only m coefficients λ_i are non-zero, and if $v \perp w_\beta$ for m different basis vectors w_β then $v = 0$. Under the additional proviso that one of the two bases, say $\{v_\alpha\}$, is orthogonal we conclude that Φ is non degenerate: $x = \sum \lambda_i v_{\alpha_i} + \sum \mu_j w_{\beta_j}$ and $x \perp E$ implies $x \perp V$, and hence $\sum \mu_j w_{\beta_j} \perp v_{\alpha_i}$ for i sufficiently large. Hence $\mu_j = 0$ by the previous condition. But then $x = \sum \lambda_i v_{\alpha_i} \in V$ and $x \perp W$ implies $x = 0$. Since $V^\perp \cap W = V \cap W^\perp = (0)$, the construction can of course be interpreted as giving dual pairings $\langle V, W \rangle$ for arbitrarily prescribed dimensions for V and W. If we choose $\mathfrak{a} \neq \mathfrak{b}$ and V and W totally isotropic for Φ then we obtain a space $(V + W, \Phi)$ which does not admit an orthogonal basis. This is a consequence of the following more general theorem:

THEOREM 24: *Let (E, Φ) be a semisimple space spanned by an orthogonal basis. If V_1 is a totally isotropic subspace of E then there is a subspace $U_1 \subset V_1$ with* $\dim U_1 = \dim V_1$, *and E admits a Witt decomposition* (cf. I.2)

$$E = (U_1 \oplus U_2) \overset{\perp}{\oplus} E_0$$

and in particular $\dim E/V_1 \geqslant \dim V_1$. *Furthermore*

(i) *If* $E = V_1 \oplus V_2$ *with* V_1 *and* V_2 *totally isotropic then* $\dim V_1 = \dim V_2$ *and we may choose* $U_2 \subset V_2$ *in the above decomposition.*

(ii) *If* $\dim V_1 \leqslant \aleph_0$ and $V_1^{\perp\perp} = V_1$ *we may choose* $U_1 = V_1$.

Proof: For finite dimensional V_1 the assertion is well known. We assume that $\dim V_1$ is infinite. Let $\{e_\nu\}_{\nu \in L}$ be an orthogonal basis for E. We consider all those finite sets of basis vectors e_ν which span spaces F with $F \cap V_1 \neq (0)$, furthermore sets $\mathscr{F}$ of such spaces F with the property that $F_1 \cap F_2 = (0)$ for $F_1, F_2 \in \mathscr{F}$, $F_1 \neq F_2$. The sets $\mathscr{F}$ are partially ordered by $\subset$. By Zorn's lemma there is a maximal element $\mathscr{F}_0$. We have an orthogonal decomposition $E = (\underset{\mathscr{F}_0}{\oplus} F) \oplus E_1$, E_1 spanned by the e_ν not occurring in the F in $\mathscr{F}_0$. Since V_1 is infinite dimensional we notice that card $\mathscr{F}_0$ is necessarily infinite. Let $\underset{\mathscr{F}_0}{\oplus} F = E_2$. We have $\dim E_2 \geqslant \dim V_1$; for if $\dim E_2 < \dim V_1$ we could decompose a basis $\{v_\iota\}$ of V_1 as follows: $v_\iota = e_{1\iota} + e_{2\iota}$, $e_{1\iota} \in E_1$, $e_{2\iota} \in E_2$, and the projections $\{e_{2\iota}\}$ would be linearly dependent. Therefore there would be a linear combination $0 \neq \sum \lambda_i v_{\iota_i} \in E_1$ with $\sum \lambda_i v_{\iota_i} \in F_* \subset E_1$, F_* spanned by finitely many basis vectors e_ν. This contradicts the maximality of $\mathscr{F}_0$. Hence we must have $\dim E_2 \geqslant \dim V_1$. Since card$\mathscr{F}_0$ is infinite and the $F \in \mathscr{F}_0$ are finite dimensional, we have $\dim E_2 = \text{card}\mathscr{F}_0$. Hence card$\mathscr{F}_0 \geqslant \dim V_1$. But by the definition of the sets F and $\mathscr{F}$ we have card$\mathscr{F} \leqslant \dim V_1$ for all $\mathscr{F}$. Thus card$\mathscr{F}_0 = \dim V_1$. We pick a vector $v_F \neq 0$, $v_F \in F \cap V_1$, for each $F \in \mathscr{F}_0$. Since F is semisimple, there exists an isotropic $v'_F \in F$ with $\Phi(v_F, v'_F) = 1$. We set $U_1 = k(v_F)_{F \in \mathscr{F}_0}$, $U_2 \in k(v'_F)_{F \in \mathscr{F}_0}$. For $F \in \mathscr{F}_0$ we have $F = k(v_F, v'_F) \overset{\perp}{\oplus} G_F$. Setting $E_0 = E_1 \oplus \underset{\mathscr{F}_0}{\oplus} G_F$ we obtain the desired decomposition $E = (U_1 \oplus U_2) \overset{\perp}{\oplus} E_0$ of our theorem. Further since $U_2 \cap V_1 = (0)$, $\dim E/V_1 \geqslant \dim U_2 = \dim U_1 = \dim V_1$.

Now to prove (i) assume in addition that $E = V_1 \oplus V_2$, V_2 a totally isotropic complement of V_1. We have just proved that $\dim V_2 \geqslant \dim V_1$. Hence $\dim V_1 = \dim V_2 = \dim E$.

In order to prove that we may choose $U_1 \subset V_1$ and $U_2 \subset V_2$ we repeat the earlier device. This time we consider finite dimensional spaces F spanned by some e_ν, $\nu \in L$, such that $(F \cap V_1) \oplus (F \cap V_2)$ is not totally isotropic. There is a maximal set $\mathscr{F}_0$ of such spaces F (with respect to $\subset$), $F_1 \cap F_2 = (0)$ for $F_1, F_2 \in \mathscr{F}_0$, $F_1 \neq F_2$; and we have an orthogonal decomposition of E, $E = (\underset{\mathscr{F}_0}{\oplus} F) \oplus E_1$. Suppose that we had $\dim E/E_1 < \dim V_1 = \dim E = \dim E_1$. Setting $W = (E_1 \cap V_1) \oplus (E_1 \cap V_2)$ we have in that case that $\dim E_1/W = \dim E/E_1 < \dim E_1$. By the first part of our theorem W cannot be totally isotropic. There is a $v \in E_1 \cap V_1$ and a $v' \in E_1 \cap V_2$ with $\Phi(v, v') \neq 0$. The vectors v and v' are contained in a space $F_0 \subset E_1$ spanned by some finitely many e_ν. F_0 qualifies for membership in $\mathscr{F}_0$ thus contradicting the maximality of $\mathscr{F}_0$. This shows that we must have $\dim E/E_1 = \dim V_1$. The proof now goes through as before. This time we may pick nonzero vectors $v_F \in F \cap V_1$ and $v'_F \in F \cap V_2$ for all $F \in \mathscr{F}_0$. This completes the proof of (i).

The assertion (ii) is readily reduced to the denumerable case. Let $\{v_i\}_{i \geq 1}$ be a basis of V_1; $v_i = \sum \alpha_{i\iota} e_\iota$, $\{e_\iota\}$ an orthogonal basis of E. Hence $V_1 \subset E_1$, E_1 spanned by the e_ι with $\alpha_{i\iota} \neq 0$, $i \geq 1$. We now apply the theorem of I.2 to E_1. This completes the proof of Theorem 24.

An important corollary of Theorem 24 is the following:

Corollary: Let (E, Φ) be a semisimple space spanned by an orthogonal basis, F some subspace of E. We have $\dim(\operatorname{rad} F) \leq \dim E/F$.

We remark that by the reasoning applied in proving (ii) every subspace H of E is contained in an orthogonal summand E_1 of E, $\dim E_1 = \dim H$. This trivial observation has the following consequences.

Theorem 25: *Let (E, Φ) be a semisimple space spanned by an orthogonal basis. Then*

(i) *E has no subspaces of the form $A \oplus B$ with $A^\perp \cap B = (0)$ and* $\dim A < \dim B$.

(ii) *All maximal totally isotropic subspaces of E are of the same dimension.*

(iii) *If H is a subspace of E with $\aleph_0 \leq \dim H < \dim E$, then* $\dim E = \dim E/H = \dim H^\perp \geq \dim H \geq \dim E/H^\perp$.

(iv) *If H is $\perp$-dense in E, i.e. $H^{\perp\perp} = E$, then* $\dim H = \dim E$.

To prove (i) write $E = E_1 \overset{\perp}{\oplus} E_2$ with $\dim A = \dim E_1$ and $A \subset E_1$, hence $E_2 \subset A^\perp$. If $\dim A < \dim B$, then B must meet E_2 so $A^\perp \cap B \neq (0)$.

The reasoning for (ii) is similar. The relationships in (iii) and (iv) follow from an examination of the decomposition $E = (H \oplus H_0) \overset{\perp}{\oplus} E_0$ with $\dim H = \dim(H \oplus H_0)$.

By (ii) we see that the $V \oplus W$ in our earlier example has no orthogonal basis when $\mathfrak{a} < \mathfrak{b}$, independently of how Φ is defined on V and W.

We now turn from spaces having orthogonal bases to the other extreme, namely infinite dimensional spaces which possess no infinite non-trivial orthogonal decomposition.

Definition 2: Let (E, Φ) be semisimple. (E, Φ) is called solid if and only if every orthogonal decomposition $\bigoplus_I E_\iota$ of E has $E_\iota = (0)$ with the exception of finitely many ι.

Corollary: *(E, Φ) is solid if and only if there is no decomposition $E = F \oplus F^\perp$ with F of denumerable dimension.*

Indeed every such decomposition $E = F \oplus F^\perp$ gives a non-trivial infinite decomposition as F admits an orthogonal basis. Conversely let $E = \oplus E_\iota$ with infinitely many $E_\iota \neq (0)$. Since each non-trivial E_ι is semisimple we find an orthogonal summand of denumerable dimension by picking one suitable vector e_{α_i} from each of $\aleph_0$ non-trivial summands E_{α_i}.

Finite dimensional spaces are of no interest in this connection. Spaces of denumerable dimension are never solid since they have orthogonal bases. Non-trivial examples of solid spaces are furnished by Hilbert spaces over the reals. The following example is of a different kind. Let k be an arbitrary field, H an infinite dimensional k-space, H^* its algebraic dual. Define Φ on $E=H^*\oplus H$ by $\Phi(h^*, h)=h^*(h)$, $h^*\in H^*$, $h\in H$ and $\Phi(H^*, H^*)=\Phi(H, H)=(0)$; then E is solid. In view of the corollary above, the assertion follows from the following more general:

THEOREM 26: *Let* $(E, \Phi)=(H^*\oplus H)\overset{\perp}{\oplus} G$ *be infinite dimensional,* H^* *the algebraic dual of* H, G *finite dimensional,* H^* *and* H *both totally isotropic and* $\Phi(h^*, h)=h^*(h)$, $h^*\in H^*$, $h\in H$. *If* A *and* B *are infinite dimensional subspaces of* E *with* $E=A\overset{\perp}{\oplus} B$ *then* $\dim A\geqslant \|k\|^{\aleph_0}$ *and* $\dim B\geqslant \|k\|^{\aleph_0}$.

We remark that the case $\dim A=\|k\|^{\aleph_0}$ does take place: Let $H=H_0\oplus H_1$, $\dim H_0=\aleph_0$. Then $E=(H_0^*\oplus H_0)\overset{\perp}{\oplus}(H_1^*\oplus H_1)\overset{\perp}{\oplus} G$ is a space of the type in Theorem 26, and $\dim H_0^*+H_0=\|k\|^{\aleph_0}$.

Proof of Theorem 26: We endow E with the topology $\tau=\tau_\Phi H^*$ and consider $\sigma: E\to E'$ defined as follows. For $x\in H^*$, $y\in H$ and $z\in G$, let $\sigma: x\to \Phi_x|_H$, $y\to\Phi_y|_{H^*}$, $z\to\Phi_z|_G$ where $\Phi_x(h)=\Phi(x, h)$ etc. σ is injective and as usual we make the identifications $\sigma(H^*)=H^*$, $\sigma(H)=H$ and $\sigma(G)=G$. On the other hand it was proved earlier that the restrictions of σ are isomorphisms as follows, $\sigma: H^*\cong H'$ and $\sigma: H\cong H^{*\prime}$ ($'$ with respect to $\tau=\tau_\Phi H^*$). Since G is finite dimensional we see that $\sigma: E\cong E'$ is an (algebraic) isomorphism, and the canonical pairing $\langle E, E'\rangle$ is induced by Φ, i.e. $\langle x, y\rangle=\Phi(x, \sigma^{-1}y)$. (We remark that σ is also a topological isomorphism when E' is supplied with the Mackey topology $\tau_c(E', E)$, for $\tau_c(E', E)$ is seen to be precisely the image topology of $\tau_\Phi H^*$ under σ). Let F be an arbitrary subspace of (E, τ). The τ-closure of F is F^{00}, ^{0} with respect to any pairing $\langle E, E'\rangle$. Hence F is τ-closed if and only if F is $\perp$-closed as $F^0=\sigma(F^\perp)$ and $F^\perp=(\sigma(F))^0$ in our case. We have thus shown: *If the space* E *of Theorem* 26 *is endowed with the topology* $\tau=\tau_\Phi H^*$ *then a subspace* $F\subset E$ *is* τ*-closed if and only if it is* $\perp$*-closed.* Assume then that $E=A\overset{\perp}{\oplus} B$ and $\dim A<\|k\|^{\aleph_0}$. A is semisimple and $\perp$-closed, hence τ-closed, hence discrete by Corollary 2 to Theorem 9. Therefore any subspace $S\subset A$ is τ-closed and therefore $\perp$-closed in E. Since $S^{\perp_A}=S^\perp+B$, $S^{\perp_A\perp_A}=S$, i.e. S is orthogonally closed in A with respect to $\Phi|_A$. Thus we see that $(A, \Phi|_A)$ is a semisimple space in which every subspace is orthogonally closed. Hence A is finite dimensional, (cf I.1), which was excluded.

COROLLARY: *The spaces of Theorem* 26 *are solid.* The last theorem permits some observations.

1) Let F and G be vectorspaces over some field, $\dim F=\aleph_0$, $\dim G=2^{\aleph_0}$. We introduce bases $F=k(f_r)_{r\in P}$, $G=k(g_\lambda)_{\lambda\in R}$, r running through the rationals and λ

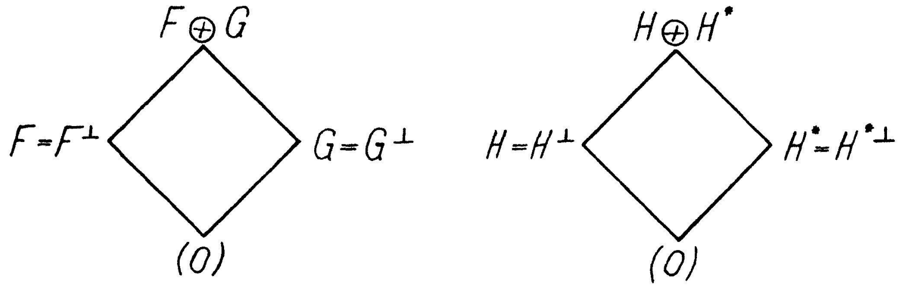
running through the reals. We define a symmetric bilinear form Φ on $F\oplus G$ by declaring the summands F and G to be totally isotropic and taking

$$\phi(f_r, e_\lambda) = \begin{cases} 0 & \text{if} \quad r < \lambda \\ 1 & \text{if} \quad r \geqslant \lambda \end{cases}.$$

It is easily verified that $(F\oplus G, \Phi)$ is semisimple. Our space contains an orthogonal summand of denumerable dimension: $F\oplus G = E_1 \overset{\perp}{\oplus} E_2$ where

$$E_1 = k(f_n, e_n)_{-\infty<n<\infty}, \qquad n \quad \text{an integer}$$
$$E_2 = k(f_{r_n} - f_n, e_{\lambda_n} - e_n)_{-\infty<n<\infty, n-1<\lambda_n<n, n<r_n<n+1}.$$

Furthermore if H is of denumerable dimension, and if $(H^*\oplus H, \Phi)$ is defined as in Theorem 26, we see that the Kaplansky-lattices generated by H^* and H in $H^*\oplus H$ and by F and G in $F\oplus G$ under the operations $+, \cap, ^\perp$ are isomorphic:



Both lattices define the same cardinal numbers (dimensions of quotient spaces of neighboring spaces). Nevertheless the two spaces are not isomorphic, one being solid, the other not. In the denumerable case isomorphism of two such lattices would guarantee isomorphism of the spaces (symplectic bases). For the importance of these lattices see [8].

2) Let V be a totally isotropic subspace of (E, Φ) with $V^\perp = V$. If $\dim E = \aleph_0$ then V always admits a totally isotropic algebraic complement W. This is not true in general. Consider the k-vectorspace $H^*\oplus H$, H spanned by an infinite basis $\{h_\iota\}_{\iota\in I}$. Let $K\subset H^*$ be spanned by the conjugate family of functions h_ι^*, $h_\iota^*(h_\kappa) = \delta_{\iota\kappa}$, $(\iota, \kappa\in I)$. Thus $H^*\oplus H = (L\oplus K)\oplus H$, L some algebraic complement of K in H^*. We define a bilinear form Φ on $H^*\oplus H$ as follows: $\Phi(L, L) = \Phi(H, H) = (0)$, $\Phi(h_\iota^*, h_\kappa^*) = \delta_{\iota\kappa}$, $\iota, \kappa \in I$; $\Phi(L, K) = (0)$ and $\Phi(h^*, h) = h^*(h)$ for $h^*\in H^*$, $h\in H$. $(H^*\oplus H, \Phi)$ is semisimple, $H^\perp = H$, and we have the decomposition:

$$H^* \oplus H = K \overset{\perp}{\oplus} (L \oplus k(h_\iota - h_\iota^*)_I).$$

In particular if $\dim H = \aleph_0$ then $\dim K = \aleph_0$ and $(H^* + H, \Phi)$ is not solid. If H admitted a totally isotropic algebraic complement W, $H^* + H = W \oplus H$, then $w \to \Phi_w|_H, (w \in W)$, would be an (algebraic) isomorphism $W \cong H^*$, and $H \oplus W$ would be a space of the type discussed in Theorem 26, hence solid.

We remark that there are of course spaces which fall somewhere between the solid case and the case of orthogonal bases: Theorem 25 (i) says that our earlier examples $V \oplus W$ (with $\mathfrak{a} < \mathfrak{b}$) and $H^* + H$ (of Theorem 26) cannot even be embedded in a space with an orthogonal basis. Thus by taking orthogonal sums of infinitely many copies of such spaces we obtain examples of this intermediate kind. As we shall prove at another time, there are spaces admitting of no orthogonal bases which can nevertheless be embedded into spaces with orthogonal bases.

Clearly if E is solid and $E = H \overset{\perp}{\oplus} G$ then H and G are solid. We now prove the converse.

THEOREM 27: *Let (E, Φ) be any semisimple space splitting into an orthogonal sum $E = H \oplus G$. Assume that E admits a denumerable orthogonal summand, i.e. $E = F \oplus F^\perp$ with* $\dim F = \aleph_0$. *Then one of H and G possesses an orthogonal summand of denumerable dimension.*

Proof: Since F is semisimple and of denumerable dimension, F is spanned by an orthogonal basis $f_i = h_i + g_i$, $h_i \in H$, $g_i \in G (i \geq 1)$. Let $H_0 = k(h_i)_{i \geq 1}$, $G_0 = k(g_i)_{i \geq 1}$. Splitting off radicals we set $H_0 = H_1 \oplus \operatorname{rad} H_0$ and $G_0 = G_1 \oplus \operatorname{rad} G_0$. Since $F \subset G_0 + H_0$, we see that at least one of H_1 and G_1 is infinite dimensional; for otherwise F would contain a totally isotropic subspace F_0 with $\dim F/F_0$ finite, contradicting the semisimplicity of F. We assume that $\dim H_1$ is infinite dimensional, i.e. $\dim H_1 = \aleph_0$. H_1 has an orthogonal basis, $H_1 = k(h_i')_{i \geq 1}$, $\|h_i'\| \neq 0$. We shall prove as a first step that we can introduce a new basis $\{h_i''\}_{i \geq 1}$ for H_1 such that

$$h_i'' = h_i' - \sum_{1}^{i-1} \lambda_{ij} h_j', \quad h_1'' = h_1' \tag{1}$$

with the following property: For every $n \geq 1$ there exists an m such that

$$h_i'' \in k(h_j)_{j \geq n} \qquad \text{for all} \qquad i > m, \tag{2}$$

h_i the components in $f_i = g_i + h_i$ above.

In order to prove this we first express every h_i' in terms of the h_j: $h_i' = \sum_{\text{finite}} \lambda_{ij}^{(1)} h_j$. Put $h_i^{(1)} = h_i'$. Let $h_{n_1}^{(1)}$ be the first $h_i^{(1)}$ with $\lambda_{i1}^{(1)} \neq 0$, i.e. $\lambda_{11}^{(1)} = \lambda_{21}^{(1)} = \cdots = \lambda_{n_1-1,1}^{(1)} = 0$ and $\lambda_{n_1,1}^{(1)} \neq 0$. Take $h_i^{(2)} = h_i^{(1)} = h_i'$ for $i \leq n_1$ and $h_i^{(2)} = h_i^{(1)} - \lambda_{i1}/\lambda_{n_1 1} h_{n_1}^{(1)}$ for $i > n_1$. Then for $i > n_1$, $h_i^{(2)} \in [h_j]_{j \geq 2}$. Proceeding by induction, we assume that we have already formed the vectors $h_1^{(1)}, \ldots, h_{n_1-1}^{(1)}, h_{n_1}^{(1)}, h_{n_1+1}^{(2)}, \ldots$ with $h_i^{(m)} \in [h_j]_{j \geq m}$ for all $i > n_{m-1}$, $n_{m-1} > n_{m-2} > \ldots$ and $h_i^{(m)} = \sum \lambda_{ij}^{(m)} h_j$. Let $h_{n_m}^{(m)}$ be the first $h_i^{(m)}$ with $\lambda_{im}^{(m)} \neq 0$ and $i > n_{m-1}$. Take

$$h_i^{(m+1)} = \begin{cases} h_i^{(m)}, & i \leqslant n_m \\ h_i^{(m)} - \dfrac{\lambda_{im}}{\lambda_{n_m m}} h_{n_m}^{(m)}, & i > n_m . \end{cases}$$

Then $h_i^{(m+1)} \in [h_j]_{j \geqslant m+1}$ for $i > n_m$ and $n_m > n_{m-1} > n_{m-2} > \dots$. We relabel the sequence $h_1^{(1)}, \dots, h_{n_1-1}^{(1)}, h_{n_1}^{(1)}, h_{n_1+1}^{(2)}, \dots, h_{n_2}^{(2)}, h_{n_2+1}^{(3)}, \dots$ which we obtain in this fashion with h_1'', $h_2'', \dots$. The h_i'' have the property that $h_i'' \in [h_j]_{j>n}$ for all i sufficiently large. Furthermore (1) is satisfied and the $\{h_i''\}$ span all of $k(h_i') = H_1$.

Now let h be an arbitrary vector in H. h is orthogonal to almost all of the h_i''. Indeed, since $h \in H \subset F \oplus F^\perp$ we have $h - \sum_1^n \beta_i f_i \perp F$ for suitable n. For $j > n$ we obtain $\Phi(h, h_j) = \Phi(h, h_j + g_j) = \Phi(h, f_j) = \Phi(h - \sum \beta_i f_i, f_j) = 0$ since $H \perp G$ and the f_i mutually orthogonal. Furthermore by (2) there exists an m such that $h_i'' \in k(h_j)_{j>n}$ for all $i > m$. Thus $\Phi(h, h_i'') = 0$ for $i > m$. We have:

for every $h \in H$, $h \perp h_j''$ for almost all j;
in particular the (symmetric) matrix $\Phi(h_r'', h_i'')$ is row- and column-finite. (3)

In other words the h_i'' are "almost" an orthogonal basis. We proceed to select an orthogonal family $\{h_i^*\}_{i \geqslant 1}$ of non-isotropic vectors in $k(h_i'') = H_1$ such that (2) holds for the h_i^*, i.e. for every $n \geqslant 1$ there exists an m such that

$$h_i^* \in k(h_j)_{j \geqslant n} \quad \text{for all} \quad i > m . \tag{4}$$

This we do as follows. Take $h_1^* = h_1''$, hence by (1), $\|h_1^*\| = \|h_1''\| = \|h_1'\| \neq 0$. By (3) there is an $n_2 \geqslant 1$ such that $\Phi(h_p'', h_1'') = 0$ for all $p \geqslant n_2$.

Case 1: if there is an $r_2 \geqslant n_2$ with $\|h_{r_2}''\| \neq 0$, we set $h_2^* = h_{r_2}''$.

Case 2: if $\|h_p''\| = 0$ for all $p \geqslant n_2$ we proceed as follows. By (3) there is an $m_2 \geqslant n_2$ such that $\Phi(h_p'', h_{m_2}'') = 0$ for all p with $1 \leqslant p < n_2$. Since $H = k(h_i'')_{i \geqslant 1}$ is semisimple, there must be a $p_2 \geqslant n_2$ such that $\Phi(h_{p_2}'', h_{m_2}'') \neq 0$. $\|h_{m2}'' + h_{p_2}''\| = 2\Phi(h_{m_2}'', h_{p_2}'') \neq 0$. Further since $m_2, p_2 \geqslant n_2$, $\Phi(h_1'', h_{m_2}'' + h_{p_2}'') = 0$. In this case we take $h_2^* = h_{m_2}'' + h_{p_2}''$.

These steps can be repeated, and we obtain an orthogonal family h_i^*, $i \geqslant 1$, $\|h_i^*\| \neq 0$. Every h_i^* is of the form

$$h_{r_i}'' \quad \text{or} \quad h_{m_i}'' + h_{p_i}'', \qquad i \leqslant r_i, m_i, p_i, \tag{5}$$

according to the two cases arising in each step. In view of (5), property (4) is clearly inherited from (2).

Finally we show that the h_i^* span an orthogonal summand H_2 of H. Indeed let h be an element of H. By (3) there is an m such that $\Phi(h, h_j'') = 0$ for all $j \geqslant m$. So $\Phi(h, h_j^*) = 0$ for all $j > m$ by (5). Hence $h - \sum_1^m \|h_i^*\|^{-1} \Phi(h, h_i^*)\, h_i^* \perp h_j^*$ for all j, i.e. $H = H_2 + H_2^\perp$. This concludes the proof of the theorem.

COROLLARY: *The solid spaces (E, Φ) form a monoid under the operation of orthogonal (external) sum.*

The last theorem provides us with the following types of spaces. Let (E, Φ) be solid, and put $-E=(E, -\Phi)$. Let V and W be spanned by the families $\{e_\iota+e'_\iota\}$ and $\{e_\iota-e'_\iota\}$ respectively, where e_ι and e'_ι run through the corresponding bases of E and $-E$. Then since $\Phi(e_\iota, e_\kappa)=-\Phi(e'_\iota, e'_\kappa)$, the (external) sum $E\overset{\perp}{\oplus}-E$ is the sum of two totally isotropic spaces: $E\overset{\perp}{\oplus}-E=V\oplus W$. However there is no sympletic basis for $E\oplus-E$ with its two halves spanning V and W respectively. For such a basis $\{v_\iota, w_\iota\}_I$, $\Phi(v_\iota, w_\kappa)=\delta_{\iota\kappa}$ would give an orthogonal basis $\{v_\iota+w_\iota, v_\iota-w_\iota\}_I$ for $E\overset{\perp}{\oplus}-E$, hence $E\overset{\perp}{\oplus}-E$ would not be solid. Therefore E would not be solid by Theorem 27. But this contradicts our choice of E.

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