More isoperimetric inequalities proved and conjectured.

More isoperimetric inequalities proved and conjectured[1])

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Dedicated to Michel Plancherel on his seventieth birthday

Introduction

0.1. We consider a simply connected plane domain D and various functionals depending on the shape and size of D, especially the following[2]):

A the area of D,

L the length of the perimeter of D,

I the polar moment of inertia of D with respect to its center of gravity,

P the torsional rigidity of an elastic beam with cross-section D.

We shall pay especial attention to those combinations of these functionals that depend on the shape of D alone and are independent of its size. Important examples are L^2A^{-1}, IA^{-2}, PIA^{-4}, etc.

0.2. For computing the torsional rigidity, Saint-Venant devised an approximate formula which amounts to the assertion that PIA^{-4} is approximately 1/40 for all „simple" cross-sections D. Closer examination showed that PIA^{-4} has 0 as lower, and ∞ as upper, bound when D varies unrestrictedly, but remains between finite positive bounds when D is *convex* [6, p. 11, 111, 250, 263]. Some time ago, I stated the conjecture that the true values of these bounds are 1/45 and 1/27, respectively [4]. In the first part of this paper, I shall prove one half of this conjecture:

$$PIA^{-4} < 1/27 \tag{1}$$

for all convex domains, and 1/27 *is the best possible constant.* That is, we cannot replace 1/27 by a smaller number if we wish the inequality (1) to remain valid for all convex domains. For an essential contribution to the proof of (1) I am indebted to Professor H. Hadwiger in Bern.

[1]) Sponsored by the Office of Naval Research.

[2]) Cf. **6**, p. 1–3. Numerals in heavy print refer to the bibliography.

0.3. The isoperimetric quotient L^2A^{-1} has two important properties:

(I) Of all domains, the circle has the minimum L^2A^{-1}.

(II) Of all polygons with a given number n of sides, the regular polygon with n sides has the minimum L^2A^{-1}.

These properties are not uncorrelated: (II) implies (I) insofar as we can derive (I) from (II) by passing to the limit and using the continuity of the functionals involved. (Observe that neither (I) nor (II) refers to the uniqueness of the minimum in question: this omission is intentional.) The observation of analogous cases suggests that, in some yet unclarified sense, also (I) implies (II).

I mention only two relatively elementary analogous cases: (I) remains true if we substitute either IA^{-2} or L^4I^{-1} for L^2A^{-1}.[3]) Various facts suggest the conjecture that in all "naturally" arising cases in which the analogue of (I) is true also the analogue of (II) is true. Here is a narrower, but definite, conjecture: (II) remains true if we substitute for L^2A^{-1} either IA^{-2} or L^4I^{-1} or any one of the nine quantities displayed in the table on p. 249 of **6** (for all of which the analogue of (I) has been abready established). This conjecture is supported by analogy and the verification of several, although not very extensive, particular cases [**6**, p. 158—159, 248—249, 259—268]. Here is a more extensive particular case: *Of all convex polygons with a given number n of sides only the regular polygon attains the minimum of IA^{-2}.* This will be proved in the second part of this paper. For an ingenious remark that led me to the proof given in the sequel I am indebted to Professor N. G. de Bruijn in Amsterdam. Some similar theorems that support the above conjecture by analogy will also be treated.

First Part

1.1. Let I_1 and I_2 denote the moments of inertia of D (covered with matter of surface density 1) about the principal axes of inertia through the center of gravity of D. Of course

$$I_1 + I_2 = I \ . \tag{1}$$

By a theorem of E. Nicolai [2]

$$P \leqq \frac{4 I_1 I_2}{I_1 + I_2} \ . \tag{2}$$

[3]) The proof for IA^{-2} is quite simple and elementary [3] but the only extant proof for L^4I^{-1} consists of two halfs both of which involve the theory of analytic functions [**6**, p. 123–126; **5**, v. II, p. 21, problem 124].

We introduce the abbreviation

$$4 I_1 I_2 = J \tag{3}$$

and derive from (1), (2), and (3)

$$PIA^{-4} \leqq JA^{-4} \ . \tag{4}$$

1.2. The inequality (4) leads us to investigate its right hand side. We introduce rectangular coordinates x and y, choosing the center of gravity of D as origin and the principal axes of inertia through the origin as coordinate axes. Then

$$\iint x dx dy = \iint y dx dy = \iint xy dx dy = 0 \ , \tag{5}$$

$$\begin{aligned} J &= 4 I_1 I_2 = 4 \iint x^2 dx dy \iint y^2 dx dy \\ &= 2 \iiiint (x_1 y_2 - x_2 y_1)^2 dx_1 dy_1 dx_2 dy_2 \ ; \end{aligned} \tag{6}$$

these double integrals (and those following in the next section) are extended over the domain D. We see from (6) that an *affinity leaves* JA^{-4} *invariant* (since it multiplies A by the determinant of a linear substitution, and J by the fourth power of the same determinant). Thus, JA^{-4} has the value 1/27 for the equilateral triangle, and so for all triangles; it has the value $(2\pi)^{-2}$ for the circle, and so for all ellipses.

In order to acquire an inductive basis, I computed JA^{-4} for a regular polygon with n sides, and found that this value steadily decreases as n increases from 3 to ∞. This led to the conjecture: *Of all convex domains, the triangle yields the maximum, and the ellipse the minimum, of* JA^{-4}. I was aware that this conjecture harmonizes with certain results of W. Blaschke concerning affine geometry in the large. Eventually, I proved one half of my conjecture: the circle yields the minimum of JA^{-4}; cf. sect. 2.5. Unfortunately, this (easier) half of the conjecture is irrelevant to Saint-Venant's problem.

1.3. H. Hadwiger, to whom I communicated the foregoing remarks, showed very simply that my conjecture is equivalent to a particular case of Blaschke's result and, therefore, correct.

Write p_i for the point (x_i, y_i), dp_i for $dx_i dy_i$, $|p_i p_j p_k|$ for the area of the triangle with vertices p_i, p_j, and p_k, and finally s for the center of gravity of D (the origin). Then, by (6),

$$J = 8 \iiiint |p_1 p_2 s|^2 dp_1 dp_2 \ . \tag{7}$$

Define

$$K = \iiiiiint |p_1 p_2 p_3|^2 dp_1 dp_2 dp_3 \ . \tag{8}$$

Obviously

$$|p_1p_2p_3| = |p_2p_3s| + |p_3p_1s| + |p_1p_2s| \ . \tag{9}$$

Yet, by (5),

$$\begin{aligned} &4\iiiint\!\!\iint |p_2p_3s|\,|p_3p_1s|\,dp_1dp_2dp_3 \\ &= \iiiint\!\!\iint (x_2y_3 - x_3y_2)(x_3y_1 - x_1y_3)dx_1dy_1dx_2dy_2dx_3dy_3 \\ &= 0 \ . \end{aligned} \tag{10}$$

We see from (7), (8), (9), and (10) that

$$K = 3AJ/8 \ . \tag{11}$$

Yet, by a result of Blaschke [**1**, p. 60] the triangle yields the maximum and the ellipse the minimum of KA^{-5} and, by (11), the same holds for JA^{-4}.

Thus, the maximum of the right hand side of (4) is 1/27, and so the theorem stated in sect. 0.2 is proved, except for the discussion of the case of equality.

As the value of 1/27 for JA^{-4} is only attained by triangles, we have to discuss PIA^{-4} for triangles only; I leave aside this point which can be handled in various ways [4].

Second Part

2.1. We begin with preparations for the proof of the theorem stated in sect. 0.3. We consider a triangle with sides a, b, and c, we let A denote its area, O the vertex of its angle α (which is opposite a) and I_0 its polar moment of inertia with respect to O. (More explicitely, I_0 is computed with respect to an axis that is perpendicular to the plane of the triangle and passes through O; the triangle is regarded as covered with matter of surface density 1.) A straightforward computation yields

$$I_0 = A\left(\frac{A}{\tan\alpha} + \frac{a^2}{6}\right). \tag{1}$$

Now

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos\alpha \\ &\geqq 2bc(1-\cos\alpha) = 4A(1-\cos\alpha)/\sin\alpha \end{aligned} \tag{2}$$

with equality if, and only if, $b = c$. By combining (1) and (2), we obtain

$$I_0 \geqq A^2\frac{2+\cos\alpha}{3\sin\alpha} \tag{3}$$

where inequality holds unless b and c, the sides including α, are equal.

2.2. We shall also need a property of the function

$$f(x) = \frac{\sin x}{2 + \cos x}\,. \tag{4}$$

We find that

$$f''(x) = -\frac{2(1 - \cos x)\sin x}{(2 + \cos x)^3} < 0$$

for $0 < x < \pi$ and so, in this interval, $f(x)$ is strictly convex from above. Therefore, if $\alpha_1, \alpha_2, \ldots \alpha_n$ are contained in the above interval and

$$\alpha_1 + \alpha_2 + \cdots + \alpha_n = \sigma\ , \tag{5}$$

we have

$$f(\alpha_1) + f(\alpha_2) + \cdots + f(\alpha_n) \leqq n f(\sigma/n) \tag{6}$$

and inequality holds unless $\alpha_1 = \alpha_2 = \cdots = \alpha_n = \sigma/n$.

2.3. We have now completed the preliminaries and we begin the proof of the theorem stated in sect. 0.3. We have to consider a convex polygon P with n sides. We let A denote its area, O its center of gravity, and I its polar moment of inertia with respect to O. This point O lies inside P. We connect O with the n vertices of P and divide so P into n triangles with a common vertex at O. Let $A_1, A_2, \ldots A_n$ denote the areas of these triangles, $\alpha_1, \alpha_2, \ldots \alpha_n$ their angles at O, and $I_1, I_2, \ldots I_n$ their polar moments of inertia with respect to O. Obviously

$$\Sigma A_\nu = A\ , \tag{7}$$

$$\Sigma \alpha_\nu = 2\pi\ , \tag{8}$$

$$\Sigma I_\nu = I\ ; \tag{9}$$

in these summations, as in the following summations in the present sect. 2.3, ν ranges from 1 to n.

Now, by (9) and (3) (observe the different notations)

$$I \geqq \Sigma A_\nu^2 \frac{2 + \cos \alpha_\nu}{3 \sin \alpha_\nu} \tag{10}$$

and equality holds only if all line-segments connecting O with the vertices of P are equal. Starting from (7) and using Cauchy's inequality, we obtain

$$\begin{aligned} A^2 &= (\Sigma A_\nu)^2 \\ &\leqq \Sigma A_\nu^2 \frac{2 + \cos \alpha_\nu}{\sin \alpha_\nu}\, \Sigma \frac{\sin \alpha_\nu}{2 + \cos \alpha_\nu} \\ &\leqq 3I\, \Sigma f(\alpha_\nu) \\ &\leqq 3 I n f(2\pi/n) \end{aligned} \tag{11}$$

from which we conclude that

$$I \geqq A^2 \frac{2 + \cos(2\pi/n)}{3n \sin(2\pi/n)} . \tag{12}$$

The work in (11) uses also (10), (4), and (6), cf. (5) and (8). Equality in (12) cannot hold unless it holds three times in (11). Yet equality at the last step there requires

$$\alpha_1 = \alpha_2 = \cdots = \alpha_n = 2\pi/n \ ,$$

and equality at the foregoing step requires that all line-segments from O to the vertices of P be equal. In short, equality in (12) requires that P be a regular polygon of n sides with center at O. Yet for such a polygon equality actually holds, and so the desired theorem is proved[4]).

2.4. We shall discuss various cases that support the conjecture stated in sect. 0.3 by analogy, that is, such cases in which both (I) and (II) possess an analogue.

Here is such a case. Let D be a convex domain, a a point inside D, C the curve surrounding D, and h the distance of the point a from a tangent to C at a point where the line-element is ds. (We may suppose C sufficiently "smooth".) We define

$$\int \frac{ds}{h} = B_a \ ; \tag{13}$$

the integral is extended along C. We let a vary and seek the minimum of B_α; this minimum, which we call B, is an interesting functional depending on the shape of D, but not on its size.

Observing that

$$2A = \int h ds \ , \tag{14}$$

we obtain from (13), (14), and Schwarz's inequality that

$$2AB_\alpha \geqq (\int ds)^2 = L^2$$

and hence, by the definition of B, that

$$2B \geqslant L^2 A^{-1} \tag{15}$$

[6, p. 70—71, 92—93]. Equality in (15) is attained when D is a circle, or a polygon circumscribed about a circle, and in some other cases. We

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[4]) The ingenious idea of de Bruijn, proposed in a conversation, that led the author eventually to this proof is scarcely visible from the final form here presented. The author hopes to discuss this idea, which is of great heuristic interest, at another opportunity.

easily see now, cf. sect. 0.3, that *both* (I) *and* (II) *remain valid if we substitute* B *for* L^2A^{-1}: here is one more case in which the analogue of (I) cannot be divorced from the analogue of (II).

2.5. *Both (I) and (II) remain valid if we substitute* JA^{-4} *for* L^2A^{-1}, *provided that the statement (II) is restricted to convex domains.*

To prove this, we choose the coordinate system as in sect. 1.2, let c be an appropriate positive constant, and set

$$x' = x \ , \quad y' = cy \ . \tag{16}$$

When the point (x, y) describes D, the point (x', y') describes D', a domain affine to D; D and D' have the same center of gravity and through it the same principal axes of inertia. Let A', I', I_1', and I_2' be so related to D' as A, I, I_1, and I_2 are to D, respectively. Obviously,

$$I_1' = cI_1 \ , \quad I_2' = c^3 I_2 \ , \quad A' = cA \ . \tag{17}$$

we determine c so that

$$I_1' = I_2' = I'/2 \tag{18}$$

which is obviously possible; cf. (1) of the first part. By (17), (18), and (3) of the first part, we have

$$\begin{aligned} JA^{-4} &= 4I_1 I_2 A^{-4} \\ &= 4I_1' I_2' A'^{-4} = (I' A'^{-2})^2 \ . \end{aligned} \tag{19}$$

That is, the set of all values of JA^{-4} coincides with the set of those particular values of $(IA^{-2})^2$ that are due to those particular domains for which $I_1 = I_2$, or the principal ellipse of inertia is a circle. Yet the circle and the regular polygons are among these particular domains. Thus, the minimum of IA^{-2} is attained by such a particular domain in both cases with which we are here concerned: first, when all domains are admitted [**3**] and second, when only convex polygons with n sides are admitted, cf. sect. 2.1—2.3. This proves the assertion stated at the beginning of this section.

2.6. We consider ϱ, the radius of the largest circle contained in D, and R, the radius of the smallest circle containing D [**6**, p. 112]. The statement (*I*) remains valid if we substitute $A\varrho^{-2}$ or R^2A^{-1} for L^2A^{-1}. This is perfectly trivial: no domain containing a given circle can have a lesser, and no domain contained in that circle can have a greater, area than the circle itself. Now, also statement (*II*) remains valid if we substitute $A\varrho^{-2}$ or R^2A^{-1} for L^2A^{-1}. This, however, is not so trivial. The following two theorems, although well known and easy to prove, are by no means vacuous: Of all polygons with n sides containing a given circle, the circumscribed regular polygon is of minimum area; and of all polygons with n sides contained in a given circle the inscribed regular polygon is of maximum area.

Also in the less elementary cases with which our conjecture (sect. 0.3) deals, the analogue of (II) seems to be more difficult than the analogue of (I).

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