Motions with Maximal Displacements.

Motions with Maximal Displacements

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To Paul Finsler on his sixtieth birthday

This note deals with the surprisingly strong implications of a nearly trivial remark. We consider an abstract Finsler space, that is, a space in which geodesics with the usual geometric properties, apart from differentiability, exist. The exact requirements are found below. The remark is this: *If for a motion Φ of a Finsler space a point z exists at which the displacement*, or the distance $xx\Phi$ from a point x to its image $x\Phi$ under Φ, *attains a maximum which is not too large, then the shortest geodesic arc from z to $z\Phi^2$ passes through $z\Phi$.*

Among the facts which we deduce from this observation we mention the following: A closed group of motions of a compact space (without any differentiability conditions) is a Lie group. In a compact space without conjugate points and with an abelian fundamental group no geodesic has multiple points, and the closed geodesics in a given free homotopy class have the same length and cover the space simply.

1. The axioms. Proof of the remark.

The space is assumed to be a G-space, see [1] or [2]. The axioms for a G-space R are:

I *R is metric.* The distance of x and y is denoted by xy.

II *R is finitely compact*, i. e. the Theorem of Bolzano-Weierstrass holds.

III *R is convex in Menger's sense*, see [3]. If we introduce the notation (xyz) to indicate that x, y, z are distinct and $xy + yz = xz$, the last condition means: if $x \neq z$ then y with (xyz) exists. $S(p, \varrho)$ will denote the set of points x satisfying $px < \varrho$.

IV *Prolongation is locally possible:* every point p has a neighborhood $S(p, \varrho_p)$, $\varrho_p > 0$, such that for any two distinct points x, y in $S(p, \varrho_p)$ a point z with (xyz) exists.

V *Prolongation is unique:* If (xyz_1), (xyz_2) and $yz_1 = yz_2$ then $z_1 = z_2$.

It follows from I, II, III that any two points y, z can be connected by a *segment* $T(y, z)$, i. e. a curve $x(t)$, $\alpha \leq t \leq \beta = \alpha + yz$ such that $x(\alpha) = y$, $x(\beta) = z$ and $x(t_1)x(t_2) = | t_1 - t_2 |$, see [3] or [7, p. 12]. A *geodesic* is a curve $x(t)$, $-\infty < t < \infty$, with the property that for every real t_0 a positive $\varepsilon(t_0)$ exists such that $x(t_1)x(t_2) = | t_1 - t_2 |$ for $| t_i - t_0 | \leq \varepsilon(t_0)$ $i = 1,2$. Axioms I to IV imply the *existence of geodesics:* a representation $x(t)$, $\alpha \leq t \leq \beta$, $\alpha < \beta$, of a segment can be extended to all real t to represent a geodesic. This extension is *unique* if V holds.

The function ϱ_p in IV may be erratic but it can be replaced by a *continuous* function: if $\varrho(p) = \sup \varrho_p$, where ϱ_p satisfies IV at p, then $S(p, \varrho(p))$ also satisfies IV. If $\varrho(p) = \infty$, then for any two distinct points x, y a point z with (xyz) exists. There fore $\varrho(q) = \infty$ for any other point q and if $x(t)$ represents a geodesic, then

$$x(t_1)x(t_2) = | t_1 - t_2 |$$

for any t_1, t_2. We call a geodesic with this property a *straight line.* Thus for $\varrho(p) = \infty$ all geodesics are straight lines, and the G-space is called *straight.* In the terminology of the calculus of variations the straight spaces are the simply connected spaces without conjugate points.

If the space is not straight, then $0 < \varrho(p) < \infty$ and

$$| \varrho(p) - \varrho(q) | \leq pq \ .$$

For if $\varrho(p) > \varrho(q)$ and $\varrho(p) > pq$ then the triangle inequality yields $S(q, \varrho(p) - pq) \subset S(p, \varrho(p))$ hence $\varrho(p) - \varrho(q) \leq pq$. The number $\varepsilon(t_0)$ occurring in the definition of a geodesic $x(t)$ may be chosen as $\varrho(x(t_0))$.

Axiom V implies that the segment $T(x, y)$ is unique if a point z with (xyz) exists, see [1, p. 216]. In particular $T(x, y)$ is always unique for $x, y \in S(p, \varrho(p))$.

The following is now an exact formulation of the remark mentioned in the introduction:

(1) *If* Φ *is a motion of the G-space R which is not the identity E and if* $zz\Phi = \sup_{x \in R} xx\Phi < \varrho(z)/2$, *then* $(zz\Phi z\Phi^2)$.[1])

Proof. Because $zz\Phi < \varrho(z)/2$ there is a point u such that $(zz\Phi u)$ and $zz\Phi = z\Phi u$, briefly $z\Phi$ is a midpoint of z and u. Then $z\Phi^2$ is a midpoint of $z\Phi$ and $u\Phi$ and the only one, because $\varrho(z\Phi) = \varrho(z)$. The relation

$$zz\Phi \geq uu\Phi \geq z\Phi u\Phi - z\Phi u = zu - z\Phi u = zz\Phi$$

[1]) It is also true, and has many applications(see [2]), that $0 < z z\Phi = \inf_{x \in R} x x\Phi < \varrho(z)/2$ implies $(z z\Phi z\Phi^2)$.

shows that u is a midpoint of $z\Phi$ and $u\Phi$, hence $u = z\Phi^2$, which proves the assertion.

2. Applications to compact spaces.

In the compact case the following additional statements can be made:

(2) For any motion $\Phi \neq E$ of a compact G-space R a point z of maximal displacement α (i. e. $\alpha = zz\Phi = \sup_{x \in R} xx\Phi$) exists. If k is the first integer for which $k\alpha \geq \varrho(z)/2$, then $zz\Phi^k = k\alpha$. If $k > 1$ then a geodesic $x(t)$ exists such that $x(i\alpha) = z\Phi^i$, $i = 0, \pm 1, \pm 2, \ldots$ and $x(t)$ represents a segment for $i\alpha \leq t \leq (i + k)\alpha$.

Proof. The existence of z is obvious and for $k = 1$ there is nothing to prove. If $k > 1$ then $k\alpha < \varrho(z)$ and with $z_i = z\Phi^i$ it follows from (1) that (zz_1z_2), hence $(z_{i-1}z_iz_{i+1})$ for all i. Since $\varrho(z_i) = \varrho(z)$ the segment $T(z_{i-1}, z_{i+1})$ is unique and passes through z_i. The existence of $x(t)$ follows, and $x(t)$ represents a segment for $i\alpha \leq t \leq (i + k)\alpha$ because $k\alpha < \varrho(z_i)$. In particular $x(0)x(k\alpha) = zz\Phi^k = k\alpha$.

We use the standard metric $\delta(\Phi, \Psi) = \sup_{x \in R} x\Phi x\Psi$ for motions Φ, Ψ of a compact space. Sinde $\varrho(x)$ is continuous and positive it has on a compact space R a positive minimum $\varrho(R)$. An immediate consequence of (2) is

(3) *A non-trivial group of motions of a compact G-space R has at least diameter* $\varrho(R)/2$.

„Non-trivial" means that the group contains at least one motion $\Phi \neq E$, and (2) implies that $\delta(E, \Phi^k) \geq \varrho(z)/2 \geq \varrho(R)/2$ for a suitable positive k. Well known theorems on topological groups yield the further result:

(4) **Theorem.** *A closed group of motions of a compact G-space R is a Lie group. If the group Γ of all motions which R possesses is transitive on R, then R is a topological manifold and* $\dim \Gamma \leq \dim R\,(\dim R + 1)/2$.

The first statement follows from [4, Theorem 53] and the second from [5, Corollary 3′, Theorem 9 and Theorem 12]. In spite of the recent result of Gleason it is an open question whether (4) extends to non-compact spaces, since no analogue to (3) is known, even when $\inf_{x \in R} \varrho(x) > 0$.

The rotations about the z-axis of the surface $z = (x^2 + y^2)^{-1/2}$ in E^3, with the length of the shortest connection on the surface as distance, show that a one-parameter group of motions of a non-compact G-space may not have any orbits which are geodesics. (1) and (2) imply the existence of such orbits on compact G-spaces:

(5) **Theorem.** *A one-parameter group of motions of a compact G-space possesses an orbit which is a geodesic.*

We assume that the one-parameter group is given in the form $\Phi(s)$ with $\Phi(s_1)\Phi(s_2) = \Phi(s_1 + s_2)$, and prove that a geodesic $x(t)$ and a positive α exist such that $x(t) = x(0)\Phi(\alpha t)$.

Choose $\varepsilon > 0$ such that $\delta(E, \Phi(s)) < \varrho(R)/2$ for $|s| < \varepsilon$. Let $0 < u < \varepsilon$. By (2) there are points z and z' of maximal displacement under $\Phi(u)$ and $\Phi(u/2)$ respectively. Then the choice of ε and (1) imply

$$z'z'\Phi(u) = 2z'z'\Phi(u/2) \geq 2zz\Phi(u/2)$$
$$= zz\Phi(u/2) + z\Phi(u/2)z\Phi(u) \geq zz\Phi(u) \geq z'z'\Phi(u).$$

Hence z is also a point of maximal displacement for $\Phi(u/2)$ and generally for $\Phi(2^{-n}u)$. Moreover $(zz\Phi(u/2)z\Phi(u))$ and generally

$$(zz\Phi(2^{-n-1}u)z\Phi(2^{-n}u)) \ .$$

If $x(t)$ is the geodesic with $x(0) = z$ which represents for

$$0 \leq t \leq zz\Phi(u) = \beta$$

the (unique) segment $T(z, z\Phi(u))$ then (2) yields

$$x(i\, 2^{-n}\beta) = z\Phi(i\, 2^{-n}u)$$

for all i and non-negative n. A trivial continuity argument shows that $x(\beta t) = z\Phi(ut)$ or $x(t) = x(0)\Phi(\alpha t)$ for all t, where $\alpha = u/\beta$.

3. Compact spaces without conjugate points and abelian fundamental groups.

For a G-space R' which satisfies the usual differentiability hypotheses of the calculus of variations the absence of conjugate points means that the universal covering space R of R' is straight.

The relation of the theorem mentioned in the introduction to motions with maximal displacements comes from:

(6) **Theorem.** *If R is straight and Φ is a motion of R for which a point z with $0 < zz\Phi = \sup_{x \in R} xx\Phi$ exists, then $xx\Phi$ is independent of x. The points $x\Phi^i$, $i = 0, \pm 1, \pm 2, \ldots$ lie for each x on a straight line $\mathfrak{g}_x$.*

For it follows from (1) that the points $z_i = z\Phi^i$ satisfy

$$(z_{i-1}z_iz_{i+1})$$

hence lie on a straight line $\mathfrak{g}_z$. If x is any other point of R and $x_i = x\Phi^i$ then

$$n \cdot zz\Phi = zz_n \leq zx + \sum_{i=1}^{n} x_{i-1}x_i + x_n z_n = 2zx + n \cdot xx\Phi$$

or $xx\Phi \geq zz\Phi - 2zx/n$. Since n is arbitrary $xx\Phi \geq zz\Phi$, hence $xx\Phi = zz\Phi$.

Thus every point x of R is a point of „maximal" displacement for Φ, therefore (1) shows that the points x_i lie on a line $\mathfrak{g}_x$.

Clearly for any two points x, y either $\mathfrak{g}_x = \mathfrak{g}_y$ or $\mathfrak{g}_x \cap \mathfrak{g}_y = 0$, since $u \in \mathfrak{g}_x \cap \mathfrak{g}_y$ implies $u\Phi^i \in \mathfrak{g}_x \cap \mathfrak{g}_y$ hence $\mathfrak{g}_x = \mathfrak{g}_y$.

Let the universal covering space R of the G-space R' be straight. There is a wellknown correspondence between the classes of conjugate elements in the fundamental group $\mathfrak{F}$ of R' and the classes of freely homotopic curves in R', see for instance [8, § 49]. If, as in [1], $\mathfrak{F}$ is realized as the group of motions of R which lie over the identity of R' then the closed geodesics in a free homotopy class K_Φ determined by a motion $\Phi \neq E$ in $\mathfrak{F}$ correspond to the straight lines in R which are taken into themselves by Φ, the so-called axes of Φ, see [2]. If x lies on an axis of Φ then $xx\Phi$ is the length of the corresponding geodesic.

If $\Phi \neq E$ possesses a point of maximal displacement then we conclude from (6) that every point x' of R' lies on a closed geodesic of length $xx\Phi$ in K_Φ and that two such geodesics do not intersect. It is now easy to prove:

(7) **Theorem.** *Let R' be a compact G-space with an abelian fundamental group and a straight universal covering space R. Then the closed geodesics in any (non-trivial) free homotopy class of R' have the same length and cover R' simply. No geodesic in R' has multiple points.*

For let Φ be any motion in the fundamental group $\mathfrak{F}$ of R' different from the identity (such motions exist because R is non-compact, hence different from R'). There is a compact subset C of R such that

$$\bigcup C\Phi_\nu = R ,$$

where Φ_ν traverses $\mathfrak{F}$, see [2, p. 267]. The Function $yy\Phi$ attains on C a maximum at some point $z \in C$. If x is an arbitrary point of R then a $\Phi_\nu \in \mathfrak{F}$ exists such that $y = x\Phi_\nu \in C$. Because $\mathfrak{F}$ is abelian

$$xx\Phi = x\Phi_\nu x\Phi\Phi_\nu = x\Phi_\nu x\Phi_\nu\Phi = yy\Phi \leq zz\Phi ,$$

so that $zz\Phi = \sup_{x \in R} xx\Phi$.

The preceding discussion shows that the closed geodesics in K_Φ all have length $xx\Phi$ and cover R' simply. K_Φ is, owing to the arbitrariness of Φ, an arbitrary non-trivial free homotopy class in R'.

There can be no geodesic monogon with a proper vertex x'. For such a monogon would lie in some free homotopy class K_Φ, not trivial because R is straight. If x lies over x' then the points $x\Phi^i$ would not lie on a straight line. The absence of proper monogons means that the geodesics in R' have no multiple points.

Theorem (7) brings a result of E. Hopf [6] to mind, namely that a two-dimensional torus T' with a Riemannian metric is euclidean, if its universal covering plane T is straight. In that case (7) is therefore trivial. However, when the condition that the metric be Riemannian is omitted, then T' possesses a great number of essentially different metrizations for which T is straight. The geodesics in T need not satisfy Desargues' Theorem, but they always satisfy the parallel axiom.

4. A characterization of Minkowskian geometry.

The translations of the euclidean space are obvious exemples für (6). When there are enough motions satisfying (6) these motions are necessarily ordinary translations:

(8) **Theorem.** *If a straight space possesses a transitive group of motions such that for each motion Φ in Γ a point exists whose displacement unter Φ is maximal, then R is Minkowskian and Γ the group of translations of R.*

We deduce from (6) that $xx\Phi$ is constant for each Φ in Γ. Hence no motion $\Phi \neq E$ in Γ has fixed points and Γ is simply transitive on R, see [7, p. 220]. The motion in Γ that takes a into b may therefore be denoted by $(a \to b)$. Because of (6) the line $\mathfrak{g}(a, b)$ through a and b, $a \neq b$, is an axis of $(a \to b)$. The proof of (8) consists of several steps the first of which is:

(a) R satisfies the parallel axiom, for the terminology see [7].

To see this let $x(t)$ be any geodesic and y a point not on $x(t)$. Since $x(t)$ is an axis of $\Phi = (x(0) \to x(1))$ it suffices to show that $\mathfrak{g}(y, x(t))$ tends for $t \to \infty$ or $t \to -\infty$ to the axis $\mathfrak{g}_y$ of Φ through y. For the statement that the line $\mathfrak{g}_y$ is an axis of the same motion Φ as $\mathfrak{g}_x$, is symmetric and transitive, hence the statement that $\mathfrak{g}_y$ is parallel to $\mathfrak{g}_x$ also has these properties.

Let $y(t)$ represent the axis of Φ through y with $y(0) = y$ and $y\Phi = y(1)$. The limit sphere Λ $(y, \mathfrak{x})$ through y to $\mathfrak{x}$ (see [1, p. 240] or

[7, p. 98]), where $\mathfrak{r}$ is the ray $t \geq 0$ of $x(t)$, intersects $x(t)$ in a point $x(t_0)$ and $x(t_0)\Phi = x(t_0 + 1)$. Moreover $\Lambda(y\Phi, \mathfrak{r}) = \Lambda(y, \mathfrak{r})\Phi = \Lambda(x(t_0 + 1), \mathfrak{r})$. The asymptote $\mathfrak{a}$ to $\mathfrak{r}$ through y intersects $\Lambda(y\Phi, \mathfrak{r})$ in the unique foot f of y on $\Lambda(y\Phi, \mathfrak{r})$. But, see [1, p. 242],

$$1 = x(t_0)x(t_0 + 1) = yf \leq yy\Phi = 1 ,$$

hence $y\Phi$ is also a foot of y on $\Lambda(y\Phi, \mathfrak{r})$, so that $y\Phi = f$ and $\mathfrak{a} = \mathfrak{g}_y$, which proves (a).

We show next

(b) If $y(t)$, $t \geq 0$ represents a ray $\mathfrak{s}$ and $\mathfrak{g}$ is a straight line through $y = y(0)$ not containing $\mathfrak{s}$ then $y(t)\mathfrak{g} \to \infty$ for $t \to \infty$.

For an indirect proof assume the existence of a sequence t_n with $x(t_n)\mathfrak{g} < M$. If f_n is a foot of $x(t_n)$ on $\mathfrak{g}$, then $f_n \neq y$ for large n, and $q_n = x(t_n)(f_n \to y)$ has y as foot on $\mathfrak{g}$. Because $q_n y = x(t_n)f_n < M$ there is a subsequence $\{\nu\}$ of $\{n\}$ for which q_ν tends to a point q. ($qy > 0$ because of [1, Theorem (11.14)]).

The line $\mathfrak{g}(x(t_\nu), q_\nu)$ is an axis of $(f_\nu \to y)$, hence parallel to $\mathfrak{g}$. It tends therefore to the parallel $\mathfrak{g}'$ to $\mathfrak{g}$ through q. On the other hand, the line $\mathfrak{g}(q_\nu, x(t_\nu))$ tends also to the parallel through q to the line $\mathfrak{h}$ carrying $\mathfrak{s}$ (see the definition of co-ray in [1]). Since parallelism is symmetric, it would follow that $\mathfrak{g}$ and $\mathfrak{h}$ are parallel to $\mathfrak{g}'$, which is impossible because $\mathfrak{g}$ and $\mathfrak{h}$ intersect.

(c) $x_1\mathfrak{g}_2$ and $x_2\mathfrak{g}_1$ are bounded for $x_i \in \mathfrak{g}_i$ if and only if $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are parallel.

If $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are parallel then the fact that they are axes of the same motion in Φ shows that $x_1\mathfrak{g}_2$ and $x_2\mathfrak{g}_1$ are bounded. The converse follows from (b), for a proof see [2, p. 278].

(d) Γ is abelian.

If Φ and Ψ are two non-trivial motions in Γ, select an arbitrary point z. If the axes of Φ and Ψ through z coincide, it is easily seen that Φ and Ψ commute (this case can also be deduced by a limit process from the general case). We assume therefore that z, $p = z\Phi$ and $q = z\Psi$ are not collinear. Put $\mathfrak{g}(z, p) = \mathfrak{g}$, $\mathfrak{g}(z, q) = \mathfrak{h}$ and $\mathfrak{h}' = \mathfrak{h}\Phi$. Then $y' = y\Phi \in \mathfrak{h}'$ for $y \in \mathfrak{h}$. The relation $yy' = zp$ shows that $y'\mathfrak{h}$ and $y\mathfrak{h}'$ are bounded, by (c) the lines $\mathfrak{h}$ and $\mathfrak{h}'$ are parallel. Therefore $\mathfrak{h}'$ is an axis of Ψ, so that $p\Psi$ is a point u of $\mathfrak{h}'$ with $zq = pu$. On the other hand $zq = z\Phi q\Phi = pq\Phi$, hence $q\Phi = u$. Therefore $\Phi = (q \to u)$, $\Psi = (p \to u)$ and

$$\Phi\Psi = (z \to p)(p \to u) = (z \to u) = (z \to q)(q \to u) = \Psi\Phi.$$

It now follows readily from a wellknown result of Pontrjagin, see [4, p. 170], that the space is a finite dimensional Minkowski space. A simple proof which does not use the theory of topological groups is found n [7, pp. 229—231].

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