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On the decomposability of mod 2 cohomological invariants of Weyl groups

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Abstract. We compute the invariants of Weyl groups in mod 2 Milnor K-theory and more general cycle modules, which are annihilated by 2. Over a base field of characteristic coprime to the group order, the invariants decompose as direct sums of the coefficient module. All basis elements are induced either by Stiefel–Whitney classes or specific invariants in the Witt ring. The proof is based on Serre's splitting principle that guarantees detection of invariants on elementary abelian 2-subgroups generated by reflections.

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1. Introduction

Let G be a smooth affine algebraic group over a field k_0 of characteristic not 2. Motivated from the concept of characteristic classes in topology, the idea behind *cohomological invariants* as presented by J.-P. Serre in [4] is to provide tools for detecting that two torsors are not isomorphic. Loosely speaking, such an invariant assigns a value in an abelian group to an algebraic object, such as a quadratic form or an étale algebra.

The formal definition of a cohomological invariant is due to J.-P. Serre and appears in his lectures [4], where also a brief account of the history of the subject is given. First, we identify the pointed set of isomorphism classes of G-torsors over a field k with the first non-abelian Galois cohomology $H^1(k, G)$. Further, let M be a functor from the category $\mathcal{F}_{k_0}$ of finitely generated field extensions of k_0, to abelian groups. Then, a *cohomological invariant* of G with values in the coefficient space M is a natural transformation from $H^1(-, G)$ to $M(-)$ considered as functors on $\mathcal{F}_{k_0}$. Interesting examples of the functor M include Witt groups or Milnor

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K-theory modulo 2, which is the same as Galois cohomology with $\mathbb{Z}/2$-coefficients by Voevodsky's proof of the Milnor conjecture.

In general, the cohomological invariants of a given algebraic group with values in some functor M are hard to compute and there are only a few explicit computations carried out yet. One exception are the cohomological invariants of the orthogonal group over a field of characteristic not 2 with values in Milnor K-theory modulo 2. These invariants are generated by Stiefel–Whitney classes

$$w_i \colon H^1(-, O_n) \to K_i^{\mathrm{M}}(-)/2$$

introduced by Delzant [2]. Now, every finite group G embeds in a symmetric group S_n for an appropriate n, and this group in turn embeds in O_n. Pulling back the Stiefel–Whitney classes along such homomorphisms $G \to S_n \to O_n$ is a rich source of cohomological invariants of finite groups considered as group scheme of finite type over a base field k_0.

In this work, we show that most cohomological invariants of a Weyl group G over a field k_0 of characteristic coprime to $|G|$ arise in this way if the coefficient space is a cycle module M_* in the sense of Rost [12], which is annihilated by 2. More precisely, there exists a finite family of invariants $\{a_i\}_{i\in I}$ with values in $K_*^M/2$, such that every invariant a over k_0 with values in M_* decomposes uniquely as

$$a = \sum_{i\in I} a_i m_i,$$

for some constant invariants $m_i \in M_*(k_0)$. In characteristic 0, any Weyl group is a product of the irreducible ones mentioned above. Hence, invoking a product formula of J.-P. Serre yields the decomposition for cohomological invariants.

The proof of this result is constructive, in the sense that we give precise formulas for the generators $\{a_i\}_{i\in I}$. For most Weyl groups the invariants are induced by Stiefel–Whitney classes coming from embeddings of the Weyl group into certain orthogonal groups. Note that these embeddings make use of the fact that such a Weyl group can be realized as orthogonal reflection group over every field of characteristic not 2. However, if the Weyl group has factors of type D_{2n}, E_7, or E_8, then besides Stiefel–Whitney classes also specific Witt-type invariants appear, which induce invariants in mod 2 Milnor K-theory via the Milnor isomorphism. All basis elements are invariants derived from either the Stiefel–Whitney or the Witt-ring invariants.

Crucial for the derivation is Serre's splitting principle for Weyl groups: if two invariants coincide on the elementary abelian 2-subgroups generated by reflections, then these are the same. This allows the following proof strategy. Since Stiefel–Whitney classes and Witt invariants provide us with a family of invariants, we only have to show that a given invariant coincides on the elementary abelian subgroups with a combination from this list. The invariants are then computed case by case for the various types.

J.-P. Serre has recently computed with a different method the invariants of Weyl groups with values in Galois cohomology, see his 2018 Oberwolfach talk [14]. In an e-mail exchange on an earlier version of the present paper, J.-P. Serre explains how to remove many of the restrictions on the characteristic of k_0. An excerpt of his letter is reproduced in Section A. J. Ducoat provided a proof of Serre's splitting principle and attempted to compute the invariants for groups of type B_n and D_n [3]. However, many proofs are incomplete as they are "left to the reader" or "similar to previous ones". Moreover, Theorem 5 on page 4 about the invariants of $W(D_n)$ is not correct as stated, because an invariant in degree $n/2$ is missing. Therefore, we provide detailed computations also for the types B_n and D_n.

The content of this article is as follows. In Section 2, we state the main result and fix notations and conventions. Next, Section 3 contains preliminary results. The proof of the main result occupies the rest of the paper. It also includes an appendix, elucidating how to use a GAP-program to determine the invariants for E_7 and E_8.

Acknowledgements. The present manuscript has a long history. It is a condensed version of my diploma thesis at LMU Munich supervised by F. Morel. I am very grateful for his comments and insights that shaped this work in many ways. The thesis is available online and contains additional background material from algebraic geometry [7] as well as results for reflection groups that are not of Weyl type. Moreover, I thank S. Gille for massive help and discussions on earlier versions of the manuscript. He was also the one to mention the thesis during a presentation of J.-P. Serre at the 2018 Oberwolfach meeting. I am very grateful to J.-P. Serre for a highly insightful e-mail exchange and for sharing with me an early version of his report [14]. His remarks helped to both substantially raise the quality of the presentation, and also improve the contents such as removing restrictions on the characteristic in the present paper. Moreover, an earlier version also contained an irritating assumption that -1 be a square in k_0. Thanks to a more appropriate representation of $W(B_2)$ pointed out by J.-P. Serre, also this assumption could be removed in the present version. Finally, I thank the anonymous referee for the careful reading of the manuscript and valuable observations that helped to improve the presentation.

Part I. Results and methods

2. Main theorem and proof strategy

2.1. Cycle modules. We consider in this work invariants with values in a cycle module M_* in the sense of Rost, which is annihilated by 2. Recall that a cycle module over a field k_0 is a covariant functor

$$k \longmapsto M_*(k) := \bigoplus_{n \in \mathbb{Z}} M_n(k)$$

on the category $\mathcal{F}_{k_0}$ with values in graded Milnor K-theory modules. For a field extension $\iota: k \subseteq L$, the image of $z \in M_*(k)$ in $M_*(L)$ is denoted by $\iota_*(z)$. By definition, cycle modules have further structure and we refer the reader to [12] for details.

The main example of a cycle module is Milnor K-theory:

$$\begin{aligned} \mathcal{F}_{k_0} &\to \mathbb{Z}\text{-graded rings} \\ k &\mapsto K_*^{\mathsf{M}}(k) = \oplus_{n \geq 0} K_n^{\mathsf{M}}(k). \end{aligned}$$

For $a_1, \dots, a_n \in k^\times$, we denote pure symbols in $K_n^{\mathsf{M}}(k)$ by $\{a_1, \dots, a_n\}$. The graded abelian group $M_*(k)$ has the structure of a graded $K_*^{\mathsf{M}}(k)$-module for every field $k \in \mathcal{F}_{k_0}$. Hence, if M_* is annihilated by 2, it becomes a $K_*^{\mathsf{M}}(k)/2$-module. For ease of notation, we set $\mathrm{k}_*^{\mathsf{M}}(k) := K_*^{\mathsf{M}}(k)/2$ and denote the image of a symbol $\{a_1, \dots, a_n\} \in K_n^{\mathsf{M}}(k)$ in $\mathrm{k}_n^{\mathsf{M}}(k)$ by $\{a_1, \dots, a_n\}$. We say that M_* has a $\mathrm{k}_*^{\mathsf{M}}$*-structure* if M_* is annihilated by 2.

From now on cycle module means cycle module with $\mathrm{k}_*^{\mathsf{M}}$*-structure.*

2.2. Invariants with values in cycle modules. Let G and M_* be a linear algebraic group and a cycle module over k_0, respectively. Recall from Section 1 that a *cohomological invariant* of G with values in M_n is a natural transformation from $H^1(-, G)$ to $M_n(-)$. We denote the set of all invariants of degree n of G with values in M_* by $\mathsf{Inv}^n(G, M_*)$, and set

$$\mathsf{Inv}(G, M_*) := \mathsf{Inv}_{k_0}(G, M_*) := \bigoplus_{n \in \mathbb{Z}} \mathsf{Inv}^n(G, M_*).$$

For $k \in \mathcal{F}_{k_0}$, any invariant $a \in \mathsf{Inv}_{k_0}(G, M_*)$ restricts to a natural transformation of functors $H^1(-, G) \to M_*(-)$ on the full sub-category $\mathcal{F}_k$ of $\mathcal{F}_{k_0}$. We denote this restricted invariant by $\mathsf{res}_{k/k_0}(a)$ or by the same symbol a if the meaning is clear from the context. A particular example of invariants are the *constant invariants*, which are in one-to-one correspondence with elements of $M_*(k_0)$: The constant invariant $c \in M_*(k_0)$ maps every $x \in H^1(k, G)$ onto the image of c in $M_*(k)$ for all $k \in \mathcal{F}_{k_0}$. The set $\mathsf{Inv}(G, M_*)$ is a $\mathrm{k}_*^{\mathsf{M}}(k_0)$-module, so that if $a: H^1(-, G) \to \mathrm{k}_*^{\mathsf{M}}(-)$ is a Milnor K-theory invariant of degree m and $x \in M_n(k_0)$, then

$$a \cdot x: H^1(k, G) \to M_{m+n}(k), \quad T \mapsto a_k(T) x_k$$

is an invariant with values in M_* of degree $m + n$. We now define precisely what it means that an invariant can be represented uniquely as a sum of basis elements.

Definition 2.1. Let M_* be a cycle module over the field k_0, and G a linear algebraic group over k_0.

(i) A subgroup $S \subseteq \mathsf{Inv}^*_{k_0}(G, M_*)$ is a *free $M_*(k_0)$-module* with *basis*

$$a^{(i)} \in \mathsf{Inv}^{d_i}_{k_0}(G, \mathrm{k}^{\mathrm{M}}_*), \quad i \in I,$$

if

$$\bigoplus_{i \in I} M_{*-d_i}(k_0) \to S, \quad \{m_i\}_{i \in I} \mapsto \sum_{i \leqslant r} a^{(i)} \cdot m_i$$

is an isomorphism of abelian groups.

(ii) $\mathsf{Inv}(G, M_*)$ is *completely decomposable* with a finite basis

$$a_i \in \mathsf{Inv}^{d_i}_{k_0}(G, \mathrm{k}^{\mathrm{M}}_*)$$

if $\mathsf{Inv}^*_k(G, M_*)$ is a free $M_*(k)$-module with the corresponding basis

$$\mathsf{res}_{k/k_0}(a_i) \in \mathsf{Inv}^{d_i}_k(G, \mathrm{k}^{\mathrm{M}}_*), \quad i \in I,$$

for all $k \in \mathcal{F}_{k_0}$.

After these preparations, we now state the main result.

Theorem 2.2. *Let G be an irreducible Weyl group. Let k_0 be a field of characteristic coprime to $|G|$ and M_* a cycle module over k_0. Then, $\mathsf{Inv}^*_{k_0}(G, M_*)$ is completely decomposable.*

The proof of Theorem 2.2 is constructive and we describe the generators explicitly. These depend on the type of the Weyl group and will be given in the course of the computation later on. Now, we explain the strategy starting with a reminder on Weyl groups.

Let $\mathbb{E}$ be a finite-dimensional real vector space with scalar product $(-,-)$ and orthogonal group $O(\mathbb{E})$. Then, $s_v: \mathbb{E} \to \mathbb{E}$,

$$s_v(w) := w - \frac{2(v, w)}{(v, v)} v,$$

defines the reflection at a vector $v \in \mathbb{E}$ with $(v, v) \neq 0$.

Now, the *Weyl group* $W(\Sigma)$ associated with a crystallographic root system $\Sigma \subseteq \mathbb{E}$ is the subgroup of $O(\mathbb{E})$ generated by all reflections s_α at the roots $\alpha \in \Sigma$. By definition of a root system, the scalars $2(\alpha, \beta)/(\alpha, \alpha)$ are integers for all $\alpha, \beta \in \Sigma$ and the reflections act on the root system. The Weyl group is *irreducible* if the corresponding root system is irreducible. The irreducible root systems are classified by types $A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2$. Let Σ be such an irreducible root system. Then, there exists an Euclidean space $\mathbb{E} = \mathbb{R}^n$ for an appropriate n, such that: (i) $\Sigma \subseteq V := \bigoplus_{i\leqslant n} \mathbb{Z}[1/2]e_i$, where $e_1,\dots,e_n$ is the standard basis of $\mathbb{R}^n$, and (ii) $W(\Sigma)$ maps V into itself. This can be deduced using the realizations of these root systems in Bourbaki [1, PLATES I-VIII]. If now k_0 is a field of characteristic not 2 then $W(\Sigma)$ acts via scalar extension on $V_{k_0} := k_0 \otimes_{\mathbb{Z}[1/2]} V$ and can so be realized as orthogonal reflection group over k_0 considering V_{k_0} has regular bilinear space with the scalar product induced by the restriction of the standard scalar product of $\mathbb{E} = \mathbb{R}^n$ to V.

The strategy of proof for an irreducible Weyl group G, is as follows. We leverage different embeddings of the Weyl group G into an orthogonal group O_n over the field k_0. Now, the invariants of O_n with values in $\mathrm{k}_*^{\mathrm{M}}$ are generated by the Stiefel–Whitney classes, see [4]. Considering embeddings $W \hookrightarrow O_n$ gives rise to a family of invariants in $\mathsf{Inv}(G, \mathrm{k}_*^{\mathrm{M}})$ by composing the Stiefel–Whitney classes with the natural transformation

$$H^1(-, W) \to H^1(-, O_n).$$

As we shall see in Sections 5–8, these already generate $\mathsf{Inv}(G, M_*)$ except if G is of type D_{2n}, E_7, or E_8. The 'missing' invariants have their source in certain Witt invariants.

Having a family of invariants with values in $\mathrm{k}_*^{\mathrm{M}}$ at our disposal, we deduce Theorem 2.2 for an irreducible Weyl group G by showing that this set of invariants contains a basis of $\mathsf{Inv}(G, M_*)$ in the sense of Definition 2.1. The main tool is the following adaptation of Serre's splitting principle, which is proven in [6, Corollary 4.10]. Loosely speaking, if k_0 is a field of characteristic coprime to $|G|$, then $\mathsf{Inv}(G, M_*)$ is detected by the maximal elementary abelian 2-subgroups of G generated by reflections. We let $\Omega(G)$ denote the set of conjugacy classes of maximal elementary 2-abelian subgroups of G, which are generated by reflections.

Note that the proof of Theorem 2.2 for Weyl groups of type G_2 in Section 3.3 is purely group theoretic, in the sense that it uses only its semi-direct decomposition and not the geometry of the corresponding root system.

Proposition 2.3 (Serre's splitting principle). *Let M_* be a cycle module over k_0 and G be a Weyl group. Let k_0 be a field of characteristic coprime to $|G|$. Then, the canonical map*

$$\big(\mathsf{res}_G^P\big)_{[P]} \colon \mathsf{Inv}(G, M_*) \to \prod_{[P]\in\Omega(G)} \mathsf{Inv}(P, M_*)^{N_G(P)}$$

is injective, where $N_G(P)$ is the normalizer of the maximal elementary 2-abelian subgroup P of G, which is generated by reflections.

We point out that the assumption that order of the irreducible Weyl group G and the characteristic of k_0 are coprime seems to be not necessary, see Section A. This assumption comes from the article [6], where the splitting principle is proven for more general orthogonal reflection groups. This would also remove that assumption from Theorem 2.2.

Remark 2.4. For groups of type A_n, D_n, E_6, E_7, or E_8, any two roots are conjugate [8, Rem. 4, Sect. 2.9]. Hence, an induction argument shows that for these types, there is up to conjugacy only one maximal abelian 2-subgroup P generated by reflections. In particular, by Proposition 2.3, the restriction map res_G^P is injective for simply-laced groups.

The computation of the invariants of an arbitrary Weyl group follows from Theorem 2.2 by a product formula of Serre. To state the product formula precisely, we first introduce the notion of a product of invariants. Identifying $H^1(k, G' \times G)$ with $H^1(k, G') \times H^1(k, G)$, for invariants $a \in \mathsf{Inv}_{k_0}(G, \mathrm{k}_*^{\mathrm{M}})$ and $b \in \mathsf{Inv}_{k_0}(G', M_*)$, we define the product ab through

$$\begin{aligned}(ab)_k \colon H^1(k, G \times G') &\to M_*(k)\\ (T, T') &\mapsto a_k(T) b_k(T').\end{aligned}$$

Proposition 2.5 (Product formula). *Let M_* be a cycle module and G, G' algebraic groups over k_0. If $\mathsf{Inv}_{k_0}^*(G, M_*)$ is completely decomposable with finite basis $\{a_i\}_{i \in I}$, then the map*

$$\begin{aligned}\bigoplus_{i \in I} \mathsf{Inv}_k^*(G', M_*) &\to \mathsf{Inv}_k^*(G \times G', M_*)\\ \{b_i\}_{i \in I} &\mapsto \sum_{i \in I} \mathsf{res}_{k/k_0}(a_i) b_i\end{aligned}$$

is an isomorphism for all $k \in \mathcal{F}_{k_0}$. In particular, if the invariants of both G and G′ are completely decomposable, then so is $\mathsf{Inv}_{k_0}^*(G \times G', M_*)$.

Proof. We follow the outline given in [4, Part I, Exercise 16.5]. Replacing a_i by $\mathsf{res}_{k/k_0}(a_i)$ we can assume $k = k_0$.

To show surjectivity, let $a \in \mathsf{Inv}_{k_0}^*(G \times G', M_*)$. Then, for every $k \in \mathcal{F}_{k_0}$ and $T' \in H^1(k, G')$ we define an invariant $\overline{a} \in \mathsf{Inv}_k^*(G, M_*)$ by mapping $T \in H^1(\ell, G)$ to $\overline{a}_\ell(T) = a_\ell(T \times T'_\ell)$, where, T'_ℓ denotes the image of T' in $H^1(\ell, G')$ under the base change map. Since $\mathsf{Inv}(G, M_*)$ is completely decomposable, $\overline{a}$ can be uniquely expressed as

$$\sum_i \mathsf{res}_{k/k_0}(a_i) b_i(T')$$

for suitable $b_i(T') \in M_*(k)$. It remains to prove that $b_i \in \mathsf{Inv}(G', M_*)$ for all i. To achieve this goal, let $\iota \colon k \subseteq k_1$ be a field extension in $\mathcal{F}_{k_0}$ and $T' \in H^1(k, G')$. Then,

$$\iota_*\Big(\sum_{i \in I} \mathsf{res}_{k/k_0}(a_i)(T) b_i(T')\Big) = \sum_{i \in I} \mathsf{res}_{k_1/k_0}(a_i)(T_{k_1}) b_i(T'_{k_1}).$$

Since a_i's are invariants

$$\sum_{i\in I}\mathsf{res}_{k_1/k_0}(a_i)\iota_*(b_i(T')) = \sum_{i\in I}\mathsf{res}_{k_1/k_0}(a_i)b_i(T'_{k_1}).$$

As the a_i's are a basis we get $b_i(T'_{k_1}) = \iota_*(b_i(T'))$, as asserted.

To show injectivity, we assume $\sum_{i\in I} a_i b_i = 0$ and claim that $b_i = 0$ for all $i \in I$. Fix a field k and $T' \in H^1(k, G')$. Then

$$\sum_{i\in I} a_i b_i(T') \in \mathsf{Inv}^*_k(G, M_*)$$

is the constant zero invariant. Since the a_i's are a basis, we get $b_i(T') = 0$ for all $i \in I$. Since k and T' were arbitrary, this implies that the b_i's are constant zero. □

Since every Weyl group is a product of irreducible ones, we get the following corollary.

Corollary 2.6. *Let k_0 be a field of characteristic coprime to $|G|$ and M_* a cycle module over k_0. Then, $\mathsf{Inv}^*_{k_0}(G, M_*)$ is completely decomposable for all Weyl groups G.*

3. Preparations for the proof

In this section, we establish several key lemmas on cycle modules. We also discuss auxiliary results used in the type-by-type proof of Theorem 2.2 for irreducible Weyl groups.

3.1. Cycle complex computations. We start with a computation of cycle module cohomology which seems to be well known, but for which we have not found an appropriate reference. To this end, we recall first the cycle complex associated with a cycle module M_* over k_0. We refer the reader to Rost [12] for further details.

Let X be a scheme essentially of finite type over k_0. That is, X is of finite type over k_0 or the localization of such a k_0-scheme. Then, the *cycle complex* is given by

$$\bigoplus_{x\in X^{(0)}} M_n(k_0(x)) \xrightarrow{d^0_{X,n}} \bigoplus_{x\in X^{(1)}} M_{n-1}(k_0(x)) \xrightarrow{d^1_{X,n}} \bigoplus_{x\in X^{(2)}} M_{n-2}(k_0(x)) \to \cdots,$$

where $X^{(p)} \subseteq X$ denotes the set of points of codimension $p \geqslant 0$ in X and $k_0(x)$ is the residue field of $x \in X$. In general, the differentials $d^p_{X,n}$ are sums of composition of second residue maps and transfer maps. If X is an integral scheme with function field $k_0(X)$ and regular in codimension 1, then the components of $d^0_{X,n}$ are the *second* *residue maps* $\partial_x\colon M_n(k_0(X)) \to M_{n-1}(k_0(x))$. In particular, the cohomology group in dimension 0, also called *unramified cohomology* of X with values in M_n, equals

$$M_{n,\mathsf{unr}}(X) := \operatorname{Ker}\big(M_n(k_0(X)) \xrightarrow{(\partial_x)_{x\in X^{(1)}}} \bigoplus_{x\in X^{(1)}} M_{n-1}(k_0(x))\big).$$

In case $X = \mathsf{Spec}(R)$, we use affine notations and write $M_{n,\mathsf{unr}}(R)$ instead of $M_{n,\mathsf{unr}}(X)$.

Lemma 3.1. *Let M_* be a cycle module over k_0 and R a regular and integral k_0-algebra with fraction field K, which is essentially of finite type. Let $a_1,\dots,a_l \in R$ be such that $a_i - a_j \in R^\times$ for all $i \neq j$. Then,*

$$M_{n,\mathsf{unr}}\big(R[T]_{\prod_{i\leqslant l}(T-a_i)}\big) \simeq M_{n,\mathsf{unr}}(R) \oplus \bigoplus_{i\leqslant l}\{T-a_i\}\cdot M_{n-1,\mathsf{unr}}(R),$$

where we consider $\{T-a_i\}$ as an element of $K_1^{\mathsf{M}}(K(T))$ and $M_{n-1,\mathsf{unr}}(R)$ as a subset of $M_{n-1}(K(T))$.

Proof. Setting $f(T) := \prod_{i\leqslant l}(T-a_i)$, we consider the following short exact sequence of cycle complexes, where for a cohomological complex $P^\bullet$ we denote by $P^\bullet[1]$ the shifted complex with P^i in degree $i+1$:

$$\mathrm{C}^\bullet\big(R[T]/R[T]\cdot f(T), M_{n-1}\big)[1] \rightarrowtail \mathrm{C}^\bullet\big(R[T], M_n\big) \twoheadrightarrow \mathrm{C}^\bullet\big(R[T]_{f(T)}, M_n\big).$$

Using homotopy invariance, the associated long exact cohomology sequence starts with

$$0 \to M_{n,\mathsf{unr}}(R) \to M_{n,\mathsf{unr}}\big(R[T]_{f(T)}\big) \to M_{n-1,\mathsf{unr}}\big(R[T]/R[T]\cdot f(T)\big).$$

We claim that the map on the right-hand side of this exact sequence is a split surjection. Indeed, by the Chinese remainder theorem,

$$R[T]/R[T]\cdot f(T) \simeq \prod_{i\leqslant l} R[T]/R[T]\cdot(T-a_i) \simeq \prod_{i\leqslant l} R,$$

so that

$$M_{n-1,\mathsf{unr}}\big(R[T]/R[T]\cdot f(T)\big) \simeq M_{n-1,\mathsf{unr}}(R)^{\oplus l}.$$

Disentangling the definitions of the appearing maps shows that

$$M_{n-1,\mathsf{unr}}(R)^{\oplus l} \to M_{n,\mathsf{unr}}\big(R[T]_{f(T)}\big), \quad (x_1,\dots,x_l) \longmapsto \sum_{i\leqslant l}\{T-a_i\}x_i$$

defines the asserted splitting. □

By induction and homotopy invariance, Lemma 3.1 implies the well-known computation of the unramified cohomology of a Laurent ring.

Corollary 3.2. *Let M_* be a cycle module over k_0. Then,*

$$M_{n,\mathsf{unr}}\big(k_0[T_1^{\pm},\dots,T_l^{\pm}]\big) \simeq \bigoplus_{\substack{r\leqslant l\\ 1\leqslant i_1<\cdots<i_r\leqslant l}} \{T_{i_1},\dots,T_{i_r}\}\cdot M_{n-r}(k_0).$$

3.2. Invariants of $(\mathbb{Z}/2)^n$. Corollary 3.2 implies that the invariants of $(\mathbb{Z}/2)^n$ with values in a cycle module are completely decomposable. This is shown for invariants of $(\mathbb{Z}/2)^n$ with values in $\mathrm{k}_*^{\mathrm{M}}$ in Serre's lectures [4, Part I, Sect. 16]. Writing $(\alpha) \in H^1(k, \mathbb{Z}/2)$ for the class of $\alpha \in k^\times$, every index set $1 \leqslant i_1 < \cdots < i_l \leqslant n$ gives rise to an invariant

$$x_{i_1,\dots,i_l}\colon H^1(k,(\mathbb{Z}/2)^n) \simeq H^1(k,\mathbb{Z}/2)^n \to \mathrm{k}_l^{\mathrm{M}}(k)$$
$$\big[(\alpha_1),\dots,(\alpha_n)\big] \mapsto \{\alpha_{i_1},\dots,\alpha_{i_l}\}.$$

We show that they form a basis of $\mathsf{Inv}((\mathbb{Z}/2)^n, M_*)$ for every cycle module M_* with $\mathrm{k}_*^{\mathrm{M}}$-structure.

Let $k \in \mathcal{F}_{k_0}$, $a \in \mathsf{Inv}_k^*((\mathbb{Z}/2)^n, M_*)$ and write $K := k(t_1,\dots,t_n)$ for the rational function field in n variables over the field k. Then,

$$T\colon k\big(\sqrt{t_1},\dots,\sqrt{t_n}\big) \supseteq k(t_1,\dots,t_n)$$

is a versal $(\mathbb{Z}/2)^n$-torsor, so that by [4, Part I, Thm. 11.1] or [6, Thm. 3.5],

$$a_K(T) \in M_{*,\mathrm{unr}}\big(k[t_1^{\pm},\dots,t_n^{\pm}]\big).$$

By Corollary 3.2, there exist unique $m_{i_1,\dots,i_l} \in M_*(k)$ with

$$a_K(T) = \sum_{\substack{l \leqslant n\\ 1\leqslant i_1<\cdots<i_l\leqslant n}} \{t_{i_1},\dots,t_{i_l}\} m_{i_1,\dots,i_l}.$$

Then, the invariant

$$b := \sum_{\substack{l \leqslant n\\ 1\leqslant i_1<\cdots<i_l\leqslant n}} x_{i_1,\dots,i_l} m_{i_1,\dots,i_l}$$

agrees with a on the versal torsor T. Hence, the detection principle in the form of [4, Part I, 12.2] or [6, Thm. 3.7] implies that $a = b$, as asserted.

3.3. Invariants of Weyl groups of type G_2. Assume here that the base field is of characteristic not 2 or 3.

The group $W(G_2)$ is a semi-direct product of a normal subgroup L of order 3 and a subgroup $P \simeq (\mathbb{Z}/2)^2$ generated by the reflections at two orthogonal roots, see [1, Chap. VI, §4, No. 13]. Since there is up to conjugacy only one such P, Proposition 2.3 shows that the restriction map $\mathsf{res}_{W(G_2)}^P$ is injective. Since the projection $W(G_2) \simeq P \ltimes L \to P$ induces a splitting, we deduce that $\mathsf{res}_{W(G_2)}^P$ is in fact an isomorphism.

In view of the results for other Weyl groups it is worthwhile to note that a basis for the invariants can also be expressed in terms of the Stiefel–Whitney invariants to be introduced in Section 3.6 below. As in Section 5.1 below, we see that the restriction of the Stiefel–Whitney classes in degrees 1 and 2 to P correspond to the invariants $x_1 + x_2$ and $x_{1,2}$. Finally, considering the morphism $W(G_2) \to O_1 = \{\pm 1\}$ sending one of the two classes of reflections to -1 and the other to 1 yields the invariant x_1 (or x_2).

3.4. Torsor computations. Henceforth, we switch freely between the interpretation of $H^1(k, O_n)$ via cocycles on the one hand and via quadratic forms on the other hand. For this purpose, we recall how to view $H^1(k, O_n)$ in terms of non-abelian Galois cohomology [13]. Let $c \in Z^1(\Gamma, O_n)$ be a cocycle. That is, c is a continuous map from the absolute Galois group Γ of a separable closure k_s/k to $O_n(k_s)$ and satisfies the cocycle condition $c_{\sigma\tau} = c_\sigma \cdot \sigma(c_\tau)$. To construct a quadratic form q_c over k, we first define an action $\star$ of Γ on k_s^n via $\sigma \star v = c_\sigma(\sigma(v))$. Then, we let $v_1, \dots, v_n \in k_s^n$ denote a k basis of the vector space

$$V^{\star\Gamma} = \{v \in k_s^n : \sigma \star v = v \text{ for all } \sigma \in \Gamma\}. \tag{3.1}$$

Now, we let q_c be the quadratic form whose associated bilinear form b_{q_c} is determined by $b_{q_c}(e_i, e_j) = \langle v_i, v_j \rangle$, where $\langle \cdot, \cdot \rangle$ denotes the standard scalar product in k_s^n. In other words, q_c is the restriction to $V^{\star\Gamma}$ of the quadratic form associated with the standard scalar product $\langle \cdot, \cdot \rangle$. We will come back frequently to the following three pivotal examples, where $V = k_s^2$.

Example 3.3. Consider the group homomorphism $(\mathbb{Z}/2)^2 \to O_2$,

$$e_1 \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_2 \mapsto \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

Let $(\alpha, \beta) \in (k^\times/k^{\times 2})^2$ be a $(\mathbb{Z}/2)^2$-torsor over k. Then, $v_1 = (\sqrt{\alpha}, -\sqrt{\alpha})^\top$, $v_2 = (\sqrt{\beta}, \sqrt{\beta})^\top$ defines a basis of $V^{\star\Gamma}$ and the induced bilinear form is the diagonal form $q_{(\alpha,\beta)} = \langle 2\alpha, 2\beta \rangle$.

Example 3.4. Consider the group homomorphism $\mathbb{Z}/2 \to O_2$,

$$e_1 \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Let $\alpha \in k^\times/k^{\times 2}$ be a $\mathbb{Z}/2$-torsor. Applying the above example with $\beta = 1$, we see that the induced bilinear form is the diagonal form $q_{(\alpha)} = \langle 2\alpha, 2 \rangle$.

Example 3.5. Consider the group homomorphism $(\mathbb{Z}/2)^2 \to O_2$,

$$e_1 \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_2 \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Let $(\alpha, \beta) \in (k^\times/k^{\times 2})^2$ be a $(\mathbb{Z}/2)^2$-torsor over k. Then, $v_1 = (1, 1)^\top$, $v_2 = (\sqrt{\alpha\beta}, -\sqrt{\alpha\beta})^\top$ defines a basis of $V^{\star\Gamma}$. The induced bilinear form is the diagonal form $q_{(\alpha,\beta)} = \langle 2, 2\alpha\beta \rangle$.

3.5. An embedding of S_{2^n} into O_{2^n}. Next, we describe a specific embedding

$$(\mathbb{Z}/2)^n \to O_{2^n}$$

on the torsor level. For any $l \leqslant 2^n - 1$ let $b(l) \subseteq [0, n-1]$ be the position of the bits in the binary representation. That is,

$$l = \sum_{i \in b(l)} 2^i.$$

Furthermore, let f_S be the flipping the bits at all positions in $S \subseteq [0, n-1]$. In other words, $f_S \colon [0, 2^n - 1] \to [0, 2^n - 1]$,

$$f_S(l) := b^{-1}(b(l) \Delta S),$$

where $R \Delta S = (R \setminus S) \cup (S \setminus R)$ is the symmetric difference. In this notation, the group homomorphism $\phi\colon (\mathbb{Z}/2)^n \to S_{2^n} \subseteq O_{2^n}$

$$\phi\Big(\sum_{s \in S} e_s\Big) := f_S$$

induces a map $\phi_*\colon H^1(k, (\mathbb{Z}/2)^n) \to H^1(k, O_{2^n})$, which we now describe explicitly.

Lemma 3.6. *Let $\epsilon_0, \dots, \epsilon_{n-1} \in k^\times/k^{\times 2}$. Then,*

$$\phi_*(\epsilon_0, \dots, \epsilon_{n-1}) = \langle 2^n \rangle \otimes \langle\!\langle -\epsilon_0 \rangle\!\rangle \otimes \langle\!\langle -\epsilon_1 \rangle\!\rangle \otimes \cdots \otimes \langle\!\langle -\epsilon_{n-1} \rangle\!\rangle.$$

Since any two simply transitive actions on $[0, 2^n - 1]$ are conjugate in S_{2^n}, Lemma 3.6 is more useful than it may seem at first.

Proof. Consider a cocycle representation

$$c \in Z^1\big(\Gamma, (\mathbb{Z}/2)^n\big)$$

of the torsor $(\epsilon_0, \dots, \epsilon_{n-1}) \in (k^\times/k^{\times 2})^n$. That is, the ith component of c_σ equals 1 if and only if $\sigma(\sqrt{\epsilon_i}) = -\sqrt{\epsilon_i}$. To determine the quadratic form defined by the induced cocycle $\sigma \mapsto \phi(c_\sigma)$, we assert that a basis of the k-vector space $V^{\star\Gamma}$ from (3.1) is given by $\{v_0, \dots, v_{2^n - 1}\}$, where v_p has components

$$(v_p)_\ell = (-1)^{|b(p) \cap b(\ell)|} \prod_{i \in b(p)} \sqrt{\epsilon_i}.$$

First, $v_p \in V^{\star\Gamma}$, since writing $c_\sigma = \sum_{i \in S} e_i$ for some $S = S(\sigma) \subseteq [0, n-1]$ shows that

$$\sigma\Big((-1)^{|b(p) \cap b(\ell)|} \prod_{i \in b(p)} \sqrt{\epsilon_i}\Big) = (-1)^{|b(p) \cap b(\ell)| + |b(p) \cap S|} \prod_{i \in b(p)} \sqrt{\epsilon_i} = (v_p)_{f_S(\ell)}.$$

Moreover, to prove the linear independence of the $\{v_p\}_p$, we note that

$$b(v_p, v_p) = \sum_{u \leqslant 2^n-1} (v_p)_u (v_p)_u = 2^n \prod_{i \in b(p)} \epsilon_i.$$

Hence, it suffices to show that $b(v_p, v_q) = 0$, if $p \neq q$. By assumption, there is at least one $i \in b(p) \Delta b(q)$, so that pairing any $L \subseteq [0, n-1] \setminus \{i\}$ with $L \cup \{i\}$ shows that

$$\begin{aligned} b(v_p, v_q) &= \prod_{i \in b(p)} \sqrt{\epsilon_i} \cdot \prod_{i \in b(q)} \sqrt{\epsilon_i} \cdot \sum_{L \subseteq [0,n-1]} (-1)^{|b(p) \cap L| + |b(q) \cap L|} \\ &= \prod_{\substack{i \in b(p) \\ j \in b(q)}} \sqrt{\epsilon_i \epsilon_j} \sum_{L \subseteq [0,n-1] \setminus \{i\}} \big((-1)^{|b(p) \cap L| + |b(q) \cap L|} + (-1)^{|b(p) \cap L| + |b(q) \cap L| + 1} \big), \end{aligned}$$

vanishes as claimed. □

3.6. Stiefel–Whitney invariants. The *total Stiefel–Whitney class* is defined by

$$\begin{aligned} w_*\colon H^1(k, O_n) &\to \mathrm{k}_*^{\mathrm{M}}(k) \\ \langle \alpha_1, \ldots, \alpha_n \rangle &\mapsto \prod_{i \leqslant n} \big(1 + \{\alpha_i\}\big), \end{aligned}$$

where $\langle \alpha_1, \ldots, \alpha_n \rangle$ is the class in $H^1(k, O_n)$ of the diagonal form. They generate the invariants of the orthogonal group O_n with values in $\mathrm{k}_*^{\mathrm{M}}$ as Serre shows in [4, Part I, Sect. 17].

Theorem 3.7. *Let k_0 be a field of characteristic not 2. Then, the Stiefel–Whitney invariants form a basis in the sense of Definition 2.1 of* $\mathrm{Inv}(O_n, \mathrm{k}_*^{\mathrm{M}})$ *for all $n \geqslant 1$.*

By [4, Rem. 17.4] the product of Stiefel–Whitney classes is given by

$$w_r w_s = \{-1\}^{b^{-1}(b(r) \cap b(s))} w_{r+s-b^{-1}(b(r) \cap b(s))}, \tag{3.2}$$

where $b(\cdot)$ denote the binary representation of Section 3.5.

Example 3.8. Later, we will meet some examples where it is easier to do the computations with a slight variant of the Stiefel–Whitney classes. Therefore, we introduce *modified* Stiefel–Whitney classes $\widetilde{w_d} \in \mathrm{Inv}^d(O_n, \mathrm{k}_*^{\mathrm{M}})$: For even n, we put

$$\widetilde{w_d}(q) := w_d\big(\langle 2 \rangle \otimes q\big)$$

for all $d \leqslant n$ and for odd n, we set inductively $\widetilde{w_0} = 1$ and

$$\widetilde{w_{d+1}}(q) = w_{d+1}\big(\langle 2 \rangle \otimes q\big) - \{2\}\widetilde{w_d}(q).$$

Then, we obtain for even $\mathsf{rank}(q)$ that

$$\widetilde{w_d}\big(\langle 2\rangle \otimes q\big) = w_d(q) = \widetilde{w_d}\big(\langle 1\rangle + \langle 2\rangle \otimes q\big).$$

Alternatively, one could also give a more direct definition of modified Stiefel–Whitney classes not depending on the parity of q by setting $\widetilde{w_d}(q)$ as $w_d(q)$ if d is odd and as $w_d(q) + \{2\}w_{d-1}(q)$ if d is even.

Finally, we recall another kind of invariants.

Example 3.9 (Witt-ring invariants). The image of an n-dimensional quadratic form in the Witt ring G yields an invariant $\mathsf{Inv}^*(O_n, W)$. Since the definition of invariants only makes use of the functor property, this concept makes sense, even though G is not a cycle module. Albeit of limited use in the setting of quadratic forms, the aforementioned invariant becomes a refreshing source of invariants for groups G embedding into O_n. Indeed, for Weyl groups G of type D_{2n}, E_7, E_8, we construct embeddings such that the restrictions become invariants with values in a suitable power of the fundamental ideal $I \subseteq W$. Since the Milnor morphism

$$\begin{aligned} f_n^{\mathsf{Mil}}\colon \mathrm{k}_n^{\mathsf{M}} &\to I^n/I^{n+1}\\ \{\alpha_1\}\cdots\{\alpha_n\} &\mapsto \langle\!\langle \alpha_1\rangle\!\rangle \otimes \cdots \otimes \langle\!\langle \alpha_n\rangle\!\rangle \end{aligned}$$

with $\langle\!\langle a\rangle\!\rangle := \langle 1, -a\rangle$ induces an isomorphism between mod 2 Milnor K-theory and the graded Witt ring [11, Theorem 4.1], we obtain elements in $\mathsf{Inv}^*(G, \mathrm{k}_*^{\mathsf{M}})$.

3.7. A technical lemma. The following technical lemma simplifies the computations of invariants.

Lemma 3.10. *Let R be a commutative ring, I a finite index set, M an R-module and G a finite group acting on I. The operation of G on I induces an operation of G on the R-module $N := \oplus_{i\in I} M$ by permutation of coordinates. Let $I = I_1 \sqcup I_2 \sqcup \cdots \sqcup I_k$ be its orbit decomposition. Then, $N^G \cong \oplus_{i\leqslant k} N_i$, where for $i \leqslant k$,*

$$N_i := \Big\{ \sum_{j\in I_i} \iota_j(m) : m \in M \Big\} \cong M.$$

Here, $\iota_j\colon M \to N$ denotes the inclusion along the jth coordinate.

Proof. Since $(\sum_{j\neq i} N_j) \cap N_i = \{0\}$ and $\oplus_{i\leqslant k} N_i \subseteq N^G$ hold for every i, it remains to show that the N_i generate N^G. To prove this, note that any $x \in N$ can be written uniquely as

$$x = \sum_{i\in I} \iota_i(m_i)$$

for certain $m_i \in M$. We prove by induction on the number of non-zero m_i that any $x \in N^G$ lies in the module generated by the N_i. We may suppose $I = [1; |I|]$,

$m_1 \neq 0$ and denote by I_1 the orbit containing 1. Now, comparing the $g(1)$th entry of x and of $g.x$ yields that $m_{g(1)} = m_1$ for every $g \in G$. In particular, we can split of a sum

$$\sum_{j \in I_1} \iota_j(m_j) = \sum_{j \in I_1} \iota_j(m_1) \in N_1$$

from x. Applying induction to $x - \sum_{j \in I_1} \iota_j(m_1)$ concludes the proof. □

In particular, Lemma 3.10 yields the following orbit decomposition.

Corollary 3.11. *Let R_* be a commutative, graded ring, $I^1, \dots, I^r$ be finite index sets, M_* be a graded R_*-module and G a finite group acting on each of the I^ℓ. The operation of G on the I^ℓ induces an operation of G on the graded R_*-module*

$$N_* := \oplus_{\ell \leqslant r} \oplus_{I^\ell} M_{*-d_\ell},$$

where the d_ℓ are certain non-negative integers. Let $I^\ell = I_1^\ell \sqcup I_2^\ell \sqcup \cdots \sqcup I_{n_\ell}^\ell$ be the orbit decomposition. Then, $N^G \cong \oplus_{\ell \leqslant r} \oplus_{i \leqslant n_\ell} N_{\ell,i}$, where for $\ell \leqslant r$, $i \leqslant n_\ell$, we put

$$(N_{\ell,i})_* := \left\{ \sum_{j \in I_i^\ell} \iota_j(m) : m \in M_{*-d_\ell} \right\} \cong M_{*-d_\ell}.$$

Part II. Computation of the invariants of irreducible Weyl groups

Throughout this part k_0 denotes a field of characteristic not 2. When we compute the invariants of an irreducible Weyl group $W = W(\Sigma)$, where Σ is an irreducible root system we assume also that the characteristic of k_0 and the order of G are coprime.

We use in the following the description of irreducible root systems given in Bourbaki [1, PLATES I-VIII] for irreducible root systems of type $\neq G_2$ (recall that for Weyl groups of type G_2 we have already computed the invariants in Section 3.3). We have

$$\Sigma \subseteq \bigoplus_{i \leqslant n} e_i \mathbb{Z}[1/2] \subseteq \mathbb{R}^n$$

for an appropriate n. Taking the tensor product $k_0 \otimes_{\mathbb{Z}[1/2]}$ we get an embedding of Σ into k_0^n, such that all $\alpha \in \Sigma$ are anisotropic for the standard scalar product of k_0^n. Hence the associated reflections generate a finite subgroup of $O_n(k_0)$ which is isomorphic to G. In the following we will identify G with this subgroup of $O_n(k_0)$.

We provide a family of elements $\{x_i\}_{i \in I} \subseteq \mathsf{Inv}(G, \mathsf{k}_*^{\mathsf{M}})$, forming a basis of $\mathsf{Inv}(G, M_*)$ for all cycle modules over k_0. For this we have to show that given $k \in \mathcal{F}_{k_0}$ and an invariant $a \in \mathsf{Inv}_k^*(G, M_*)$, then there exist unique $c_i \in M_*(k)$ such that

$$a = \sum_{i \in I} \mathsf{res}_{k/k_0}(x_i) c_i.$$

To verify this claim, we may assume $k = k_0$ and let $e_1, \dots, e_n$ denote the standard basis elements of the k_0-vector space k_0^n.

If $a_1, \dots, a_n \in \Sigma$ are pairwise orthogonal, then $P(a_1, \dots, a_n)$ denotes the elementary 2-abelian subgroup generated by the corresponding reflections $s_{a_1}, \dots, s_{a_n}$. For $1 \leqslant i_1 < \dots < i_l \leqslant n$, we write $x_{a_{i_1},\dots,a_{i_l}}$ for the invariant

$$H^1\big(-, (\mathbb{Z}/2)\cdot s_{a_1} \times \dots \times (\mathbb{Z}/2)\cdot s_{a_n}\big) \xrightarrow{\simeq} H^1\big(-, (\mathbb{Z}/2)^n\big) \xrightarrow{x_{i_1,\dots,i_l}} \mathrm{k}_l^{\mathrm{M}}(-),$$

see Corollary 3.2 for the definition of the invariant $x_{i_1,\dots,i_l}$.

4. Weyl groups of type A_n

The invariants of Weyl groups of type A_n with values in $\mathrm{k}_*^{\mathrm{M}}$ are induced by the Stiefel–Whitney classes $\{w_i\}_i$, see [4, Part I, Sect. 25]. The proof carries over essentially verbatim to invariants with values in cycle modules M_* with $\mathrm{k}_*^{\mathrm{M}}$-structure using the splitting principle in the form of Proposition 2.3 and the computation of $\mathsf{Inv}((\mathbb{Z}/2)^n, M_*)$ in Corollary 3.2. The result is as follows. Here, we identify $H^1(k, S_n)$ with the set of isomorphism classes of étale algebras of dimension n over k, and denote for such an algebra E by q_E its trace form.

Proposition 4.1. *Let $n \geqslant 1$. Then, $\mathsf{Inv}(S_n, M_*)$ is completely decomposable with basis $\{E \mapsto w_i(q_E)\}_{i \leqslant \lfloor n/2 \rfloor}$.*

5. Weyl groups of type B_n/C_n

First, we note that the Weyl group $W(C_n)$ is isomorphic to the Weyl group $W(B_n)$. Hence, determining the invariants for $W(B_n)$ will also yield the determinants for $W(C_n)$.

5.1. Invariants of B_2. First, we consider $W(B_2)$, which is isomorphic to the dihedral group of order 8. In particular, $G := W(B_2) = \langle \sigma, \tau \rangle \subseteq S_4$ admits the permutation representation defined by

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}, \quad \tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}.$$

Considering G as orthogonal reflection group over k_0 yields an embedding $\phi\colon G \subseteq O_2$ of algebraic groups over k_0 given by

$$\sigma \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \tau \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Now, ϕ determines an action of G on $k_0[X,Y]$ given by

$$^{\sigma}X = Y, \quad ^{\sigma}Y = -X, \quad ^{\tau}X = Y, \quad ^{\tau}Y = X.$$

In particular,

$$k_0[X,Y]^G = k_0[X^2+Y^2, X^2Y^2] \cong k_0[A,B],$$

where $A := X^2+Y^2$, $B := 4X^2Y^2$. Fix the notation

$$E := k_0(X,Y), \quad K := k_0(X^2+Y^2, X^2Y^2).$$

Now, the group G acts freely on the open subscheme

$$U := D\big(XY(X-Y)(X+Y)\big) = D\big(X^2Y^2(X^2-Y^2)^2\big) \subseteq \mathbb{A}^2,$$

where for a polynomial f, we denote by $D(f) \subseteq \mathbb{A}^2$ the open subset given by $f \neq 0$.

By [4, Part I, Thm. 12.3] or [6, Thm. 3.7], the evaluation at the versal torsor $\mathsf{Spec}(E) \to \mathsf{Spec}(K)$ yields an injection $\mathsf{Inv}(G, M_*) \to M_{*,\mathsf{unr}}(U/G)$. To check that this map is also surjective, we first compute $M_{*,\mathsf{unr}}(U/G)$. An explicit computation yields

$$\begin{aligned} U/G &\cong \mathsf{Spec}\big(k_0[X,Y,X^{-2}Y^{-2}(X^2-Y^2)^{-2}]^G\big)\\ &= \mathsf{Spec}\big(k_0[X^2+Y^2, X^2Y^2, X^{-2}Y^{-2}, (X^2-Y^2)^{-2}]\big)\\ &\cong \mathsf{Spec}\big(k_0[A,B,B^{-1},(B-A^2)^{-1}]\big). \end{aligned}$$

To compute $M_{*,\mathsf{unr}}(U/G)$, note that $V := D(A) \subseteq U/G$ is isomorphic to the spectrum of

$$k_0\big[A,B,B^{-1},A^{-1},(B-A^2)^{-1}\big] \cong k_0\big[A,B',(B')^{-1},A^{-1},(B'-1)^{-1}\big],$$

where the isomorphism is induced by mapping B' to B/A^2. Now, by applying Lemma 3.1 twice and homotopy invariance,

$$\begin{aligned} M_{*,\mathsf{unr}}(V) \cong M_*(k_0) &\oplus \{B/A^2-1\}M_{*-1}(k_0) \oplus \{A\}M_{*-1}(k_0)\\ &\oplus \{B\}M_{*-1}(k_0) \oplus \{A\}\{B/A^2-1\}M_{*-2}(k_0)\\ &\oplus \{A\}\{B\}M_{*-2}(k_0). \end{aligned}$$

$M_{*,\mathsf{unr}}(U/G)$ can be computed as the kernel of the boundary

$$\partial = \partial^A_{(A)}\colon M_*(V) \to M_{*-1}(\mathbb{G}_m).$$

Thus, for every $t \in M_*(k_0)$,

$$\begin{aligned}
\partial(t) &= 0,\\
\partial\big(\{B/A^2-1\}t\big) &= \partial\big(\{B-A^2\}t\big) = \{B\}\partial(t) = 0,\\
\partial\big(\{B\}t\big) &= \{B\}\partial(t) = 0,\\
\partial\big(\{A\}t\big) &= t,\\
\partial\big(\{A\}\{B/A^2-1\}t\big) &= \partial\big(\{A\}\{B-A^2\}t\big) = \{B\}\partial\big(\{A\}t\big) = \{B\}t.\\
\partial\big(\{A\}\{B\}t\big) &= \{B\}\partial\big(\{A\}t\big) = \{B\}t.
\end{aligned}$$

Writing M_* short for $M_*(k_0)$, we conclude that $M_{*,\mathsf{unr}}(U/G)$ is given by

$$\begin{aligned}
&M_* \oplus \{B-A^2\}M_{*-1} \oplus \{B\}M_{*-1} \oplus \{A\}\{B(B-A^2)\}M_{*-2}\\
&\qquad \cong M_* \oplus \{B-A^2\}M_{*-1} \oplus \{B\}M_{*-1} \oplus \{A\}\{B-A^2\}M_{*-2}.
\end{aligned}$$

It remains to construct invariants mapping to the three non-constant basis elements of $M_{*,\mathsf{unr}}(U/G)$. Pulling back $w_1, w_2 \in \mathsf{Inv}(O_2, \mathrm{k}^{\mathrm{M}}_*)$ along the embedding ϕ gives invariants in $\mathsf{Inv}(G, \mathrm{k}^{\mathrm{M}}_*)$ that — by abuse of notation — we again denote by w_1, w_2. We first compute the value $w_1(E/K)$ of w_1 at the versal torsor E/K constructed above. To do this, we note that the determinant of $\phi(\sigma^i\tau)$ is -1, while the determinant of $\phi(\sigma^i)$ is 1. Now, $XY(X^2-Y^2) \in E$ maps to its negative by each reflection and is fixed by all the σ^i. Thus,

$$w_1(E/K) = \{X^2Y^2(X^2-Y^2)^2\} = \{B(A^2-B)\}.$$

Another invariant comes from the embedding $G \subseteq S_4$. We may define $v_1 := \mathsf{res}^G_{S_4}(\widetilde{w_1})$. Again, we compute $v_1(E/K)$. We note that $\widetilde{w_1} \in \mathsf{Inv}^1(S_4, \mathrm{k}^{\mathrm{M}}_*)$ may be computed as follows. Start with an arbitrary $x \in H^1(k, S_4)$; then

$$\widetilde{w_1}(x) = \mathsf{sgn}_*(x) \in H^1(k, \mathbb{Z}/2) \cong k^\times/k^{\times 2} \cong \mathrm{k}^{\mathrm{M}}_1(k).$$

The kernel of sgn consists exactly of the elements $\{\mathrm{id}, \tau, \sigma^2, \sigma^2\tau\}$ with σ, τ as above. Since XY is fixed by this kernel and is mapped to its negative by σ, the value of v_1 at the versal torsor is $\{X^2Y^2\} = \{B\}$. Consequently, it remains to find an invariant mapping to the basis $\{A\}\{B^2-A\}$ of $M_{*,\mathsf{unr}}(U/G)$.

Finally, we compute the value of $w_2 \in \mathsf{Inv}^2(G, \mathrm{k}^{\mathrm{M}}_*)$ at E/K. First consider the elementary abelian 2-subgroup generated by reflections $P := \langle \tau, \tau' \rangle$, where $\tau' = \sigma^2\tau$. Thus,

$$\phi(\tau) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \phi(\tau') = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

Recalling that the action of G on E is defined via ϕ, we now consider the versal P-torsor

$$E/E^P = k_0(X,Y)/k_0(X^2+Y^2, XY).$$

Then, $\tau \in P = \mathrm{Gal}(E/E^P)$ acts via $\tau(X) = Y$, $\tau(Y) = X$ and τ' via $\tau'(X) = -Y$, $\tau'(Y) = -X$. Thus, this $(\mathbb{Z}/2)^2$-torsor over E^P is equivalently described by the pair

$$\big((X-Y)^2, (X+Y)^2\big) \in \big((E^P)^\times/(E^P)^{\times 2}\big)^2.$$

We conclude that the value of $\mathrm{res}^P_{O_4} w_2$ at this P-torsor is

$$\{(X-Y)^2\}\{(X+Y)^2\} \in \mathrm{k}_2^{\mathrm{M}}(E^P).$$

By the computations above, the value of $\mathrm{res}^G_{O_4}(w_2)$ at E/K is of the form

$$\alpha_1 + \{B - A^2\}\alpha_2 + \{A\}\alpha_3 + \{B\}\{B(B-A^2)\}\alpha_4 \in \mathrm{k}_2^{\mathrm{M}}(K)$$

for some $\alpha_1 \in \mathrm{k}_2^{\mathrm{M}}(k_0)$, $\alpha_2, \alpha_3 \in \mathrm{k}_1^{\mathrm{M}}(k_0)$, $\alpha_4 \in \mathrm{k}_0^{\mathrm{M}}(k_0)$. Now, consider the diagram

$$\begin{tikzcd}
H^1(K,G) \arrow[r, "w_2"] \arrow[d, "\mathrm{res}^{E^P}_K(E)"'] & \mathrm{k}_2^{\mathrm{M}}(K) \arrow[d] \\
H^1(E^P,G) \arrow[r, "w_2"] & \mathrm{k}_2^{\mathrm{M}}(E^P) \\
H^1(E^P,P). \arrow[u, "\mathrm{ind}^G_P"] &
\end{tikzcd}$$

The square commutes by the definition of invariants. Denote by $E \in H^1(K,G)$ the G-torsor E/K and by $F \in H^1(E^P,P)$ the P-torsor E/E^P. Interpreting the torsors as cocycles yields

$$\mathrm{ind}^G_P(F) = \mathrm{res}^{E^P}_K(E) \in H^1(E^P,G).$$

Observing that XY is a square in E^P, this means

$$\{(X-Y)^2\}\{(X+Y)^2\} = \alpha_1 + \{B-A^2\}\alpha_2 + \{A\}\{A^2-B\}\alpha_4.$$

Applying the identity $\{\beta\}\{\beta'\} = \{\beta+\beta'\}\{-\beta\beta'\}$ to the left-hand side gives

$$\{2A\}\{B-A^2\},$$

so that we may choose $\alpha_1 = 0$, $\alpha_2 = \{2\}$, and $\alpha_4 = 1$. We conclude that the injection $\mathrm{Inv}(G, M_*) \to M_{*,\mathrm{unr}}(U/G)$ is surjective. This finishes the computation of $\mathrm{Inv}(G, M_*)$ and we obtain the following.

Proposition 5.1. *The invariants* $\mathrm{Inv}(W(B_2), M_*)$ *are completely decomposable with basis consisting of the invariants* $\{1, v_1, w_1, w_2\}$.

We conclude this section with a corollary of the proof.

Corollary 5.2. *Let $P_1 = P(e_1, e_2)$ and $P_2 = P(e_1 - e_2, e_1 + e_2)$. Then,*

$$\begin{aligned}
\mathrm{res}^{P_1}_{W(B_2)}(v_1) &= x_{\{e_1\}} + x_{\{e_2\}},\\
\mathrm{res}^{P_1}_{W(B_2)}(w_1) &= x_{\{e_1\}} + x_{\{e_2\}},\\
\mathrm{res}^{P_1}_{W(B_2)}(w_2) &= x_{\{e_1,e_2\}},
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{res}^{P_2}_{W(B_2)}(v_1) &= 0,\\
\mathrm{res}^{P_2}_{W(B_2)}(w_1) &= x_{\{e_1-e_2\}} + x_{\{e_1+e_2\}},\\
\mathrm{res}^{P_2}_{W(B_2)}(w_2) &= x_{\{e_1+e_2,e_1-e_2\}} + \{2\}\cdot\big(x_{\{e_1-e_2\}} + x_{\{e_1+e_2\}}\big).
\end{aligned}$$

5.2. Invariants of B_n. After dealing with the case $n = 2$, we now compute the invariants of Weyl groups of type B_n for general n. The root system B_n is the disjoint union

$$\Delta_1 \sqcup \Delta_2 \subseteq \mathbb{R}^n,$$

where $\Delta_1 = \{\pm e_i : 1 \leqslant i \leqslant n\}$ are the short roots and $\Delta_2 = \{\pm e_i \pm e_j : 1 \leqslant i < j \leqslant n\}$ are the long roots. This root system induces an orthogonal reflection group over any k_0 satisfying the above requirements. Furthermore,

$$W(B_n) \cong S_n \ltimes (\mathbb{Z}/2)^n$$

as abstract groups. Put $m := [n/2]$ and for $i \leqslant m$ define

$$a_i := e_{2i-1} - e_{2i} \quad \text{and} \quad b_i := e_{2i-1} + e_{2i}.$$

For each $L \leqslant m$ the elements of $X_L := \{a_1, b_1, \ldots, a_L, b_L, e_{2L+1}, e_{2L+2}, \ldots, e_n\}$ are mutually orthogonal. Defining $P_L := P(X_L)$, we prove by induction on m that

$$\Omega(G) = \{[P_0], \ldots, [P_m]\}.$$

The claim is clear for $n = 2$. In the general case, let P be any maximal elementary abelian 2-subgroup generated by reflections. First assume that P contains a short root, say e_n. Now, observe that $\langle e_n\rangle^\perp \cap B_n = B_{n-1}$ and use induction. If P contains a long root, we may assume this root to be a_1. Then, $\langle a_1\rangle^\perp \cap B_n = \{\pm b_1\} \cup B_{n-2}$, where we consider B_{n-2} to be embedded in $\mathbb{R}^n$ using the last $n-2$ coordinates. In particular, we may again use the induction hypothesis.

To determine $\mathrm{Inv}(B_n, M_*)$, we introduce additional pieces of notation. We denote P_L-torsors over a field k by

$$(\alpha_1, \beta_1, \ldots, \alpha_L, \beta_L, \epsilon_{2L+1}, \ldots, \epsilon_n) \in (k^\times/k^{\times 2})^n.$$

From the $(\mathbb{Z}/2)^n$-section, we know that $\mathsf{Inv}(P_L, M_*)$ is completely decomposable with basis $\{x_I\}_{I\subseteq[1;n]}$. Since this parameterization is inconvenient in the present setting, we change the index set by putting

$$\Lambda_L^d := \{(A, B, C, E) \subseteq [1; L]^3 \times [2L+1; n] : \\ A, B, C \text{ pw. disjoint, } |A| + |B| + 2|C| + |E| = d\}.$$

We reindex the basis of $\mathsf{Inv}(P_L, M_*)$ by defining for every $(A, B, C, E) \in \Lambda_L^d$:

$$x^L_{A,B,C,E} \colon H^1(k, P_L) \to \mathrm{k}^{\mathrm{M}}_*(k)$$
$$(\alpha_1, \beta_1, \dots, \alpha_L, \beta_L, \epsilon_{2L+1}, \dots \epsilon_n) \mapsto \prod_{a\in A}\{\alpha_a\} \prod_{b\in B}\{\beta_b\} \prod_{c\in C}\{\alpha_c\}\{\beta_c\} \prod_{e\in E}\{\epsilon_e\}.$$

In the same spirit, we also write

$$P(A, B, C, E) := P\big(\{a_p\}_{p\in A} \cup \{b_q\}_{q\in B} \cup \{a_r, b_r\}_{r\in C} \cup \{e_s\}_{s\in E}\big).$$

For $d \leqslant n$, we now construct the specific $W(B_n)$-invariant

$$u_d := \rho^*(\widetilde{w_d}) \in \mathsf{Inv}^d\big(W(B_n), M_*\big),$$

where $\widetilde{w_d} \in \mathsf{Inv}^d(S_n, \mathrm{k}^{\mathrm{M}}_*)$ denotes the dth modified Stiefel–Whitney class and

$$\rho \colon W(B_n) \cong S_n \ltimes (\mathbb{Z}/2)^n \to S_n$$

is the canonical projection. Then, the map $W(B_n) \to S_n$ sends both s_{a_i}, s_{b_i} to $(2i-1, 2i)$ and s_{e_i} to the neutral element. Let $k \in \mathcal{F}_{k_0}$ and $(\alpha_1, \beta_1, \dots, \alpha_L, \beta_L, \epsilon_{2L+1}, \dots, \epsilon_n)$ be a P_L-torsor over k. Using Example 3.5 and $\{2\}\{2\} = 0$, gives that the value of the total modified Stiefel–Whitney class at this torsor is $\prod_{i\leqslant L}(1 + \{\alpha_i\beta_i\})$. Hence,

$$\mathrm{res}^{P_L}_{W(B_n)}(u_d) = \sum_{(A,B,\varnothing,\varnothing)\in\Lambda_L^d} x^L_{A,B,\varnothing,\varnothing}. \tag{5.1}$$

Next, we construct an invariant v_d such that

$$\mathrm{res}^{P_L}_{W(B_n)}(v_d) = \sum_{(\varnothing,\varnothing,C,E)\in\Lambda_L^d} x^L_{\varnothing,\varnothing,C,E}. \tag{5.2}$$

To that end, we note that $W(B_n)$ embeds into S_{2n} via

$$\sigma \prod_{i\in I} s_{e_i} \mapsto \sigma \cdot (\sigma + n) \prod_{i\in I} (i, i+n),$$

where $I \subseteq [1; n]$, $\sigma \in S_n$ and $\sigma + n \in S_{2n}$ is given by

$$k \mapsto \begin{cases} k & \text{if } k \leqslant n, \\ n + \sigma(k-n) & \text{if } k > n. \end{cases}$$

We define the modified Stiefel–Whitney invariants $\widetilde{w_d} \in \mathsf{Inv}^d(S_{2n}, \mathrm{k}_*^{\mathsf{M}})$ as before and put

$$v'_d := \mathsf{res}_{S_{2n}}^{W(B_n)}(\widetilde{w_d}) \in \mathsf{Inv}^d(W(B_n), \mathrm{k}_*^{\mathsf{M}})$$

for $d \leqslant n$. Then, we define v_d recursively, by setting $v_0 := 0$ and then

$$v_d := v'_d + \sum_{k \leqslant d-1} u_{d-k} v_k.$$

To show that the so-defined invariant satisfies (5.2), we first note that already when restricting v'_d to P_L, we obtain an agreement with the right-hand side of (5.3) up to mixed lower-order expressions.

Lemma 5.3.

$$\mathsf{res}_{W(B_n)}^{P_L}(v'_d) = \sum_{(\varnothing,\varnothing,C,E) \in \Lambda_L^d} x^L_{\varnothing,\varnothing,C,E} + \sum_{k \leqslant d-1} \{-1\}^{d-k} \sum_{(A,B,C,E) \in \Lambda_L^k} x^L_{A,B,C,E}. \tag{5.3}$$

Proof. Observe that the map $W(B_n) \to S_{2n}$ sends $s_{e_i} \mapsto (i, i+n)$ and

$$s_{a_i} \mapsto (2i-1, 2i)(2i-1+n, 2i+n), \quad s_{b_i} \mapsto (2i-1, 2i+n)(2i, 2i-1+n)$$

Hence, by Lemma 3.6, the composition $P_L \to W(B_n) \to S_{2n} \to O_{2n}$ maps a P_L-torsor to the quadratic form

$$\langle\langle -\alpha_1, -\beta_1 \rangle\rangle \oplus \cdots \oplus \langle\langle -\alpha_L, -\beta_L \rangle\rangle \oplus \langle 2, 2\epsilon_{2L+1}, \ldots, 2, 2\epsilon_n \rangle.$$

We claim that the total modified Stiefel–Whitney class evaluated at this quadratic form equals

$$\prod_{i \leqslant L} \big(1 + \{-1\}(\{\alpha_i\} + \{\beta_i\}) + \{\alpha_i\}\{\beta_i\}\big) \prod_{2L+1 \leqslant i \leqslant n} (1 + \{\epsilon_i\}). \tag{5.4}$$

To see this, we compute it suffices to check that

$$w(\langle 2 \rangle \otimes \langle\langle \alpha, \beta \rangle\rangle) = 1 + \{-1\}\{-1\} + \{\alpha\}\{\beta\}.$$

To see this, we compute

$$\begin{aligned} w(\langle 2 \rangle \otimes \langle\langle -\alpha, -\beta \rangle\rangle) &= (1 + \{2\})(1 + \{2\alpha\})(1 + \{2\beta\})(1 + \{-2\beta\} + \{-\alpha\}) \\ &= (1 + \{\alpha\} + \{2\}\{\alpha\})(1 + \{\alpha\} + \{2\beta\}\{-\alpha\}) \\ &= 1 + \{\alpha\}\{\alpha\} + \{2\}\{\alpha\} + \{2\beta\}\{-\alpha\} \\ &= 1 + \{-1\}\{\alpha\} + \{-1\}\{\beta\} + \{\alpha\}\{\beta\}. \end{aligned}$$

Thus, translating (5.4) into the new notation, we obtain that

$$\mathsf{res}_{W(B_n)}^{P_L}(v'_d) = \sum_{(\varnothing,\varnothing,C,E) \in \Lambda_L^d} x^L_{\varnothing,\varnothing,C,E} + \sum_{k \leqslant d-1} \{-1\}^{d-k} \sum_{(A,B,C,E) \in \Lambda_L^k} x^L_{A,B,C,E}. \qquad \square$$

In light of Lemma 5.3, to establish (5.2), it remains to understand the product structure between u_{d-k} and v_k. To that end, we restrict the products to P_L.

Lemma 5.4. *We have*

$$\sum_{(A,B,\varnothing,\varnothing)\in\Lambda_L^d} x^L_{A,B,\varnothing,\varnothing} \sum_{(\varnothing,\varnothing,C,E)\in\Lambda_L^f} x^L_{\varnothing,\varnothing,C,E} = \sum_{\substack{(A,B,C,E)\in\Lambda^{d+f} \\ 2|C|+|E|=f}} x^L_{A,B,C,E}.$$

Proof. First, since $x^L_{A,B,\varnothing,\varnothing}x^L_{\varnothing,\varnothing,C,E} = \{-1\}^{|A\cap C|+|B\cap C|}x^L_{A-C,B-C,C,E}$,

$$\begin{aligned}
&\sum_{(A,B,\varnothing,\varnothing)\in\Lambda_L^d} x^L_{A,B,\varnothing,\varnothing} \sum_{(\varnothing,\varnothing,C,E)\in\Lambda_L^f} x^L_{\varnothing,\varnothing,C,E} \\
&\quad = \sum_{k\geqslant 0} \sum_{\substack{(A,B,\varnothing,\varnothing)\in\Lambda_L^d \\ (\varnothing,\varnothing,C,E)\in\Lambda_L^f \\ |A\cap C|+|B\cap C|=k}} \{-1\}^k x^L_{A-C,B-C,C,E} \\
&\quad = \sum_{\substack{(A,B,C,E)\in\Lambda_L^{d+f} \\ 2|C|+|E|=f}} x^L_{A,B,C,E} + \sum_{k\geqslant 1} \sum_{\substack{(A,B,\varnothing,\varnothing)\in\Lambda_L^d \\ (\varnothing,\varnothing,C,E)\in\Lambda_L^f \\ |A\cap C|+|B\cap C|=k}} \{-1\}^k x^L_{A-C,B-C,C,E}.
\end{aligned}$$

To show that the second sum vanishes, fix $k \geqslant 1$ and $(A', B', C, E) \in \Lambda_L^{d+f-k}$. Then, define

$$\begin{aligned}
S &:= \big\{(A,B) : (A,B,\varnothing,\varnothing) \in \Lambda_L^d \text{ and } A - C = A' \text{ and } B - C = B'\big\} \\
&= \big\{(A' \cup U, B' \cup V) : U, V \subseteq C \text{ and } U \cap V = \varnothing \text{ and } |U| + |V| = k\big\}.
\end{aligned}$$

Using this description, we conclude $|S| = 2^k\binom{|C|}{k}$. Since $k \geqslant 1$, this is even and we obtain the desired vanishing of the second sum. □

In the rest of this section, we show that $\mathsf{Inv}(W(B_n), M_*)$ is completely decomposable and that the products $\{u_{d-r}v_r\}_{\max(0,2d-n)\leqslant r\leqslant d,\, d\leqslant n}$ yield a basis.

Before determining the structure of $\mathsf{Inv}(W(B_n), M_*)$, it is helpful to know something about the image of the restriction maps $\mathsf{Inv}(W(B_n), M_*) \to \mathsf{Inv}(P_L, M_*)$. Let d, k, ℓ, L be non-negative integers, $L \leqslant m$. Then, the invariant

$$\phi^d_{L,k,\ell} := \sum_{\substack{(A,B,C,E)\in\Lambda_L^d \\ |C|=k,|E|=\ell}} x^L_{A,B,C,E}$$

is non-trivial if and only if there exists $(A, B, C, E) \in \Lambda_L^d$ with $|C| = k$ and $|E| = \ell$.

Lemma 5.5. *The image of the restriction map* $\mathsf{Inv}(W(B_n), M_*) \to \mathsf{Inv}(P_L, M_*)$ *is contained in the free submodule with basis*

$$\{\phi^d_{L,k,\ell} : 2k + \ell \leqslant d \leqslant n,\ 2(d - k - \ell) \leqslant 2L \leqslant n - \ell\}.$$

Proof. Let us first show that $\phi^d_{L,k,\ell} \neq 0$ iff $2k + \ell \leqslant d \leqslant n$ and $2(d - k - \ell) \leqslant 2L \leqslant n - \ell$. First, the conditions $2k + \ell \leqslant d$ and $2L + \ell \leqslant n$ are necessary. Furthermore, from the pairwise disjointness of A, B, C, we conclude $|A| + |B| + |C| \leqslant L$. This is equivalent to $d - (2k + \ell) + k \leqslant L$. Thus, $d - k - \ell \leqslant L$ is also necessary. To check sufficiency, suppose, we are given L, k, ℓ, d satisfying the restrictions. Then,

$$\big([1; d - \ell - 2k], \varnothing, [d - \ell - 2k + 1; d - \ell - k], [2L + 1; 2L + \ell]\big) \in \Lambda^d_L.$$

Thus, $\phi^d_{L,k,\ell} \neq 0$. Next, we check that the image of the restriction map is indeed contained in the submodule generated by the $\phi^d_{L,k,\ell} M_*(k_0)$.

Observe that all of the following elements normalize P_L:

$$\{s_{e_{2i-1}-e_{2j-1}} s_{e_{2i}-e_{2j}}\}_{i,j \leqslant L}, \quad \{s_{e_i - e_j}\}_{i,j \geqslant 2L+1} \quad \text{and} \quad \{s_{e_{2i}}\}_{i \leqslant L}.$$

Let $N_L \subseteq N_{W(B_n)}(P_L)$ be the subgroup generated by these elements. We claim that N_L permutes the $x^L_{A,B,C,E}$. Applying $s_{e_{2i-1}-e_{2j-1}} s_{e_{2i}-e_{2j}}$ for $i, j \leqslant L$ to a P_L-torsor

$$(\alpha_1, \beta_1, \dots, \alpha_L, \beta_L, \epsilon_{2L+1}, \dots, \epsilon_n)$$

interchanges $\alpha_i \leftrightarrow \alpha_j$ and $\beta_i \leftrightarrow \beta_j$. Thus, $x^L_{A,B,C,E}$ maps to $x^L_{A',B',C',E}$ where $A'/B'/C'$ is obtained from $A/B/C$ by applying the transposition (i, j) to the respective sets. Similarly, we see that swapping the ith and the jth coordinate for $i, j \geqslant 2L + 1$ maps $x^L_{A,B,C,E}$ to $x^L_{A,B,C,E'}$ where E' is obtained from E by applying to it the transposition (i, j). Finally, changing the $(2i)$th sign maps $x^L_{A,B,C,E}$ to $x^L_{A',B',C,E}$, where

$$A' = \big(A - \{i\}\big) \cup \big(B \cap \{i\}\big) \quad \text{and} \quad B' = \big(B - \{i\}\big) \cup \big(A \cap \{i\}\big).$$

That is, if $i \in A$ we remove it from A and put it into B and vice versa.

Iteratively applying these operations to an arbitrary $(A_0, B_0, C_0, E_0) \in \Lambda^d_L$ shows that its orbit under N_L equals

$$\big\{(A, B, C, E) \in \Lambda^d_L : |C| = |C_0|, |E| = |E_0|\big\}.$$

Now, the lemma follows from Corollary 3.11. □

By Proposition 2.3, the injection $\mathsf{Inv}(W(B_n), M_*) \to \prod_{L \leqslant m} \mathsf{Inv}(P_L, M_*)$ has its image inside $\prod_{L \leqslant m} \mathsf{Inv}(P_L, M_*)^{N_L}$ and Lemma 5.5 gives a good description of this object. However, this map is not surjective. One reason is the following: If an element $(z_L)_L$ of the right hand side comes from a $W(B_n)$-invariant, then certainly the restrictions of z_L and $z_{L'}$ to $P_L \cap P_{L'}$ must coincide. To address this, we prove the following refined lemma.

Lemma 5.6. *The image of* $\mathsf{Inv}(W(B_n), M_*) \to \prod_{L \leqslant m} \mathsf{Inv}(P_L, M_*)$ *lies in the subgroup generated by* $\{s \cdot M_{*-|s|}(k_0) : s \in S\}$, *where*

$$S := \Big\{ \Big(\sum_{2k+\ell=r} \phi^d_{L,k,\ell} \Big)_L : \max(0, 2d-n) \leqslant r \leqslant d \leqslant n \Big\} \subseteq \prod_{L \leqslant m} \mathsf{Inv}(P_L, \mathrm{k}^{\mathrm{M}}_*).$$

Proof. Let $\tilde{z} \in \mathsf{Inv}(W(B_n), M_*)$ be a homogeneous invariant and $z = (z_L)_L \in \prod_{L \leqslant m} \mathsf{Inv}(P_L, M_*)$ be the image of $\tilde{z}$ under the restriction maps. By Lemma 5.5,

$$z = \Big(\sum_{d,k,\ell} \phi^d_{L,k,\ell} m_{L,d,k,\ell} \Big)_L$$

for some $m_{L,d,k,\ell} \in M_{*-d}(k_0)$, where the sums are over all those d, k, ℓ such that $\phi^d_{L,k,\ell} \neq 0$.

First goal, we show that $m_{L,d,k,\ell}$ is independent of L in the sense that

$$m_{L,d,k,\ell} = m_{L',d,k,\ell},$$

if $\phi^d_{L,k,\ell} \neq 0$ and $\phi^d_{L',k,\ell} \neq 0$. We then denote by $m_{d,k,\ell}$ the common value. Observe that $(A_0, B_0, C_0, E_0) \in \Lambda^d_{L'} \cap \Lambda^d_L$, where

$$(A_0, B_0, C_0, E_0) := \big([1; d-\ell-2k], \varnothing, [d-\ell-2k+1; d-\ell-k], [n-\ell+1; n]\big).$$

Hence, since z comes from an invariant of $W(B_n)$,

$$\mathrm{res}^{P(A_0,B_0,C_0,E_0)}_{P_L}(z_L) = \mathrm{res}^{P(A_0,B_0,C_0,E_0)}_{P_{L'}}(z_{L'}).$$

Comparing coefficients of x_{A_0,B_0,C_0,E_0}-components on both sides yields that $m_{L,d,k,\ell} = m_{L',d,k,\ell}$.

Now, let us have a look at the second obstruction. We want to prove

$$m_{d,k,\ell} = m_{d,k',\ell'},$$

if $2k + \ell = 2k' + \ell'$ and if there exist L, L' such that $\phi^d_{L',k',\ell'} \neq 0$ and $\phi^d_{L,k,\ell} \neq 0$. It suffices to prove this in the case $k' - k = 1$. Since there exist L, L' satisfying $\phi^d_{L',k',\ell'}, \phi^d_{L,k,\ell} \neq 0$, we can choose some L such that $\phi^d_{L+1,k',\ell'}, \phi^d_{L,k,\ell} \neq 0$.

Let y be the restriction of $\tilde{z}$ to

$$P\big([1; d-\ell-2k], \varnothing, [L-k+1; L], [2L+3; 2L+\ell]\big) \times W(B_2),$$

where B_2 is embedded via the $(2L+1)$th and the $(2L+2)$th coordinates. By Proposition 2.5,

$$y = \sum_{\substack{A \subseteq [1; d-\ell-2k] \\ C \subseteq [L-k+1; L] \\ E \subseteq [2L+3; 2L+\ell]}} x^L_{A,\varnothing,C,E}\, y_{A,C,E}$$

for uniquely determined $y_{A,C,E} \in \mathsf{Inv}^{*-|A|-2|C|-|E|}(W(B_2), M_*)$. Furthermore, by the results of Section 5.1,

$$y_{A,C,E} = m^{(0)}_{A,C,E} + w_1 m^{(1a)}_{A,C,E} + v_1 m^{(1b)}_{A,C,E} + w_2 m^{(2)}_{A,C,E}$$

for uniquely determined

$$m^{(0)}_{A,C,E} \in M_{*-|A|-2|C|-|E|}(k_0),\ m^{(1a)}_{A,C,E}, m^{(1b)}_{A,C,E} \in M_{*-|A|-2|C|-|E|-1}(k_0)$$

and

$$m^{(2)}_{A,C,E} \in M_{*-|A|-2|C|-|E|-2}(k_0).$$

Restricting y further to

$$P\big([1; d-\ell-2k], \varnothing, [L-k+1; L], [2L+1; 2L+\ell]\big)$$

and considering the $x_{[1;d-2k-\ell],\varnothing,[L-k+1;L],[2L+1;2L+\ell]}$-component, Corollary 5.2 yields that

$$m_{d,k,\ell} = m^{(2)}_{([1;d-\ell-2k],[L-k+1;L],[2L+3;2L+\ell])}.$$

On the other hand, restricting y to

$$P\big([1; d-\ell-2k], \varnothing, [L-k+1; L+1], [2L+3; 2L+\ell]\big)$$

and considering the $x_{[1;d-2k-\ell],\varnothing,[L-k+1;L+1],[2L+3;2L+\ell]}$-component, we obtain from Corollary 5.2 that

$$m_{d,k',\ell'} = m^{(2)}_{([1;d-\ell-2k],[L-k+1;L],[2L+3;2L+\ell])}.$$

This proves the lemma. □

From Lemma 5.4, we deduce the following decomposition of $\mathsf{Inv}(W(B_n), M_*)$.

Corollary 5.7. *The group $\mathsf{Inv}(W(B_n), M_*)$ is completely decomposable with basis*

$$\big\{u_{d-r}v_r : \max(0, 2d-n) \leqslant r \leqslant d \leqslant n\big\}.$$

6. Weyl groups of type F_4

The root system F_4 is the disjoint union $\Delta_1 \sqcup \Delta_2 \sqcup \Delta_3 \subseteq \mathbb{R}^4$ with short routes

$$\Delta_1 := \{\pm e_i \pm e_j : 1 \leqslant i < j \leqslant 4\}$$

and long roots

$$\Delta_2 := \{\pm e_i : 1 \leqslant i \leqslant 4\}, \qquad \Delta_3 := \{1/2(\pm e_1 \pm e_2 \pm e_3 \pm e_4)\}.$$

Moreover, $\Omega(W(F_4)) = \{[P_0],[P_1],[P_2]\}$, where

$$P_0 := P(e_1,e_2,e_3,e_4), \quad P_1 := P(a_1,b_1,e_3,e_4), \quad P_2 := P(a_1,b_1,a_2,b_2).$$

Indeed, the set of long roots of F_4 is the root system D_4, which up to conjugacy has a unique maximal set of pairwise orthogonal vectors, namely a_1, b_1, a_2, b_2. On the other hand, if we have a maximal set of pairwise orthogonal roots containing a short root, say e_4, then $\langle e_4\rangle^{\perp} \cap F_4 = B_3$. We have determined before that up to conjugacy B_3 contains two maximal sets of pairwise orthogonal roots; namely $\{e_1,e_2,e_3\}$ and $\{a_1,b_1,e_3\}$.

Furthermore, the inclusion $P_2 \subseteq W(B_4) \subseteq W(F_4)$ shows that the restriction map

$$\mathsf{Inv}(W(F_4), M_*) \to \mathsf{Inv}(W(B_4), M_*)$$

is injective. Recall that $\mathsf{Inv}(W(B_4), M_*)$ is a free $M_*(k_0)$-module with the basis

$$\{1, u_1, v_1, u_2, v_1u_1, v_2, v_2u_1, v_3, v_4\}.$$

Before constructing specific invariants, we first point to another restriction in degree 2. Since

$$\mathsf{res}^{P_2}_{W(F_4)}(v_1) = \mathsf{res}^{P_2}_{W(F_4)}(v_3) = 0,$$

the image of the restriction $\mathsf{res}^{P_2}_{W(F_4)}$ is contained in the free submodule $S \subseteq \mathsf{Inv}^*(P_2, M_*)$ with basis $\{1, y_1, y_2, y_2', y_3, y_4\}$, where

$$y_1 = \mathsf{res}^{P_2}_{W(B_4)}(u_1), \quad y_2 = \mathsf{res}^{P_2}_{W(B_4)}(u_2), \quad y_2' = \mathsf{res}^{P_2}_{W(B_4)}(v_2),$$
$$y_3 = \mathsf{res}^{P_2}_{W(B_4)}(v_2u_1), \quad \text{and} \quad y_4 = \mathsf{res}^{P_2}_{W(B_4)}(v_4).$$

Now, let $a \in \mathsf{Inv}(P_2, M_*)$ be any invariant which is induced by an invariant from $\mathsf{Inv}(W(F_4), M_*)$. Then, we can find unique $m_d \in M_{*-d}(k_0)$, $m_2, m_2' \in M_{*-2}(k_0)$ such that

$$a = \sum_{\substack{d\leq 4\\ d\neq 2}} \Big(\sum_{(A,B,C)\in\Lambda^d} x_{A,B,C} \Big) m_d + \Big(\sum_{(A,B,\varnothing)\in\Lambda^2} x_{A,B,\varnothing} \Big) m_2 + \Big(\sum_{(\varnothing,\varnothing,C)\in\Lambda^2} x_{\varnothing,\varnothing,C} \Big) m_2'.$$

Now, $s_{1/2(e_1+e_2+e_3+e_4)}$ lies in the normalizer of P_2, as it leaves a_1, a_2 fixed and swaps b_1 with $-b_2$. Since a comes from $\mathsf{Inv}(W(F_4), M_*)$, the action of $s_{1/2(e_1+e_2+e_3+e_4)}$ leaves a invariant. Hence,

$$a = \sum_{\substack{d\leq 4\\ d\neq 2}} \Big(\sum_{(A,B,C)\in\Lambda^d} x_{A,B,C} \Big) m_d + \big(x_{\{a_1,a_2\}} + x_{\{b_1,b_2\}} + x_{\{a_1,b_1\}} + x_{\{a_2,b_2\}}\big) m_2$$
$$+ \big(x_{\{a_1,b_2\}} + x_{\{a_2,b_1\}}\big) m_2'.$$

Comparing coefficients yields $m_2 = m_2'$.

Thus, the image of the restriction $\mathrm{Inv}(W(F_4), M_*) \to \mathrm{Inv}(P_2, M_*)$ is contained in the free submodule with basis $\{1, y_1, y_2 + y_2', y_3, y_4\}$. Therefore, the image of the restriction $\mathrm{Inv}(W(F_4), M_*) \to \mathrm{Inv}(W(B_4), M_*)$ is contained in the free $M_*(k_0)$-module with basis $\{1, u_1, v_1, u_2 + v_2, v_1u_1, v_2u_1, v_3, v_4\}$.

Now, we need to construct F_4-invariants which restrict to these elements. First observe that $D_4 \subseteq F_4$ and that $W(F_4)$ stabilizes D_4. Thus, any $g \in W(F_4)$ maps the simple system $S = \{e_1 - e_2, e_2 - e_3, e_3 - e_4, e_3 + e_4\}$ to another simple system $S' \subseteq D_4$. Since all simple systems are conjugate there exists a *unique* $h \in W(D_4)$ mapping S' to S. This procedure induces a permutation of the 3 outer vertices $\{e_1 - e_2, e_3 - e_4, e_3 + e_4\}$ of the Coxeter graph, thereby giving rise to a group homomorphism $\psi\colon W(F_4) \to S_3$.

Then, we define $v_1 := \psi^*(\widetilde{w_1})$, where $\widetilde{w_1} \in \mathrm{Inv}(S_3, \mathrm{k}^{\mathrm{M}}_*)$ is the first modified Stiefel–Whitney class. To determine the restriction of v_1 to P_L note that the map $W(F_4) \to S_3$ sends $W(D_4)$ to the identity and s_{e_4} to the transposition $(2,3)$. Since $s_{e_i} = g_i s_{e_4} g_i^{-1}$, where $g_i \in W(D_4)$ denotes the element switching the 4th and the ith coordinate ($i \leqslant 3$), we conclude that all s_{e_i} are sent to $(2,3)$. Thus, the value of $\mathrm{res}^{P_L}_{W(F_4)}(v_1)$ at the P_L-torsor $(\alpha_1, \beta_1, \dots, \alpha_L, \beta_L, \epsilon_{2L+1}, \dots, \epsilon_4)$ is

$$\sum_{i \geqslant 2L+1} \{\epsilon_i\}.$$

The embedding $W(F_4) \subseteq O_4$ as orthogonal reflection group yields invariants

$$\mathrm{res}^{W(F_4)}_{O_4}(w_d) \in \mathrm{Inv}^d(W(F_4), \mathrm{k}^{\mathrm{M}}_*),$$

where $w_d \in \mathrm{Inv}^d(O_4, \mathrm{k}^{\mathrm{M}}_*)$ is the dth unmodified Stiefel–Whitney class. Again, if 2 is not a square in k_0, then these invariants do not have a nice form, when restricted to the P_L. Therefore, we change them a little and define invariants $\widehat{w_d}$. The image of a P_L-torsor $(\alpha_1, \dots, \alpha_L, \beta_1, \dots, \beta_L, \epsilon_{2L+1}, \dots, \epsilon_4)$ in $H^1(k, O_4)$ under the map $P_L \subseteq W(F_4) \subseteq O_4$ may be computed by using Example 3.3 and is given by

$$\langle 2\alpha_1, 2\beta_1, \dots, 2\alpha_L, 2\beta_L, \epsilon_{2L+1}, \dots, \epsilon_4 \rangle.$$

We would like to have

$$\mathrm{res}^{P_L}_{W(F_4)}(\widehat{w_d}) = \sum_{(A,B,C,E) \in \Lambda^d_L} x^L_{A,B,C,E}.$$

Since the restriction of w_1 to P_L is already given by $\sum_{(A,B,C,E)\in\Lambda^1_L} x^L_{A,B,C,E}$, we put $\widehat{w}_1 := w_1$. Now, for $d = 2$,

$$\mathrm{res}^{P_L}_{O_4}(w_2) = \sum_{(A,B,C,E)\in\Lambda^2_L} x^L_{A,B,C,E} + \sum_{(A,B,\varnothing,\varnothing)\in\Lambda^1_L} \{2\} x^L_{A,B,\varnothing,\varnothing},$$

so that $\widehat{w_2} := w_2 - \{2\}(w_1 - v_1)$ has the desired property. The restriction of w_3 to P_L is

$$\mathrm{res}^{P_L}_{O_4}(w_3) = \sum_{(A,B,C,E)\in\Lambda^3_L} x^L_{A,B,C,E} + \sum_{\substack{(A,B,\varnothing,E)\in\Lambda^2_L\\ |E|=1}} \{2\}x^L_{A,B,\varnothing,E},$$

so that we set $\widehat{w_3} := w_3 - \{2\}(w_1 - v_1)v_1$. Finally, the restriction of w_4 to P_L is

$$\mathrm{res}^{P_L}_{O_4}(w_4) = \sum_{(A,B,C,E)\in\Lambda^4_L} x^L_{A,B,C,E} + \sum_{\substack{(A,B,C,E)\in\Lambda^3_L\\ 2|C|+|E|=2}} \{2\}x^L_{A,B,C,E}$$

so that we set $\widehat{w_4} := w_4 - \{2\}w_2(w_1 - v_1)$. Furthermore, define

$$u_1 := w_1 - v_1 \in \mathrm{Inv}^1(W(F_4), \mathrm{k}^{\mathrm{M}}_*).$$

Now, we restrict the so-constructed invariants to $W(B_4)$. We claim that:

(a) $u_1, v_1 \in \mathrm{Inv}^1(W(F_4), \mathrm{k}^{\mathrm{M}}_*)$ restrict to $u_1, v_1 \in \mathrm{Inv}^1(W(B_4), \mathrm{k}^{\mathrm{M}}_*)$;

(b) $u_1v_1, (\widehat{w_2}-u_1v_1) \in \mathrm{Inv}^2(W(F_4), \mathrm{k}^{\mathrm{M}}_*)$ restrict to $u_1v_1, u_2+v_2 \in \mathrm{Inv}^2(W(B_4), \mathrm{k}^{\mathrm{M}}_*)$; and

(c) $u_1\widehat{w_2}, (\widehat{w_3} - u_1\widehat{w_2}) \in \mathrm{Inv}^3(W(F_4), \mathrm{k}^{\mathrm{M}}_*)$ restrict to u_1v_2, v_3.

Finally, $\widehat{w_4} \in \mathrm{Inv}^4(W(F_4), \mathrm{k}^{\mathrm{M}}_*)$ restricts to $v_4 \in \mathrm{Inv}^4(W(B_4), \mathrm{k}^{\mathrm{M}}_*)$. To prove these claims, we only need to consider the restrictions to $\mathrm{Inv}(P_L, \mathrm{k}^{\mathrm{M}}_*)$, where the identities are clear by construction. Thus, $\mathrm{Inv}(W(F_4), M_*)$ is a free $M_*(k_0)$-module with basis

$$\{1, \widehat{w_1}, v_1, \widehat{w_2}, \widehat{w_1}v_1, \widehat{w_3}, \widehat{w_2}v_1, \widehat{w_4}\}.$$

The construction of the $\widehat{w_d}$ also yields the following result.

Proposition 6.1. $\mathrm{Inv}(W(F_4), M_*)$ *is completely decomposable with basis*

$$\{1, w_1, v_1, w_2, v_1w_1, w_3, v_1w_2, w_4\}.$$

Remark 6.2. Alternatively, to the approach above, one could also rely on transfer-restriction arguments to characterize the invariants of $W(B_4)$, which extend to $W(F_4)$ as those whose restriction to $W(D_4)$ is fixed under the action of $W(F_4)/W(D_4)$.

7. Weyl groups of type D_n

The root system D_n, $n \geqslant 2$ consists of the elements

$$D_n = \{\pm e_i \pm e_j : 1 \leqslant i < j \leqslant n\}.$$

Let $m := [n/2]$, $a_i := e_{2i-1} - e_{2i}$, and $b_i := e_{2i-1} + e_{2i}$. By Remark 2.4, this root system defines an orthogonal reflection group over k_0 with $|\Omega(W(D_n))| = 1$. More precisely, $P := P(a_1, b_1, \dots, a_m, b_m)$ is a maximal elementary abelian 2-group generated by reflections. Furthermore, $W(D_n)$ is a subgroup of $S_n \ltimes (\mathbb{Z}/2)^n \cong W(B_n)$ in the precise sense that

$$W(D_n) = \Big\{\sigma \cdot \prod_{i\in I} s_{e_i} \in S_n \ltimes (\mathbb{Z}/2)^n : |I| \text{ even}\Big\}.$$

Remark 7.1. We note that for odd n the invariants of $W(D_n)$ can be deduced from those of $W(B_n)$, since $W(B_n) = \{\pm 1\} \times W(D_n)$. For instance, since $W(D_3) \cong W(A_3)$, this gives the invariants for $W(B_3)$.

Similarly to the B_n-section, we define

$$\Lambda^d := \big\{(A, B, C) \subseteq [1, m]^3 : A, B, C \text{ are pw. disjoint, } |A| + |B| + 2|C| = d\big\}$$

and $x_{A,B,C} \colon H^1(k, P) \to \mathrm{k}_d^{\mathrm{M}}(k)/2$

$$x_{A,B,C}(\alpha_1, \beta_1, \dots, \alpha_m, \beta_m) = \prod_{a\in A}\{\alpha_a\} \cdot \prod_{b\in B}\{\beta_b\} \cdot \prod_{c\in C}\{\alpha_c\}\{\beta_c\}.$$

As in the B_n-section, we now construct specific invariants. First, for $d \leqslant m$ the group homomorphism $\rho \colon W(D_n) \subseteq W(B_n) \to S_n$ induces the invariant

$$u_d := \rho^*(\widetilde{w_d}) \in \mathrm{Inv}^d(W(D_n), \mathrm{k}_*^{\mathrm{M}})$$

with $\mathrm{res}^P_{W(B_n)}(u_d) = \sum_{(A,B,\varnothing)\in\Lambda^d} x_{A,B,\varnothing}$.

Furthermore, from Section 5 we have an embedding $W(D_n) \subseteq W(B_n) \subseteq S_{2n}$. Starting with a $W(D_n)$-torsor $x \in H^1(k, W(D_n))$, we may consider its image $q_x \in H^1(k, O_{2n})$ induced by the map $W(D_n) \to S_{2n} \to O_{2n}$. Observe that $W(D_n) \to S_{2n}$ sends

$$s_{a_i} \mapsto (2i-1, 2i)(2i-1+n, 2i+n), \quad s_{b_i} \mapsto (2i-1, 2i+n)(2i, 2i-1+n).$$

Thus, starting with a P-torsor $(\alpha_1, \beta_1, \dots, \alpha_m, \beta_m)$, we may apply Lemma 3.6 to see that under the composition $P \to W(D_n) \to S_{2n} \to O_{2n}$ this torsor maps to

$$\langle\!\langle -\alpha_1, -\beta_1\rangle\!\rangle \oplus \cdots \oplus \langle\!\langle -\alpha_m, -\beta_m\rangle\!\rangle \ \big(\oplus \langle 1, 1\rangle\big),$$

where the expression in parentheses appears only for odd n. We would like to have an element $v \in \mathrm{Inv}(W(D_n), \mathrm{k}_*^{\mathrm{M}})$ such that $\mathrm{res}^P_{W(D_n)}(v)$ is given by

$$\begin{gathered} H^1(k, P) \to \mathrm{k}_*^{\mathrm{M}}(k) \\ (\alpha_1, \beta_1 \dots, \alpha_m, \beta_m) \mapsto \big(1 + \{\alpha_1\}\{\beta_1\}\big) \cdots \big(1 + \{\alpha_m\}\{\beta_m\}\big). \end{gathered}$$

To achieve this goal, we proceed recursively as in Section 5. First, we compute the value of the total Stiefel–Whitney class $w \in \mathsf{Inv}(O_4, \mathrm{k}_*^{\mathrm{M}})$ at a 2-fold Pfister form:

$$\begin{aligned} w(\langle\langle -\alpha, -\beta\rangle\rangle) &= (1+\{\alpha\})(1+\{\beta\})(1+\{\alpha\}+\{\beta\}) \\ &= 1+\{-1\}\{\alpha\}+\{-1\}\{\beta\}+\{\alpha\}\{\beta\}. \end{aligned}$$

Hence, setting $v' := \mathsf{res}_{O_{2n}}^{W(D_n)}(w)$, we obtain as in Lemma 5.3 that

$$\mathsf{res}_{W(D_n)}^{P}(v'_d) = \sum_{(\varnothing,\varnothing,C)\in\Lambda^d} x_{\varnothing,\varnothing,C}^{L} + \sum_{k\leqslant d-1} \{-1\}^{d-k} \sum_{(A,B,C)\in\Lambda^k} x_{A,B,C}.$$

Hence, proceeding recursively by setting $v_0 := 0$ and then

$$v_d := v'_d + \sum_{k\leqslant d-1} u_{d-k} v_k$$

yields the desired invariant. Moreover, $\mathsf{res}_{W(D_n)}^{P}(v_d) = \sum_{(\varnothing,\varnothing,C)\in\Lambda^d} x_{\varnothing,\varnothing,C}$ and, by Lemma 5.4,

$$\mathsf{res}_{W(D_n)}^{P}(u_d)\mathsf{res}_{W(D_n)}^{P}(v_e) = \sum_{\substack{(A,B,C)\in\Lambda^{d+e} \\ 2|C|=e}} x_{A,B,C}. \tag{7.1}$$

Now, suppose that $n = 2m$ is even. In this case, we need to construct one further invariant. Since $W(D_n) \cong S_n \ltimes (\mathbb{Z}/2)^{n-1}$, we have an embedding $S_n \subseteq W(D_n)$ such that $|W(D_n)/S_n| = 2^{n-1}$. More precisely, $|W(D_n)/S_n|$ consists of the cosets $g_I S_n$, where $g_I := \prod_{i\in I} s_{e_i}$ and where $I \subseteq [1;n]$ has even cardinality. The left action of $W(D_n)$ on these cosets induces a map

$$W(D_n) \to S_{2^{n-1}} \to O_{2^{n-1}}.$$

Thus, any $k \in \mathcal{F}_{k_0}$ and $y \in H^1(k, W(D_n))$ induce a quadratic form $q_y \in H^1(k, O_{2^{n-1}})$ and thereby an invariant $\omega \in \mathsf{Inv}(W(D_n), W)$. In fact, we claim that $\omega \in \mathsf{Inv}(W(D_n), I^m)$, where $I(k) \subseteq W(k)$ is the fundamental ideal.

To prove this, we start by showing that $\mathsf{res}_{W(D_n)}^{P}(\omega) \in \mathsf{Inv}(P, I^m)$. It is convenient to understand the map $W(D_n) \to S_{2^{n-1}}$ on the subgroup P.

Lemma 7.2. *Let $L = \{\{2i-1, 2i\} : i \leqslant m\}$ and define $f : 2^{[1;n]} \to 2^L$,*

$$f(I) := \{\{2i-1, 2i\} : \textit{either } 2i-1 \in I \textit{ or } 2i \in I, \textit{ but not both}\}.$$

Then,

(1) *The action of P on $W(D_n)/S_n$ has the 2^{m-1} orbits $\mathcal{O}_{\mathcal{J}} := \{g_I S_n \mid f(I) = \mathcal{J}\}$, $\mathcal{J} \subseteq L$, $|\mathcal{J}|$ even.*

(2) *Let $\mathcal{O}_{\mathcal{J}}$ be an arbitrary orbit from* (1). *Put $A_{\mathcal{J}} := \{i \leqslant m : \{2i-1, 2i\} \in \mathcal{J}\}$ and $B_{\mathcal{J}} := \{i \leqslant m : \{2i-1, 2i\} \notin \mathcal{J}\}$. Then, $P(\{a_i\}_{i \in B_{\mathcal{J}}} \cup \{b_j\}_{j \in A_{\mathcal{J}}})$ acts trivially on $\mathcal{O}_{\mathcal{J}}$ and the action of $P_{\mathcal{J}} := P(\{a_i\}_{i \in A_{\mathcal{J}}} \cup \{b_j\}_{j \in B_{\mathcal{J}}})$ on $\mathcal{O}_{\mathcal{J}}$ is simply transitive.*

Proof. (1) Let $I \subseteq [1;n]$. If $\{2i-1, 2i\} \notin f(I)$, then

$$s_{a_i} g_I = g_I s_{a_i} \quad \text{and} \quad s_{b_i} g_I = g_{I \Delta \{2i-1,2i\}} s_{a_i},$$

where Δ is the symmetric difference. On the other hand, if $\{2i-1, 2i\} \in f(I)$, then

$$s_{a_i} g_I = g_{I \Delta \{2i-1,2i\}} s_{a_i} \quad \text{and} \quad s_{b_i} g_I = g_I s_{a_i}.$$

(2) By the proof of part (1), $P(\{a_i\}_{i \in B_{\mathcal{J}}} \cup \{b_j\}_{j \in A_{\mathcal{J}}})$ acts trivially on $\mathcal{O}_{\mathcal{J}}$. Since

$$|P(\{a_i\}_{i \in A_{\mathcal{J}}} \cup \{b_j\}_{j \in B_{\mathcal{J}}})| = 2^m = |\mathcal{O}_{\mathcal{J}}|,$$

assertion (2) follows after verifying that $P(\{a_i\}_{i \in A_{\mathcal{J}}} \cup \{b_j\}_{j \in B_{\mathcal{J}}})$ acts freely on $\mathcal{O}_{\mathcal{J}}$. So suppose, $I \subseteq [1;n]$, $M \subseteq A_{\mathcal{J}}$ and $N \subseteq B_{\mathcal{J}}$ is such that $f(I) = \mathcal{J}$ and $g := \prod_{i \in M} s_{a_i} \cdot \prod_{j \in N} s_{b_j}$ fixes $g_I S_n$. The proof of part (1) gives that

$$g g_I S_n = g_{I'} S_n,$$

where

$$I' = I \Delta \big(\cup_{i \in M \cup N} \{2i-1, 2i\} \big).$$

Observing that $I' = I$ if and only if $M = N = \varnothing$ concludes the proof. □

Using Lemma 7.2, we conclude the following. Consider an arbitrary

$$y = (\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_m) \in H^1(k, P)$$

and let $q_y \in H^1(k, O_{2^{n-1}})$ be the quadratic form induced by the composition

$$P \to W(D_n) \to S_{2^{n-1}} \to O_{2^{n-1}}.$$

The decomposition of the action of P into orbits $\mathcal{O}_{\mathcal{J}}$ induces a decomposition of q_y as $q_y \cong \oplus_{\mathcal{J}} q_{\mathcal{J}}$. More precisely, the action of P on $\mathcal{O}_{\mathcal{J}}$ induces a map $P \to S_{2^m}$ and $q_{\mathcal{J}}$ is defined to be the image of $y \in H^1(k, P)$ under the composition

$$P \to S_{2^m} \to O_{2^m}.$$

By Lemma 7.2, this composition factors through the projection $P \to P_{\mathcal{J}}$. Now, by Lemma 3.6, its remark and Lemma 7.2,

$$q_{\mathcal{J}} \cong \langle 2^m \rangle \otimes \bigotimes_{i \in A_{\mathcal{J}}} \langle\langle -\alpha_i \rangle\rangle \otimes \bigotimes_{j \in B_{\mathcal{J}}} \langle\langle -\beta_j \rangle\rangle. \tag{7.2}$$

Thus, the image of $q_y = \oplus_{\mathcal{J}} q_{\mathcal{J}}$ in $W(k)$ lies in $I^m(k)$, so that $\mathrm{res}^P_{W(D_n)}(\omega) \in \mathrm{Inv}(P, I^m)$.

Now, we pass from P to $W(D_n)$. First, ω induces an invariant

$$\bar{\omega} \in \mathsf{Inv}^0\big(W(D_n), I^*/I^{*+1}\big)$$

through the projection $W \to (I^*/I^{*+1})_0 = W/I$. Since the image of $\mathsf{res}^P_{W(D_n)}(\omega)$ lies in $I^m \subseteq I$, we conclude that $\mathsf{res}^P_{W(D_n)}(\bar{\omega}) = 0$. As P is up to conjugation the only maximal elementary abelian 2-subgroup of $W(D_n)$ generated by reflections, Corollary 2.3 gives that $\bar{\omega} = 0 \in \mathsf{Inv}^0(W(D_n), I^*/I^{*+1})$, i.e., $\omega \in \mathsf{Inv}(W(D_n), I)$. Iterating this procedure m times shows that $\omega \in \mathsf{Inv}(W(D_n), I^m)$.

By Example 3.9, there exists an invariant $e_m\colon I^m(k) \to \mathrm{k}_2^{\mathsf{M}}(k)$ satisfying

$$e_m\big(\langle\!\langle \alpha_1 \rangle\!\rangle \otimes \cdots \otimes \langle\!\langle \alpha_m \rangle\!\rangle\big) = \prod_{i \leqslant m} \{\alpha_i\}. \tag{7.3}$$

Then,

$$e_m(y) := e_m\big(\langle 2^m \rangle \otimes \omega(y)\big) + \{-1\} \sum_{k \leqslant d-1} u_{d-1-k} v_k$$

defines an element of $\mathsf{Inv}^m(W(D_n), \mathrm{k}_*^{\mathsf{M}})$ and, in the vein of Lemma 5.3, we now determine its restriction to P.

Lemma 7.3.

$$\mathsf{res}^P_{W(D_n)}(e_m) = \sum_{\substack{(A,B,\varnothing) \in \Lambda^m \\ |A| \text{ even}}} x_{A,B,\varnothing}. \tag{7.4}$$

Proof. First, by identity (7.1), it suffices to show that the restriction of the invariant $e'_m(y) := e_m(\langle 2^m \rangle \otimes \omega(y))$ to P is given by

$$\sum_{\substack{(A,B,\varnothing) \in \Lambda^m \\ |A| \text{ even}}} x_{A,B,\varnothing} + \{-1\} \sum_{(A,B,C) \in \Lambda^m_{d-1}} x_{A,B,C}. \tag{7.5}$$

Then, by identities (7.2) and (7.3), evaluating $\mathsf{res}^P_{W(D_n)}(e'_m)$ at the torsor

$$(\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_m) \in H^1(k, P)$$

gives that

$$\begin{aligned}
\sum_{\substack{(A,B,\varnothing) \in \Lambda^m \\ |A| \text{ even}}} \prod_{i \in A} \{-\alpha_i\} \prod_{j \in B} \{-\beta_j\} &= \sum_{\substack{(A,B,\varnothing) \in \Lambda^m \\ |A| \text{ even}}} \sum_{\substack{U \subseteq A \\ V \subseteq B}} \{-1\}^{m-|U|-|V|} \prod_{i \in U} \{\alpha_i\} \prod_{j \in V} \{\beta_j\} \\
&= \sum_{\substack{U,V \subseteq [1,m] \\ U \cap V = \varnothing}} N_{U,V} \{-1\}^{m-|U|-|V|} \prod_{i \in U} \{\alpha_i\} \prod_{j \in V} \{\beta_j\},
\end{aligned}$$

where

$$N_{U,V} := \big|\{A \subseteq [1,m] : A \supset U,\, A \cap V = \varnothing,\, |A| \text{ even}\}\big|.$$

To conclude the proof, we distinguish on the value of $|U| + |V|$. First, the contributions coming from $|U|+|V| = m$ give precisely the leading-order expression in (7.5). Next, suppose that $|U| + |V| = m - k$ with $k \geqslant 1$. Then, $N_{U,V} = 2^{k-1}$, so that the corresponding contribution vanishes mod 2 if and only if $k \geqslant 1$. Now, we conclude the proof by noting that the contributions for $k = 1$ yield precisely the summation expression in (7.5). □

Now, we derive a central set of constraints for the image of the restriction map $\mathsf{Inv}(W(D_n), M_*) \to \mathsf{Inv}(P, M_*)$. For $d \leqslant n$ and $i \leqslant [d/2]$ put

$$\phi_i^d := \sum_{\substack{(A,B,C)\in\Lambda^d\\ |C|=i}} x_{A,B,C} \in \mathsf{Inv}^d(P, \mathrm{k}_*^{\mathrm{M}})$$

and $\psi_1 := \sum_{\substack{(A,B,\varnothing)\\ |A| \text{ even}}} x_{A,B,\varnothing}$.

Lemma 7.4. *The image of the restriction map* $\mathsf{Inv}(W(D_n), M_*) \to \mathsf{Inv}(P, M_*)$ *is contained in the free* $M_*(k_0)$*-module with basis*

$$S = \{\phi_i^d : d \leqslant n,\ \max(0, d - m) \leqslant i \leqslant [d/2]\} \cup R,$$

where $R = \varnothing$, *if* n *is odd and* $R = \{\psi_1\}$, *if* n *is even.*

Proof. Arguing as in the B_n-section shows that all elements of S are non-zero. Furthermore, both $s_{e_{2i-1}-e_{2j-1}} s_{e_{2i}-e_{2j}}$ and $s_{e_{2i-1}} s_{e_{2j-1}}$ normalize P.

Let us denote by $N_1, N_2 \subseteq N(P)$ the subgroups generated by the first, respectively second kind of elements and let us denote by N the subgroup generated by N_1 and N_2. At the torsor level, conjugation by the first kind of elements swaps $\alpha_i \leftrightarrow \alpha_j$ and $\beta_i \leftrightarrow \beta_j$. Thus for $(A, B, C) \in \Lambda^d$, the invariant $x_{A,B,C}$ maps to $x_{A',B',C'}$, where

$$A' = (i, j)A, \quad B' = (i, j)B, \quad \text{and} \quad C' = (i, j)C.$$

On the other hand, conjugation by the second kind of elements swaps $\alpha_i \leftrightarrow \beta_i$ and $\alpha_j \leftrightarrow \beta_j$. Thus, it maps $x_{A,B,C}$ to $x_{A',B',C}$, where

$$A' = \big(A - \{i, j\}\big) \cup \big(B \cap \{i, j\}\big) \quad \text{and} \quad B' = \big(B - \{i, j\}\big) \cup \big(A \cap \{i, j\}\big).$$

That is, if $i \in A$, we remove it from A and put it into B and vice versa; then we do the same for j. Thus, N acts on $\mathsf{Inv}(P, \mathrm{k}_*^{\mathrm{M}})$ by permuting the $x_{A,B,C}$ and hence we can apply Corollary 3.11.

In the next step, we determine the orbit of x_{A_0,B_0,C_0} under N for an arbitrary $(A_0, B_0, C_0) \in \Lambda^d$. First, suppose that n is odd or that $C_0 \neq \varnothing$ or that ($n = 2m$ is even and $d < m$). Then, we claim that the orbit of x_{A_0,B_0,C_0} under N_2 is given by

$$\big\{x_{A,B,C_0} : (A, B, C_0) \in \Lambda^d,\ A \cup B = A_0 \cup B_0\big\}.$$

It suffices to show that for any $a \in A_0$, there exists an element of N_2 mapping x_{A_0,B_0,C_0} to $x_{A_0-\{a\},B_0\cup\{a\},C_0}$. As soon as this is proven, one observes that the symmetric statement with $b \in B_0$ also holds; iterating these operations, we indeed get the claimed orbit. For n odd, $s_{e_{2a-1}}s_{e_n}$ maps x_{A_0,B_0,C_0} to $x_{A_0-\{a\},B_0\cup\{a\},C_0}$. If $C_0 \neq \varnothing$ choose $c \in C_0$; then $s_{e_{2a-1}}s_{e_{2c-1}}$ maps x_{A_0,B_0,C_0} to $x_{A_0-\{a\},B_0\cup\{a\},C_0}$. Finally, if $n = 2m$ is even and $d < m$, then there exists $i \in [1;m]$ such that $i \notin A_0 \cup B_0 \cup C_0$ and the element $s_{e_{2a-1}}s_{e_{2i-1}}$ does the trick. Thus, the orbit of x_{A_0,B_0,C_0} under N_2 equals

$$\{x_{A,B,C_0} : (A,B,C_0) \in \Lambda^d,\ A \cup B = A_0 \cup B_0\}.$$

Similarly, for any $(A_1,B_1,C_1) \in \Lambda^d$ the orbit of x_{A_1,B_1,C_1} under N_1 equals

$$\{x_{A,B,C} : (A,B,C) \in \Lambda^d,\ |A| = |A_1|,\ |B| = |B_1|,\ |C| = |C_1|\}.$$

Combining these results, the orbit of x_{A_0,B_0,C_0} under N is given by

$$\{x_{A,B,C} : (A,B,C) \in \Lambda^d,\ |C| = |C_0|\}.$$

Finally, let $C_0 = \varnothing$, $n = 2m$ be even and $d = m$. Then, the orbit of $x_{A_0,B_0,\varnothing}$ under N_2 equals

$$\{x_{A,B,\varnothing} : (A,B,\varnothing) \in \Lambda^d,\ A \cup B = A_0 \cup B_0, |B| - |B_0| \text{ is even}\}.$$

Using that for any $(A_1,B_1,C_1) \in \Lambda^d$ the orbit of x_{A_1,B_1,C_1} under N_1 is given by

$$\{x_{A,B,C} : (A,B,C) \in \Lambda^d,\ |A| = |A_1|,\ |B| = |B_1|,\ |C| = |C_1|\},$$

we see that the orbit of $x_{A_0,B_0,\varnothing}$ under N is

$$\{x_{A,B,\varnothing} : (A,B,\varnothing) \in \Lambda^d,\ |B| - |B_0| \text{ is even}\}.$$

Hence, applying Corollary 3.11 concludes the proof. □

In particular, as Lemma 5.4 gives that

$$\mathrm{res}^P_{W(D_n)}(u_{d-2i}v_{2i}) = \phi_i^d$$

and as

$$\mathrm{res}^P_{W(D_n)}(e_m) = \psi_1,$$

we obtain the following result.

Corollary 7.5. $\mathrm{Inv}(W(D_n), M_*)$ *is completely decomposable with basis*

$$\{u_{d-2i}v_{2i} : d \leqslant n, \max(0, d-m) \leqslant i \leqslant [d/2]\} \cup R,$$

where $R = \varnothing$ *for odd* n *and* $R = \{e_m\}$ *for even* n.

Remark 7.6. A relation between $W(B_n)$ and $W(D_n)$ explains why in Corollary 7.5, we only see v_d with even d. Indeed, the kernel of the determinant of the $2n$-dimensional representation of $W(B_n)$ contains $W(D_n)$. Since for odd d, all the $W(B_n)$-invariants v_d are divisible by v_1 and since v_1 is vanishing, we deduce that they all reduce to 0 on $W(D_n)$.

8. Weyl groups of type E_6, E_7, and E_8

8.1. Type E_6. Up to conjugacy, $P := P(a_1, b_1, a_2, b_2)$ is the unique maximal elementary abelian subgroup generated by reflections in $W(E_7)$. Since the injection $\mathsf{Inv}(W(E_6), M_*) \to \mathsf{Inv}(P, M_*)$ factors through $\mathsf{Inv}(W(D_5), M_*)$, the map $\mathsf{Inv}(W(E_6), M_*) \to \mathsf{Inv}(W(D_5), M_*)$ is injective and a basis of $\mathsf{Inv}(W(D_5), M_*)$ is given by $\{1, u_1, u_2, v_2, v_2u_1, v_4\}$.

So let $a \in \mathsf{Inv}(P, M_*)$ be an invariant which comes from a $W(E_6)$-invariant. Since the inclusion $P \subseteq W(E_6)$ factors through $W(D_5) \subseteq W(E_6)$, a decomposes uniquely as

$$a = \sum_{\substack{d \leqslant 4 \\ d \neq 2}} \sum_{(A,B,C) \in \Lambda^d} x_{A,B,C} m_d + \sum_{(A,B,\varnothing) \in \Lambda^2} x_{A,B,\varnothing} m_2 + \sum_{(\varnothing,\varnothing,C) \in \Lambda^2} x_{\varnothing,\varnothing,C} m_2'$$

for certain $m_d \in M_{*-d}(k_0), m_2, m_2' \in M_{*-2}(k_0)$. Now, the element

$$g := s_{\frac{1}{2}(e_1-e_2-e_3-e_4-e_5-e_6-e_7+e_8)} s_{\frac{1}{2}(-e_1+e_2+e_3+e_4-e_5-e_6-e_7+e_8)} \in W(E_6)$$

lies in the normalizer of P, since

$$g s_{a_1} g^{-1} = s_{b_2}, \quad g s_{b_1} g^{-1} = s_{b_1}, \quad g s_{a_2} g^{-1} = s_{a_2}, \quad g s_{b_2} g^{-1} = s_{a_1}.$$

The induced action of g on a P-torsor $(\alpha_1, \alpha_2, \beta_1, \beta_2)$ is thus given by swapping $\alpha_1 \leftrightarrow \beta_2$, while leaving α_2, β_1 fixed. Therefore, applying g to the invariant a yields

$$\sum_{\substack{d \leqslant 4 \\ d \neq 2}} \sum_{(A,B,C) \in \Lambda^d} x_{A,B,C} m_d + \sum_{i,j \in \{1,2\}} x_{\{a_i,b_j\}} m_2 + (x_{\{a_1,a_2\}} + x_{\{b_1,b_2\}}) m_2'.$$

Since a comes from an invariant of $W(E_6)$, it stays invariant under g and comparing coefficients, we conclude that the image of the restriction $\mathsf{Inv}(W(E_6), M_*) \to \mathsf{Inv}(W(D_5), M_*)$ lies in the free submodule with basis

$$\{1, u_1, u_2 + v_2, v_2u_1, v_4\}.$$

The embedding of $W(E_6)$ in O_8 as orthogonal reflection group gives rise to the invariants $\mathsf{res}_{O_8}^{W(E_6)}(\widetilde{w}_d) \in \mathsf{Inv}^d(O_8, \mathsf{k}_*^{\mathsf{M}})$, which we again denote by $\widetilde{w}_d$. For any $k \in \mathcal{F}_{k_0}$ and $(\alpha_1, \beta_1, \alpha_2, \beta_2) \in (k^\times/k^{\times 2})^4$, the map $P \to W(E_6) \subseteq O_8$ induces the quadratic form

$$\langle 2\alpha_1, 2\beta_1, 2\alpha_2, 2\beta_2, 1, 1, 1, 1 \rangle.$$

Thus, the total modified Stiefel–Whitney class evaluated at this torsor equals

$$\big(1 + \{\alpha_1\}\big)\big(1 + \{\alpha_2\}\big)\big(1 + \{\beta_1\}\big)\big(1 + \{\beta_2\}\big).$$

Now,

$$\operatorname{res}^P_{W(D_5)}(u_1) = \operatorname{res}^P_{W(E_6)}(\widetilde{w_1}), \quad \operatorname{res}^P_{W(D_5)}(u_2 + v_2) = \operatorname{res}^P_{W(E_6)}(\widetilde{w_2}),$$
$$\operatorname{res}^P_{W(D_5)}(v_2 u_1) = \operatorname{res}^P_{W(E_6)}(\widetilde{w_3}), \quad \operatorname{res}^P_{W(D_5)}(v_4) = \operatorname{res}^P_{W(E_6)}(\widetilde{w_4}).$$

Hence, $\{\widetilde{w_d}\}_{d\leqslant 4}$ form a basis of $\mathsf{Inv}(W(E_6), M_*)$ as $M_*(k_0)$-module.

8.2. Type E_7. Up to conjugacy, $P := P(a_1, b_1, a_2, b_2, a_3, b_3, a_4)$ is the unique maximal elementary abelian subgroup generated by reflections in $W(E_7)$. Looking at the root systems, we see that there is an inclusion $W(D_6) \times \langle s_{a_4}\rangle \subseteq W(E_7)$. Invoking the same factorization argument as before, the restriction map

$$\mathsf{Inv}\big(W(E_7), M_*\big) \to \mathsf{Inv}\big(W(D_6) \times \langle s_{a_4}\rangle, M_*\big)$$

is injective. We first recall that $\mathsf{Inv}(W(D_6) \times \langle s_{a_4}\rangle, M_*)$ is a free $M_*(k_0)$-module with basis:

(0) 1

(1) $u_1, x_{\{a_4\}}$

(2) $u_2, v_2, u_1 x_{\{a_4\}}$

(3) $(u_3 - e_3), e_3, u_1 v_2, u_2 x_{\{a_4\}}, v_2 x_{\{a_4\}}$

(4) $u_2 v_2, v_4, (u_3 - e_3) x_{\{a_4\}}, e_3 x_{\{a_4\}}, u_1 v_2 x_{\{a_4\}}$

(5) $v_4 u_1, u_2 v_2 x_{\{a_4\}}, v_4 x_{\{a_4\}}$

(6) $v_6, v_4 u_1 x_{\{a_4\}}$

(7) $v_6 x_{\{a_4\}}$.

Defining $g := s_{\frac{1}{2}(e_1-e_2-e_3-e_4-e_5-e_6-e_7+e_8)} s_{\frac{1}{2}(-e_1+e_2+e_3+e_4-e_5-e_6-e_7+e_8)} \in W(E_7)$ as in the E_6-case yields that

$$g s_{a_1} g^{-1} = s_{b_2}, \quad g s_{b_1} g^{-1} = s_{b_1}, \quad g s_{a_2} g^{-1} = s_{a_2}, \quad g s_{b_2} g^{-1} = s_{a_1},$$
$$g s_{a_3} g^{-1} = s_{a_3}, \quad g s_{b_3} g^{-1} = s_{a_4}, \quad g s_{a_4} g^{-1} = s_{b_3}.$$

The action of g on a P-torsor $(\alpha_1, \beta_1, \ldots, \alpha_3, \beta_3, \alpha_4) \in (k^\times/k^{\times 2})^7$ is thus given by swapping $\alpha_1 \leftrightarrow \beta_2$, $\beta_3 \leftrightarrow \alpha_4$ while leaving $\beta_1, \alpha_2, \alpha_3$ fixed. Arguing just as in the E_6-case, we see that the image of $\mathsf{Inv}(W(E_7), M_*) \to \mathsf{Inv}(W(D_6) \times \langle s_{a_4}\rangle, M_*)$ lies in the free $M_*(k_0)$-module with basis

(0) 1

(1) $u_1 + x_{\{a_4\}}$

(2) $v_2 + u_2 + u_1 x_{\{a_4\}}$

(3) $u_1 v_2 + (u_3 - e_3) + u_2 x_{\{a_4\}}, e_3 + v_2 x_{\{a_4\}}$

(4) $v_4 + (u_3 - e_3)x_{\{a_4\}}, u_2v_2 + u_1v_2x_{\{a_4\}} + e_3x_{\{a_4\}}$

(5) $v_4x_{\{a_4\}} + u_2v_2x_{\{a_4\}} + v_4u_1$

(6) $v_4u_1x_{\{a_4\}} + v_6$

(7) $v_6x_{\{a_4\}}$.

Now, we provide specific $W(E_7)$-invariants. First, the embedding $W(E_7) \subseteq O_8$ gives us invariants $\mathrm{res}_{O_8}^{W(E_7)}(\widetilde{w_d}) \in \mathrm{Inv}^d(W(E_7), \mathrm{k}_*^{\mathrm{M}})$, which we again denote by $\widetilde{w_d}$. Then,

$$\begin{aligned}
\mathrm{res}^P_{W(E_7)}(\widetilde{w_1}) &= \mathrm{res}^P_{W(D_6)\times\langle s_{a_4}\rangle}\big(u_1 + x_{\{a_4\}}\big),\\
\mathrm{res}^P_{W(E_7)}(\widetilde{w_2}) &= \mathrm{res}^P_{W(D_6)\times\langle s_{a_4}\rangle}\big(u_2 + v_2 + u_1x_{\{a_4\}}\big),\\
\mathrm{res}^P_{W(E_7)}(\widetilde{w_3}) &= \mathrm{res}^P_{W(D_6)\times\langle s_{a_4}\rangle}\big(u_3 + u_1v_2 + u_2x_{\{a_4\}} + v_2x_{\{a_4\}}\big),\\
\mathrm{res}^P_{W(E_7)}(\widetilde{w_4}) &= \mathrm{res}^P_{W(D_6)\times\langle s_{a_4}\rangle}\big(u_2v_2 + v_4 + u_3x_{\{a_4\}} + u_1v_2x_{\{a_4\}}\big),\\
\mathrm{res}^P_{W(E_7)}(\widetilde{w_5}) &= \mathrm{res}^P_{W(D_6)\times\langle s_{a_4}\rangle}\big(v_4u_1 + v_4x_{\{a_4\}} + u_2v_2x_{\{a_4\}}\big),\\
\mathrm{res}^P_{W(E_7)}(\widetilde{w_6}) &= \mathrm{res}^P_{W(D_6)\times\langle s_{a_4}\rangle}\big(v_6 + v_4u_1x_{\{a_4\}}\big),\\
\mathrm{res}^P_{W(E_7)}(\widetilde{w_7}) &= \mathrm{res}^P_{W(D_6)\times\langle s_{a_4}\rangle}\big(v_6x_{\{a_4\}}\big).
\end{aligned}$$

So we still lack invariants in degree 3 and 4. To construct the missing invariant in degree 3, we mimic the construction of the invariant e_m in the D_n-section. Let $U \cong S_6 \times \langle s_{a_4}\rangle \subseteq W(E_7)$ be the subgroup generated by the reflections at

$$\{e_1 + e_2, e_2 - e_3, e_3 - e_4, e_4 - e_5, e_5 - e_6, e_7 - e_8\}.$$

Then, $|U\backslash W(E_7)| = 2016$ and we obtain a map $W(E_7) \to S_{2016} \to O_{2016}$. To be more precise, there is a right action of $W(E_7)$ on the right cosets $U\backslash W(E_7)$ given by right multiplication. This induces an anti-homomorphism $W(E_7) \to S_{2016}$ and precomposing this map with $g \mapsto g^{-1}$, we obtain the desired homomorphism. We need the following lemma which tells us that we are in a situation which is quite similar to the D_n-case:

Lemma 8.1. *Let $k \in \mathcal{F}_{k_0}$ and $y \in H^1(k, P)$ be a P-torsor. Let q_y be the quadratic form induced by y under the composition*

$$P \to W(E_7) \to S_{2016} \to O_{2016}.$$

Then, the image of q_y in $W(k)$ is contained in $I^3(k)$.

Proof. This can be checked by a computational algebra system, see the appendix. □

We now argue similarly to the D_n-case. In concrete terms, if y is a $W(E_7)$-torsor, and q_y is the quadratic form induced by y under the composition

$$W(E_7) \to S_{2016} \to O_{2016},$$

then the image of q_y in $W(k)$ is contained in $I^3(k)$ and we define the invariant

$$f_3'(y) := e_3\big(\langle 2^3\rangle \otimes q_y\big). \tag{8.1}$$

In the D_n-case, namely in Lemma 7.3, we could compute the restriction of the invariant e_m to the maximal elementary abelian 2-subgroup explicitly. In principle, this would also be possible in the present setting. However, the computations would be substantially more involved. Therefore, we provide a more conceptual level argument. To that end, we recall from Section 7 that if $g \in W(E_7)$ is contained in the normalizer $N_{W(E_7)}(P)$ of P in $W(E_7)$, then g acts both on the invariants

$$\{x_{A,B,C}\}_{(A,B,C)\in\Lambda^d} \in \mathsf{Inv}^d(P, M_*)$$

as well as on the indexing set Λ^d.

Lemma 8.2. *Let $d \leqslant 7$ and $g \in N_{W(E_7)}(P)$. Also, let $a \in \mathsf{Inv}^d(W(E_7), \mathrm{k}^{\mathsf{M}}_*)$ be an invariant and represent its restriction to $\mathsf{Inv}^d(P, \mathrm{k}^{\mathsf{M}}_*)$ as*

$$\mathsf{res}^P_{W(E_7)}(a) = \sum_{\ell\leqslant d}\sum_{I\in\Lambda^\ell} m_I x_I, \tag{8.2}$$

for certain coefficients $m_I \in \mathrm{k}^{\mathsf{M}}_{d-|I|}(k_0)$. Then, $m_I = m_{g(I)}$ for all $\ell \leqslant d$ and $I \in \Lambda^\ell$.

Proof. First, since the restriction is invariant under the action of g,

$$\sum_{\ell\leqslant d}\sum_{I\in\Lambda^{d-\ell}} \big(m_I - m_{g(I)}\big)x_I = 0. \tag{8.3}$$

Now, suppose that the assertion of the lemma was false, and choose a counterexample $I^* \in \Lambda^{\ell^*}$ with maximal ℓ^*. Then, we first evaluate both sides of (8.2) at the function field $E = k_0(A_1, B_1, \dots, A_3, B_3, A_4)$ in the indeterminates $A_1, B_1, \dots, A_3, B_3, A_4$ corresponding to the roots in P, and then apply the Milnor residue maps corresponding to the indeterminates associated with the index set I^*. Since ℓ^* was chosen to be maximal, the identity (8.3) reduces to $m_I - m_{g(I)} = 0$, which concludes the proof. □

In words, just as in Corollary 3.11, when representing the restrictions of invariants as in (8.2), then basis elements in the same orbit share the same coefficient.

In particular, we have seen above that in degree 1 and 2 all basis elements are in a single orbit and are therefore the restriction of the corresponding modified Stiefel–Whitney classes. Thus, applying Lemma 8.2 with $a = f_3'$, there exist $m_\ell \in \mathrm{k}^{\mathsf{M}}_{3-\ell}(k_0)$, $\ell \in \{0,1,2\}$ and $m_{A,B,C} \in \mathbb{Z}/2$, $(A,B,C) \in \Lambda^3$ such that

$$\mathsf{res}^P_{W(E_7)}(f_3') = \sum_{(A,B,C)\in\Lambda^3} m_{A,B,C}\, x_{A,B,C} + \sum_{\ell\leqslant 2} m_\ell \mathsf{res}^P_{W(E_7)}(\widetilde{w}_\ell).$$

Then, proceeding as in the definition of e_m in Section 7, we define an invariant $f_3 \in \mathsf{Inv}^3(W(E_7), \mathrm{k}^{\mathsf{M}}_*)$ by stripping of the mixed terms from f_3'. That is,

$$f_3 := f_3' - \sum_{\ell \leqslant 2} m_\ell \widetilde{w_\ell}.$$

In the appendix, we expound on how a computational algebra system shows that

$$\mathsf{res}^P_{W(E_7)}(f_3) = \mathsf{res}^P_{W(D_6)\times\langle s_{a4}\rangle}\big(u_1 v_2 + u_3 - e_3 + u_2 x_{\{a_4\}}\big). \tag{8.4}$$

Finally, we can proceed in a similar fashion in order to remove the mixed terms in the product expression.

$$\big(u_1 + x_{\{a_4\}}\big)\big(u_1 v_2 + (u_3 - e_3) + u_2 x_{\{a_4\}}\big).$$

Thus, $\mathsf{Inv}(W(E_7), M_*)$ is completely decomposable with basis $\{\widetilde{w_d}\}_{d\leqslant 7} \cup \{f_3, f_3\widetilde{w_1}\}$.

8.3. Type E_8. Up to conjugacy, $P := P(a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4)$ is the unique maximal elementary abelian subgroup generated by reflections in $W(E_8)$. By the same arguments as in the E_6/E_7-case, we obtain that the restriction map

$$\mathsf{Inv}\big(W(E_8), M_*\big) \to \mathsf{Inv}\big(W(D_8), M_*\big)$$

is injective. We first recall that $\mathsf{Inv}(W(D_8), M_*)$ is a free $M_*(k_0)$-module with the basis

$$\big\{1, u_1, u_2, v_2, u_3, v_2u_1, e_4, v_4, (u_4 - e_4), v_2u_2, v_2u_3, v_4u_1, v_4u_2, v_6, v_6u_1, v_8\big\}.$$

Again, we define $g \in W(E_8)$ as in the E_6 or E_7-case and check that it normalizes P:

$$\begin{aligned} g s_{a_1} g^{-1} &= s_{b_2}, & g s_{b_1} g^{-1} &= s_{b_1}, & g s_{a_2} g^{-1} &= s_{a_2}, & g s_{b_2} g^{-1} &= s_{a_1}, \\ g s_{a_3} g^{-1} &= s_{a_3}, & g s_{b_3} g^{-1} &= s_{a_4}, & g s_{a_4} g^{-1} &= s_{b_3}, & g s_{b_4} g^{-1} &= s_{b_4}. \end{aligned}$$

The action of g on a P-torsor $(\alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3, \alpha_4, \beta_4)$ is thus given by swapping $\alpha_1 \leftrightarrow \beta_2$, $\beta_3 \leftrightarrow \alpha_4$ while leaving $\beta_1, \alpha_2, \alpha_3, \beta_4$ fixed. Again, applying the same kind of arguments as in the E_6-case, we see that the image of the restriction map $\mathsf{Inv}(W(E_8), M_*) \to \mathsf{Inv}(W(D_8), M_*)$ is contained in the free submodule with basis

$$\begin{aligned} \big\{1, u_1, u_2 + v_2, u_3 + v_2u_1, e_4 + v_4, (u_4 - e_4) + v_2u_2,& \\ v_2u_3 + v_4u_1, v_4u_2 + v_6, v_6u_1, v_8\big\}.& \end{aligned}$$

We need to construct $W(E_8)$-invariants mapping to these basis elements. On the one hand, the inclusion $W(E_8) \subseteq O_8$ gives modified Stiefel–Whitney classes $\widetilde{w_d} \in \mathsf{Inv}^d(W(E_8), \mathrm{k}_*^{\mathsf{M}})$. Again,

$$\begin{aligned}
\mathsf{res}^P_{W(E_8)}(\widetilde{w_1}) &= \mathsf{res}^P_{W(D_8)}(u_1), & \mathsf{res}^P_{W(E_8)}(\widetilde{w_5}) &= \mathsf{res}^P_{W(D_8)}(v_2u_3 + v_4u_1),\\
\mathsf{res}^P_{W(E_8)}(\widetilde{w_2}) &= \mathsf{res}^P_{W(D_8)}(u_2 + v_2), & \mathsf{res}^P_{W(E_8)}(\widetilde{w_6}) &= \mathsf{res}^P_{W(D_8)}(v_4u_2 + v_6),\\
\mathsf{res}^P_{W(E_8)}(\widetilde{w_3}) &= \mathsf{res}^P_{W(D_8)}(u_3 + u_1v_2), & \mathsf{res}^P_{W(E_8)}(\widetilde{w_7}) &= \mathsf{res}^P_{W(D_8)}(v_6u_1),\\
\mathsf{res}^P_{W(E_8)}(\widetilde{w_4}) &= \mathsf{res}^P_{W(D_8)}(u_4 + u_2v_2 + v_4), & \mathsf{res}^P_{W(E_8)}(\widetilde{w_8}) &= \mathsf{res}^P_{W(D_8)}(v_8).
\end{aligned}$$

The situation is very similar to the E_7-case except that now, we miss a basis invariant in degree 4. Let $U \subseteq W(E_8)$ be the subgroup generated by the reflections at

$$\{e_1 + e_2, e_2 - e_3, e_3 - e_4, e_4 - e_5, e_5 - e_6, e_6 - e_7, e_7 - e_8\}.$$

By observing that $U \cong S_8$ or by using a computational algebra software, we conclude $|U\backslash W(E_8)| = 17280$. As in the E_7-case, we obtain a map

$$W(E_8) \to S_{17280} \to O_{17280}.$$

Again, we need the following lemma.

Lemma 8.3. *Let $k \in \mathcal{F}_{k_0}$ and $y \in H^1(k, P)$ be a P-torsor. Let q_y be the quadratic form induced by y under the composition*

$$P \to W(E_8) \to S_{17280} \to O_{17280}.$$

Then, the image of q_y in $W(k)$ is contained in $I^4(k)$.

Proof. Again, this can be checked by a computational algebra software, see the appendix. □

As in the D_n-case, we obtain from this an invariant $f_4 \in \mathsf{Inv}^4(W(E_8), \mathrm{k}_*^{\mathsf{M}})$. More precisely, if y is a $W(E_8)$-torsor and q_y is the quadratic form induced by y under the composition

$$W(E_8) \to S_{17280} \to O_{17280},$$

then the image of q_y in $W(k)$ is contained in $I^4(k)$ and we define $f_4'(y) := e_4(q_y)$. We then proceed as in the E_7-case and set

$$f_4 := f_4' - \sum_{\ell \leqslant 3} m_\ell \widetilde{w_\ell}$$

for suitable $m_\ell \in \mathrm{k}_\ell^{\mathsf{M}}(4 - \ell)$ in order to strip off the mixed contributions from f_4'.

The restriction of f_4 to P is determined through a computational algebra system, see the appendix. The result is $\mathsf{res}^P_{W(D_8)}(v_2u_2 + (u_4 - e_4))$. Thus, we conclude that $\mathsf{Inv}(W(E_8), M_*)$ is completely decomposable with basis $\{f_4\} \cup \{\widetilde{w_d}\}_{d \leqslant 8}$.

A. Excerpts from a letter by J.-P. Serre

[...] Hence, the only technical point which remains is the "splitting principle": if the restrictions of an invariant to every cube is 0, the invariant is 0. In your text with Gille, you prove that result under the restrictive condition that the characteristic p does not divide the order $|G|$ of the group G. The proof you give (which is basically the same as in my UCLA lectures) is based on the fact that the polynomial invariants of G (in its natural representation) make up a polynomial algebra; in geometric language, the quotient Aff^n/G is isomorphic to Aff^n. This is OK when p does not divide $|G|$, but it is also true in many other cases. For instance, it is true for all p ($\neq 2$) for the classical types (provided, for type A_n, that we choose for lattice the natural lattice for GL_{n+1}, namely $\mathbb{Z}^{n+1}$). For types G_2, F_4, E_6, E_7, it is true if $p > 3$ and for E_8 it is true for $p > 5$: this is not easy to prove, but it has been known to topologists since the 1950's (because the question is related to the determination of the mod p cohomology of the corresponding compact Lie groups). When I started working on these questions, I found natural to have to exclude, for instance, the characteristics 3 and 5 for E_8. It is only a few years ago that I realized that even these small restrictions are unnecessary: the splitting principle holds for every $p > 2$.

I have sketched the proof in my Oberwolfach report: take for instance the case of E_8; the group $G = W(E_8)$ contains $W(D_8)$ as a subgroup of odd index, namely 135; moreover, the reflections of $W(D_8)$ are also reflections of $W(E_8)$; hence every cube of $W(D_8)$ is a cube of $W(E_8)$; if a cohomological invariant of $W(E_8)$ gives 0 over every cube, its restriction to $W(D_8)$ has the same property, hence is 0 because D_8 is a classical type; since the index of $W(D_8)$ is odd, then this invariant is 0. It is remarkable that a similar proof works in every other case. [...]

B. Computations for E_7 and E_8

For the computations involving E_7 and E_8, we use the computational algebra system `GAP` and the `GAP`-package `CHEVIE` [5]. The complete source code used for the proof of Lemmas 8.1 and 8.3 together with detailed instructions on how to reproduce the results are provided on the author's GitHub page: `https://github.com/Christian-Hirsch/orbit-e78`.

B.1. Computations concerning $W(E_7)$. The proof of Lemma 8.1 requires detailed information on the action of P on $U\backslash W(E_7)$. We analyze this action, via the procedure `fullCheck(7, U, P)`.

First, `fullCheck(7, U, P)` computes the action of P on $U\backslash W(E_7)$ and also its orbits $\mathcal{O}_1, \dots, \mathcal{O}_r$. Then, for each orbit $\mathcal{O}_k$, it determines a subset $A_k \subseteq \{a_1, b_1, a_2, b_2, a_3, b_3, a_4\}$, such that $P(\{a_1, b_1, a_2, b_2, a_3, b_3, a_4\} - A_k)$ acts trivially on $\mathcal{O}_k$ and such that $P(A_k)$ acts simply transitively on $\mathcal{O}_k$. A priori, there is no reason that such a subset should exist; however — as checked by the program — it exists in the case we are considering. The return value of the procedure `fullCheck` is an array whose kth entry is the set A_k. Inspecting the return value reveals that each A_k consists of at least 3 elements and that the subsets consisting of 3 elements have the desired form.

More precisely, to call `fullCheck(7, U, P)`, we need to determine the indices of the roots generating U and P. In the following, the roots are expressed as linear combinations of the simple system of roots given by

$$v_1 = \tfrac{1}{2}(e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7 + e_8), \quad v_2 = e_1 + e_2,$$
$$v_i = e_{i-1} - e_{i-2}, \; 3 \leqslant i \leqslant 7.$$

Additionally,

$$\begin{aligned} b_2 &= v_2 + v_3 + 2v_4 + v_5, \\ b_3 &= v_2 + v_3 + 2v_4 + 2v_5 + 2v_6 + v_7, \\ -a_4 &= 2v_1 + 2v_2 + 3v_3 + 4v_4 + 3v_5 + 2v_6 + v_7. \end{aligned}$$

We claim that U and P are represented by the indices $[2, 4, 5, 6, 7, 63]$ and $[3, 2, 5, 28, 7, 49, 63]$, respectively. This can be checked by printing the basis representation of the E_7 roots: gap> p: = [3, 2, 5, 28, 7, 49, 63];

```
gap> for u in p do Print(CoxeterGroup("E", 7).roots[u]);Print("\ n");od;
[ 0, 0, 1, 0, 0, 0, 0 ]
[ 0, 1, 0, 0, 0, 0, 0 ]
[ 0, 0, 0, 0, 1, 0, 0 ]
[ 0, 1, 1, 2, 1, 0, 0 ]
[ 0, 0, 0, 0, 0, 0, 1 ]
[ 0, 1, 1, 2, 2, 2, 1 ]
[ 2, 2, 3, 4, 3, 2, 1 ]
```

We can now call the `fullCheck`-procedure.

```
gap> Aks: = fullCheck(7, [2, 4, 5, 6, 7, 63], [3, 2, 5, 28, 7, 49, 63]);
```

Verifying that all $\{A_k\}_{k \leqslant r}$ consist of at least 3 elements can be achieved via the command

```
gap> for Ak in Aks do if Length(Ak)<3 then Print("Fail");fi;od;
```

To see that those A_k with $|A_k| = 3$ correspond precisely to the elements

$$\{(A, B, C) \in \Lambda_3 : |C| = 1\} \cup \{(A, B, \varnothing) \in \Lambda_3 : |A| \text{ odd}\} \cup \{(A, B, \varnothing, a_4) : (A, B, \varnothing) \in \Lambda_2\},$$

we use the `e7Correct`-procedure. It checks that the $\{A_k\}_{k \leqslant r}$ do not contain elements which are not in the claimed set above. Since there are precisely 28 A_k with

3 elements, which is precisely the cardinality of the above set, this reasoning yields the claimed description.

```
gap> Y: = Filtered(Aks, Ak-> Length(Ak)<4);
gap> e7Correct(Y);
```

B.2. Computations concerning $W(E_8)$. Since the arguments are very similar to the E_7-case, we only explain the most important changes. First, we consider the maximal elementary abelian subgroup generated by reflections

$$P = P(a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4)$$

and the subgroup

$$U = \langle s_{e_1+e_2}, s_{e_2-e_3}, s_{e_3-e_4}, s_{e_4-e_5}, s_{e_5-e_6}, s_{e_6-e_7}, s_{e_7-e_8} \rangle.$$

In addition to the computations provided in Appendix B.1, we note that

$$b_4 = 2v_1 + 3v_2 + 4v_3 + 6v_4 + 5v_5 + 4v_6 + 3v_7 + 2v_8.$$

Then, P and U are represented by the indices $[3, 2, 5, 32, 7, 61, 97, 120]$ and $[2, 4, 5, 6, 7, 8, 97]$:

```
gap> a: = [3, 2, 5, 32, 7, 61, 97, 120];
[ 3, 2, 5, 32, 7, 61, 97, 120 ]
gap> for u in a do Print(CoxeterGroup("E", 8).roots[u]); Print("\ n"); od;
[ 0, 0, 1, 0, 0, 0, 0, 0 ]
[ 0, 1, 0, 0, 0, 0, 0, 0 ]
[ 0, 0, 0, 0, 1, 0, 0, 0 ]
[ 0, 1, 1, 2, 1, 0, 0, 0 ]
[ 0, 0, 0, 0, 0, 0, 1, 0 ]
[ 0, 1, 1, 2, 2, 2, 1, 0 ]
[ 2, 2, 3, 4, 3, 2, 1, 0 ]
[ 2, 3, 4, 6, 5, 4, 3, 2 ]
```

To understand the orbit structure, we proceed as in the E_7-case:

```
gap> Aks: = fullCheck(8, [2, 4, 5, 6, 7, 8, 97], [3, 2, 5, 32, 7, 61, 97, 120]);
gap> for Ak in Aks do if Length(Ak)<4 then Print("Fail");fi;od;
gap> Y: = Filtered(Aks, Ak->Length(Ak)<5);
gap> e8Correct(Y);
```

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