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On Borel Anosov representations in even dimensions

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Abstract. We prove that a word hyperbolic group which admits a P_{2q+1}-Anosov representation into $\mathsf{PGL}(4q+2,\mathbb{R})$ contains a finite-index subgroup which is either free or a surface group. As a consequence, we give an affirmative answer to Sambarino's question for Borel Anosov representations into $\mathsf{SL}(4q+2,\mathbb{R})$.

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1. Introduction

In this note, we address the following question of Andrés Sambarino and provide a positive answer when $d = 4q+2$ for some $q \in \mathbb{N}$.

Sambarino's Question. *Suppose that Γ is a torsion free word hyperbolic group which admits a Borel Anosov representation into $\mathsf{SL}(d,\mathbb{R})$. Is Γ necessarily free or a surface group?*

Anosov representations of fundamental groups of closed negatively curved Riemannian manifolds were introduced by Labourie [19] in his study of the Hitchin component. Guichard–Wienhard extended Labourie's definition for general word hyperbolic groups in [14]. Anosov representations define discrete subgroups of real reductive Lie groups which generalize convex cocompact subgroups of rank one Lie groups. A representation $\rho\colon \Gamma \to \mathsf{GL}(d,\mathbb{R})$ is called *P_k-Anosov*, where $1 \leqslant k \leqslant \frac{d}{2}$, if it is Anosov with respect to the pair of opposite parabolic subgroups of $\mathsf{GL}(d,\mathbb{R})$ defined as the stabilizers of a k-plane and a complementary $(d-k)$-plane (see Subsection 2.3). The representation ρ is called *Borel Anosov* if ρ is P_k-Anosov for every k. Labourie in [19] proved that every Hitchin representation into $\mathsf{PSL}(d,\mathbb{R})$ is irreducible and admits a lift into $\mathsf{GL}(d,\mathbb{R})$ which is Borel Anosov. The only known examples of Borel Anosov representations are constructed from representations of free or surface groups. By a surface group we mean the fundamental group of a closed surface of negative Euler characteristic. Hitchin representations are the only known examples of Borel Anosov representations of surface groups in even dimensions. In all odd dimensions, Barbot's construction [1] can be used to produce reducible examples.

A positive answer to Sambarino's question was given in [8] for $d = 3$ or 4. By using results of Benoist in [2, 3], we prove that a torsion free word hyperbolic group admitting a P_{2q+1}-Anosov representation into $\mathsf{GL}(4q+2,\mathbb{R})$ has to be either free or a surface group. Moreover, by using Wilton's result [23] on the existence of quasiconvex surface groups or rigid subgroups in one ended-word hyperbolic groups and a theorem of Kapovich–Leeb–Porti in [16] (see also [17, Theorem 6]), we prove the following stronger statement:

Theorem 1.1. *Let Γ be a word hyperbolic group and $\rho\colon\Gamma\to\mathsf{GL}(4q+2,\mathbb{R})$ a representation. Suppose that there exists a continuous, ρ-equivariant dynamics preserving map $\xi\colon\partial_\infty\Gamma\to\mathsf{Gr}_{2q+1}(\mathbb{R}^{4q+2})$. Then Γ is virtually free or virtually a surface group.*

The group Γ is virtually free (resp. a surface group) if it contains a finite-index subgroup which is free (resp. a surface group). The map ξ is called dynamics preserving whenever $\gamma\in\Gamma$ is an infinite order element, $\rho(\gamma)$ is P_k-proximal and $\xi(\gamma^+)$ is its attracting fixed point in $\mathsf{Gr}_{2q+1}(\mathbb{R}^{4q+2})$. An analogue of Theorem 1.1 does not hold in dimensions which are multiples of 4, see Section 4.

Corollary 1.2. *Let G_{4q+2} be either $\mathsf{GL}(4q+2,\mathbb{R})$ or $\mathsf{PGL}(4q+2,\mathbb{R})$. If Γ is a word hyperbolic group and $\rho\colon\Gamma\to\mathsf{G}_{4q+2}$ is a P_{2q+1}-Anosov representation, then Γ is virtually free or virtually a surface group.*

Let $\tau_k^+\colon\mathsf{Gr}_k(\mathbb{R}^d)\to\mathbb{P}(\wedge^k\mathbb{R}^d)$ be the Plücker embedding (see subsection 2.1). By using the connectedness properties of the boundary of a rigid hyperbolic group with the methods of the proof of Theorem 1.1 we have:

Corollary 1.3. *Let Γ be a torsion free rigid word hyperbolic group and $\rho\colon\Gamma\to\mathsf{GL}(4q+2,\mathbb{R})$ be a representation. Suppose there exists a continuous ρ-equivariant map $\xi\colon\partial_\infty\Gamma\to\mathsf{Gr}_{2q+1}(\mathbb{R}^{4q+2})$. Then the map ξ is nowhere dynamics preserving and $\tau_{2q+1}^+\circ\xi$ is not spanning.*

The map ξ is called nowhere dynamics preserving if for every infinite order element $\gamma\in\Gamma$ the restriction of ξ on $\{\gamma^-,\gamma^+\}$ is not dynamics preserving.

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2. Background

In this section, we provide some background on proximality, define Anosov representations and state Benoist's results that we are going to use for the proof of the main theorem.

2.1. Proximality. Let $d \geqslant 2$ and $e_1, \dots, e_d$ be the canonical basis of $\mathbb{R}^d$. For an element $g \in \mathsf{GL}(d, \mathbb{R})$ we denote by

$$\lambda_1(g) \geqslant \lambda_2(g) \geqslant \cdots \geqslant \lambda_d(g)$$

the moduli of its eigenvalues. For $1 \leqslant k \leqslant \frac{d}{2}$, we denote by P_k the stabilizer of the plane $\langle e_1, \dots, e_k \rangle$ and by P_k^- the stabilizer of the complementary $(d-k)$-plane $\langle e_{k+1}, \dots, e_d \rangle$. The Grassmannian of k-planes, $\mathsf{Gr}_k(\mathbb{R}^d)$ is identified with the quotient manifold $\mathsf{GL}(d, \mathbb{R})/P_k$. Similarly $\mathsf{Gr}_{d-k}(\mathbb{R}^d)$ is identified with $\mathsf{GL}(d, \mathbb{R})/P_k^-$. A pair of planes

$$(V^+, V^-) \in \mathsf{Gr}_k(\mathbb{R}^d) \times \mathsf{Gr}_{d-k}(\mathbb{R}^d)$$

is *transverse* if there exists $h \in \mathsf{GL}(d, \mathbb{R})$ such that

$$V^+ = h\langle e_1, \dots, e_k \rangle \quad \text{and} \quad V^- = h\langle e_{k+1}, \dots, e_d \rangle.$$

An element $g \in \mathsf{GL}(d, \mathbb{R})$ is called P_k*-proximal* if

$$\lambda_k(g) > \lambda_{k+1}(g).$$

Equivalently, g has two fixed points $x_g^+ \in \mathsf{Gr}_k(\mathbb{R}^d)$ and $V_g^- \in \mathsf{Gr}_{d-k}(\mathbb{R}^d)$ such that the pair (x_g^+, V_g^-) is transverse and for every k-plane V_0 transverse to V_g^- we have

$$\lim_n g^n V_0 = x_g^+.$$

The element g is called P_k*-biproximal* if g and g^{-1} are P_k-proximal. We denote by x_g^- the attracting fixed point of g^{-1} in $\mathsf{Gr}_k(\mathbb{R}^d)$. For $k = 1$, a P_1-proximal element $g \in \mathsf{GL}(d, \mathbb{R})$ in $\mathbb{P}(\mathbb{R}^d)$ has a unique eigenvalue, $\ell_1(g)$, of maximum modulus with multiplicity exactly one. The matrix g is called P_1*-positively proximal* if $\ell_1(g) > 0$.

The Plücker embeddings

$$\tau_k^+ : \mathsf{Gr}_k(\mathbb{R}^d) \to \mathbb{P}(\wedge^k \mathbb{R}^d) \quad \text{and} \quad \tau_k^- : \mathsf{Gr}_{d-k}(\mathbb{R}^d) \to \mathsf{Gr}_{d_k - 1}(\wedge^k \mathbb{R}^d), \ d_k = \binom{d}{k},$$

are defined as follows

$$\tau_k^+(gP_k) = \big[ge_1 \wedge \cdots \wedge ge_k\big] \quad \text{and} \quad \tau_k^-(gP_k^-) = \big[(\wedge^k g) W_k\big],$$

where

$$W_k = \langle e_{i_1} \wedge \cdots \wedge e_{i_k} : \{i_1, \dots, i_k\} \neq \{1, \dots, k\} \rangle.$$

The maps τ_k^+ and τ_k^- define embeddings of $\mathsf{Gr}_k(\mathbb{R}^d)$ and $\mathsf{Gr}_{d-k}(\mathbb{R}^d)$ into $\mathbb{P}(\wedge^k \mathbb{R}^d)$ and $\mathsf{Gr}_{d_k-1}(\wedge^k \mathbb{R}^d)$ respectively. An element $g \in \mathsf{GL}(d, \mathbb{R})$ is P_k-proximal if and only if $\tau_k^+(g)$ is P_1-proximal (see also [13, Proposition 3.3] for more details).

From now, unless specified, proximal (resp. positively proximal) will refer to P_1-proximality (resp. positive P_1-proximality) in the projective space.

2.2. Dynamics preserving maps. Let Γ be a word hyperbolic group and denote by $\partial_\infty\Gamma$ its Gromov boundary. Every infinite order element $\gamma \in \Gamma$ has exactly two fixed points γ^+ and γ^- on $\partial_\infty\Gamma$ called the attracting and repelling fixed points of γ respectively. Let $\rho\colon \Gamma \to \mathsf{GL}(d,\mathbb{R})$ be a representation and $1 \leqslant k \leqslant d-1$. Suppose there exists a continuous ρ-equivariant map $\xi\colon \partial_\infty\Gamma \to \mathsf{Gr}_k(\mathbb{R}^d)$. The map ξ is called *dynamics preserving* if for every element $\gamma \in \Gamma$ of infinite order, $\rho(\gamma)$ is P_k-proximal and $\xi(\gamma^+) = x^+_{\rho(\gamma)}$. The map ξ is called *nowhere dynamics preserving* if for every $\gamma \in \Gamma$ the restriction of ξ on $\partial_\infty\langle\gamma\rangle = \{\gamma^-,\gamma^+\}$ is not dynamics preserving.

2.3. Anosov representations. The dynamical definition of Anosov representations (see [14, 19]) involves the geodesic flow of a word hyperbolic group. Characterizations of Anosov representations into real reductive Lie groups, without involving flow spaces, have been established in several papers, see [4, 13, 15, 18]. Here we define Anosov representations by using a characterization of Kapovich–Leeb–Porti in [15] and Bochi–Potrie–Sambarino [4]. For a finitely generated group Γ we always fix a left-invariant word metric and for $\gamma \in \Gamma$, $|\gamma|_\Gamma$ is the distance of γ from the identity element of Γ. For an element $g \in \mathsf{GL}(d,\mathbb{R})$ let

$$\sigma_1(g) \geqslant \sigma_2(g) \geqslant \cdots \geqslant \sigma_d(g)$$

be the singular values of g. Recall that for each i,

$$\sigma_i(g) = \sqrt{\lambda_i(gg^t)},$$

where g^t is the transpose of g. Notice that for an element $[h] \in \mathsf{PGL}(d,\mathbb{R})$ the ratio $\frac{\sigma_i(h)}{\sigma_{i+1}(h)}$ does not depend on the choice of the representative $h \in \mathsf{GL}(d,\mathbb{R})$.

Let $\mathsf{G_d}$ be either $\mathsf{GL}(d,\mathbb{R})$ or $\mathsf{PGL}(d,\mathbb{R})$, $\rho\colon \Gamma \to \mathsf{G}_d$ a representation and $1 \leqslant k \leqslant \frac{d}{2}$. Then ρ is P_k-Anosov if and only if there exist $C, \alpha > 0$ such that

$$\frac{\sigma_k(\rho(\gamma))}{\sigma_{k+1}(\rho(\gamma))} \geqslant Ce^{\alpha|\gamma|_\Gamma}$$

for every $\gamma \in \Gamma$.

It is clear from the previous definition that for every quasiconvex subgroup H of Γ the restriction $\rho|_H$ is P_k-Anosov. The following theorem summarizes some of the properties of Anosov representations.

Theorem 2.1 ([14, 19]). *Let G_d be either $\mathsf{GL}(d,\mathbb{R})$ or $\mathsf{PGL}(d,\mathbb{R})$ and Γ be a word hyperbolic group. Suppose $1 \leqslant k \leqslant \frac{d}{2}$ and $\rho\colon \Gamma \to \mathsf{G}_d$ is a P_k-Anosov representation. Then:*

(i) *ρ is a quasi-isometric embedding, i.e. there exist constants $A, C > 0$ such that for every $\gamma \in \Gamma$*

$$\frac{1}{C}|\gamma|_\Gamma - A \leqslant \log \frac{\sigma_1(\rho(\gamma))}{\sigma_d(\rho(\gamma))} \leqslant C|\gamma|_\Gamma + A.$$

(ii) *There exist continuous ρ-equivariant maps*

$$\xi_\rho^k : \partial_\infty\Gamma \to \mathsf{Gr}_k(\mathbb{R}^d) \quad \textit{and} \quad \xi_\rho^{d-k} : \partial_\infty\Gamma \to \mathsf{Gr}_{d-k}(\mathbb{R}^d),$$

which are dynamics preserving and for distinct points $x, y \in \partial_\infty\Gamma$ the pair $(\xi_\rho^k(x), \xi_\rho^{d-k}(y))$ is transverse.

(iii) *The set of P_k-Anosov representations of Γ in G_d is open in* $\mathrm{Hom}(\Gamma, \mathsf{G}_d)$.

Notice that by the previous definition, the representation ρ is P_k-Anosov if and only if $\wedge^k\rho$ is P_1-Anosov. The Anosov limit maps of $\wedge^k\rho$ are $\tau_{d,k}^+ \circ \xi_\rho^k$ and $\tau_{d,k}^- \circ \xi_\rho^{d-k}$.

We also need the following fact which implies the continuity of the first eigenvalue among P_1-Anosov representations.

Fact 2.2. Let $\{A_t\}_{t\in[0,1]}$ be a continuous family of proximal elements of $\mathsf{GL}(d,\mathbb{R})$. Then, the function $t \mapsto \ell_1(A_t)$ is continuous.

2.4. The work of Benoist. We summarize here some results that we use from [2] and [3]. An open cone $C \subset \mathbb{R}^d$ is called *properly convex* if it does not contain an affine line. A domain $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is called *properly convex* if it is contained in some affine chart of $\mathbb{P}(\mathbb{R}^d)$ in which Ω is bounded and convex. An element $g \in \mathsf{GL}(d,\mathbb{R})$ is called positively semi-proximal if $\lambda_1(g)$ is an eigenvalue of g. A subgroup Γ of $\mathsf{GL}(d,\mathbb{R})$ is called *positively proximal* if it contains a proximal element and every proximal element of Γ is positively proximal.

Lemma 2.3 ([3, Lemma 3.2]). *Let Γ be a subgroup of* $\mathsf{GL}(d,\mathbb{R})$ *which preserves a properly convex open cone C in $\mathbb{R}^d$. Then every $\gamma \in \Gamma$ is positively semi-proximal. In particular, every proximal element $\gamma \in \Gamma$ is positively proximal.*

Benoist characterized irreducible subgroups of $\mathsf{GL}(d,\mathbb{R})$ which preserve a properly convex cone in $\mathbb{R}^d$ as follows:

Theorem 2.4 ([2, Proposition 1.1]). *Let Γ be an irreducible subgroup of* $\mathsf{GL}(d,\mathbb{R})$. *Then Γ preserves a properly convex open cone C in $\mathbb{R}^d$ if and only if Γ is positively proximal.*

We also have the following fact for subgroups of $\mathsf{GL}(d,\mathbb{R})$ which preserve properly convex domains in $\mathbb{P}(\mathbb{R}^d)$:

Fact 2.5. Let Γ be a subgroup of $\mathsf{GL}(d,\mathbb{R})$ which preserves a properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^d)$. There exists a representation $\tilde{\iota}: \Gamma \to \mathsf{GL}(d,\mathbb{R})$ and a group homomorphism $\varepsilon: \Gamma \to \mathbb{Z}/2$ such that: $\tilde{\iota}(\gamma) = (-1)^{\varepsilon(\gamma)}\gamma$ for every $\gamma \in \Gamma$ and $\tilde{\iota}(\Gamma)$ preserves a properly convex open cone C lifting Ω. Thus, if Γ is also finitely generated the group $\Gamma_2 := \bigcap\{H : [\Gamma : H] \leqslant 2\}$ has finite-index in Γ and preserves the properly convex cone C.

We will also use the following fact:

Proposition 2.6. *Let Γ be a word hyperbolic group and $\rho\colon \Gamma \to \mathsf{GL}(d,\mathbb{R})$ be a representation. If there exists a continuous ρ-equivariant non-constant map $\xi\colon \partial_\infty\Gamma \to \mathbb{P}(\mathbb{R}^d)$, then ρ is discrete and* $\ker(\rho)$ *is finite.*

Proof. Assume that there exists an infinite sequence $(\gamma_n)_{n\in\mathbb{N}}$ of distinct elements of Γ with $\lim_n \rho(\gamma_n) = I_d$. The group Γ acts on $\partial_\infty\Gamma$ as a convergence group, hence up to subsequence, there exists $\eta, \eta' \in \partial_\infty\Gamma$ with $\lim_n \gamma_n x = \eta$ for $x \neq \eta'$ and $\xi(x) = \xi(\eta)$ for $x \neq \eta'$. Since $\partial_\infty\Gamma$ is perfect, ξ has to be constant, a contradiction. □

Let F_k be the free group on k generators. We close this section with the following proposition which follows by the work of Breuillard–Green–Guralnick–Tao (see [6, Theorem 4.1]):

Proposition 2.7 ([6]). *The set of Zariski dense representations from F_2 in $\mathsf{SL}(d,\mathbb{R})$ is dense in the representation variety* $\mathrm{Hom}(F_2, \mathsf{SL}(d,\mathbb{R}))$.

3. Proof of the main result

In this section we give the proof of Theorem 1.1. First, we need the following lemma which is proved using a theorem of Kapovich–Leeb–Porti [16] (see also [7]).

Lemma 3.1. *Let Γ be a torsion free non-elementary word hyperbolic group and $\rho\colon \Gamma \to \mathsf{GL}(d,\mathbb{R})$ be a representation which admits a continuous ρ-equivariant map $\xi\colon \partial_\infty\Gamma \to \mathbb{P}(\mathbb{R}^d)$. Suppose there exists $\gamma \in \Gamma$ such that $\rho(\gamma)$ is biproximal,*

$$\xi(\gamma^+) = x^+_{\rho(\gamma)} \quad \text{and} \quad \xi(\gamma^-) = x^-_{\rho(\gamma)}.$$

Then, there exist $a, b \in \Gamma$ such that $\langle a, b\rangle$ is a free quasiconvex subgroup of Γ of rank 2 and the restricted representation $\rho\colon \langle a, b\rangle \to \mathsf{GL}(d,\mathbb{R})$ is P_1-Anosov with Anosov limit map ξ.

Proof. By Proposition 2.6, the representation ρ is discrete and faithful. Let $t \in \Gamma$ be an infinite order element such that

$$\{\gamma^+, \gamma^-\} \cap \{t^+, t^-\}$$

is empty. Note that

$$\lim_n t^n \gamma^{\pm} = t^+ \quad \text{and} \quad \lim_n t^{-n}\gamma^{\pm} = t^-,$$

so we may find $m > 0$ such that

$$\{t^m\gamma^+, t^m\gamma^-\} \cap \{\gamma^+, \gamma^-\} \quad \text{and} \quad \{t^{-m}\gamma^+, t^{-m}\gamma^-\} \cap \{\gamma^+, \gamma^-\}$$

are empty. Up to conjugating ρ we may assume that

$$x^+_{\rho(\gamma)} = [e_1], \quad x^+_{\rho(\gamma^{-1})} = [e_d]$$

and

$$V^-_{\rho(\gamma)} = \langle e_2, \dots, e_d \rangle, \quad V^-_{\rho(\gamma^{-1})} = \langle e_1, \dots, e_{d-1} \rangle.$$

Then we notice that

$$\rho(t^{\pm m}) x^+_{\rho(\gamma)} \notin \mathbb{P}\big(V^-_{\rho(\gamma)}\big) \cup \mathbb{P}\big(V^-_{\rho(\gamma^{-1})}\big)$$

and

$$\rho(t^{\pm m}) x^-_{\rho(\gamma)} \notin \mathbb{P}\big(V^-_{\rho(\gamma)}\big) \cup \mathbb{P}\big(V^-_{\rho(\gamma^{-1})}\big).$$

For example, suppose that $\rho(t^m)x^+_{\rho(\gamma)} \in \mathbb{P}(V^-_{\rho(\gamma)})$, then

$$\lim_n \rho(\gamma^n)\rho(t^m)x^+_{\rho(\gamma)} = \lim_n \xi(\gamma^n t^m \gamma^+) = \xi(\gamma^+) = [e_1]$$

has to be in $\mathbb{P}(V^-_{\rho(\gamma)})$, a contradiction. Since, $\lim_n \gamma^n t^{-m} \gamma^+ = \gamma^+$ we have

$$\lim_n \rho(\gamma^n t^{-m})\xi(\gamma^+) = x^+_{\rho(\gamma)} \quad \text{and} \quad \rho(t^{-m})x^+_{\rho(\gamma^{-1})} \notin \mathbb{P}\big(V^-_{\rho(\gamma)}\big).$$

Then, by [16, Theorem 7.40] (see also [7, Theorem A2]), there exists $N > 0$ such that the group $H = \langle \gamma^N, t^m \gamma^N t^{-m} \rangle$ is a free group of rank 2 and the restriction $\rho|_H$ is P_1-Anosov. The restriction $\rho|_H$ is also a quasi-isometric embedding hence H is a quasiconvex subgroup of Γ and its Anosov limit map is the restriction of ξ on $\partial_\infty H$ considered as a subset of $\partial_\infty \Gamma$. □

Recall that for a finitely generated group Γ, Γ_2 is defined to be the intersection of all finite-index subgroups of Γ of index at most 2.

Lemma 3.2. *Let Γ be a torsion free one-ended word hyperbolic group and $\rho\colon \Gamma * \mathbb{Z} \to \mathsf{GL}(d,\mathbb{R})$ be a representation which admits a ρ-equivariant continuous map $\xi\colon \partial_\infty(\Gamma * \mathbb{Z}) \to \mathbb{P}(\mathbb{R}^d)$. Suppose that $\delta \in \Gamma_2$ is a non-trivial element such that $\rho(\delta)$ is biproximal and $\xi(\delta^+) = x^+_{\rho(\delta)}$ and $\xi(\delta^-) = x^-_{\rho(\delta)}$. Then $\rho(\delta)$ is positively proximal.*

Proof. Let s be a generator of the free cyclic factor, $t = s\delta s^{-1} \in \Gamma$ and notice that $\rho(t)$ is proximal with $\rho(s)x^+_{\rho(\delta)} = x^+_{\rho(t)} = \xi(t^+)$ and $t^\pm \notin \partial_\infty\Gamma$. If $x \in \partial_\infty\Gamma$,

$$\lim_n \rho(t^n)\xi(x) = \lim_n \xi(t^n x) = \xi(t^+).$$

Since $\rho(t)$ preserves $V^-_{\rho(t)}$ and $\lim_n t^n x = t^+$, $\xi(x)$ cannot lie in $\mathbb{P}(V^-_{\rho(t)})$. It follows that $\xi(\partial_\infty\Gamma)$ lies in the affine chart

$$\mathbb{P}(\mathbb{R}^d) - \mathbb{P}\big(V^-_{\rho(t)}\big).$$

Let $V = \langle \xi(\partial_\infty \Gamma) \rangle$ and we consider the representation $\rho' \colon \Gamma \to \mathsf{GL}(V)$ where $\rho'(\gamma) = \rho|_V(\gamma)$, $\gamma \in \Gamma$. The map ξ is not constant, hence ρ' is discrete and faithful. The map $\xi \colon \partial_\infty \Gamma \to \mathbb{P}(V)$ is ρ'-equivariant, $\rho'(\delta)$ is proximal with attracting fixed point $\xi(\delta^+)$ and $\ell_1(\rho(\delta)) = \ell_1(\rho'(\delta))$.

Then we notice that $\xi(\partial_\infty \Gamma)$ also lies in the affine chart

$$A = \mathbb{P}(V) - \mathbb{P}\big(V \cap V^-_{\rho(t)}\big)$$

of $\mathbb{P}(V)$. Since Γ is one-ended, $\partial_\infty \Gamma$ and $\xi(\partial_\infty \Gamma)$ are connected. The convex hull of $\xi(\partial_\infty \Gamma)$ in A, say $\mathcal{C}$, is bounded and convex in A and has non-empty interior since $\xi(\partial_\infty \Gamma)$ spans V. Then $\rho'(\Gamma)$ preserves $\xi(\partial_\infty \Gamma)$ and by [8, Proposition 2.8] it also preserves $\mathcal{C}$. It follows that $\rho'(\Gamma)$ preserves the non-empty properly convex set

$$\Omega = \mathrm{Int}(\mathcal{C}) \subset \mathbb{P}(V).$$

Fact 2.5 shows that there exists a representation $\widetilde{\rho}' \colon \Gamma \to \mathsf{GL}(V)$ which preserves a properly convex cone $C \subset V$ and $\rho'(\gamma) = \widetilde{\rho}'(\gamma)$ for every $\gamma \in \Gamma_2$. By Lemma 2.3, $\rho(\delta)$ is positively proximal in $\mathbb{P}(V)$ and hence in $\mathbb{P}(\mathbb{R}^d)$. □

A torsion free word hyperbolic group Γ is called *rigid* if it does not admit a non-trivial splitting over a cyclic subgroup. For example, the fundamental group of a closed negatively curved Riemannian manifold of dimension at least 3 is rigid. By a theorem of Bowditch [5] the Gromov boundary $\partial_\infty \Gamma$ of a rigid hyperbolic group Γ does not contain local cut points.

Lemma 3.3. *Let Γ be a torsion free rigid one-ended word hyperbolic group. Let $\rho \colon \Gamma \to \mathsf{GL}(d, \mathbb{R})$ be a representation which admits a continuous ρ-equivariant map $\xi \colon \partial_\infty \Gamma \to \mathbb{P}(\mathbb{R}^d)$. Suppose that $\delta \in \Gamma_2$ is a non-trivial element such that $\rho(\delta)$ is biproximal and $\xi(\delta^+) = x^+_{\rho(\delta)}$ and $\xi(\delta^-) = x^-_{\rho(\delta)}$. Then $\rho(\delta)$ is positively proximal.*

Proof. Since $\partial_\infty \Gamma$ does not have any local cut points, the set $\partial_\infty \Gamma - \{\delta^+, \delta^-\}$ is connected. For $x \neq \delta^+, \delta^-$ we have that $\lim_n \delta^{\pm n} x = \delta^{\pm}$ and, as in Lemma 3.2, the connected set $\xi(\partial_\infty \Gamma - \{\delta^+, \delta^-\})$ is contained in

$$\mathbb{P}(\mathbb{R}^d) - \mathbb{P}\big(V^-_{\rho(\delta)}\big) \cup \mathbb{P}\big(V^-_{\rho(\delta^{-1})}\big).$$

Note that the two $(d-1)$-planes $V^-_{\rho(\delta)}$ and $V^-_{\rho(\delta^{-1})}$ are distinct, hence by the connectedness of $\partial_\infty \Gamma - \{\delta^+, \delta^-\}$ we can find a hyperplane V_0 such that $\xi(\partial_\infty \Gamma)$ is contained in $\mathbb{P}(\mathbb{R}^d) - \mathbb{P}(V_0)$. Then we consider the restriction $\rho' \colon \Gamma \to \mathsf{GL}(V)$, $V = \langle \xi(\partial_\infty \Gamma) \rangle$, whose image preserves the compact connected subset $\xi(\partial_\infty \Gamma)$ of the affine chart

$$\mathbb{P}(V) - \mathbb{P}(V \cap V_0)$$

of $\mathbb{P}(V)$. The element $\rho'(\gamma)$ is proximal in $\mathbb{P}(V)$ and $\ell_1(\rho(\gamma)) = \ell_1(\rho'(\gamma))$. We similarly conclude that $\rho'(\Gamma)$ preserves a properly convex domain Ω of $\mathbb{P}(V)$. Again, Fact 2.5 guarantees that $\rho'(\Gamma_2)$ preserves a properly convex cone of V and $\ell_1(\rho'(\delta)) > 0$. □

Now we combine the previous results to prove Theorem 1.1.

Theorem 1.1. *Let Γ be a word hyperbolic group and $\rho\colon \Gamma \to \mathsf{GL}(4q+2,\mathbb{R})$ a representation. Suppose that there exists a continuous, ρ-equivariant dynamics preserving map $\xi\colon \partial_\infty\Gamma \to \mathsf{Gr}_{2q+1}(\mathbb{R}^{4q+2})$. Then Γ is virtually free or virtually a surface group.*

Proof. We first assume that Γ is a torsion free hyperbolic group. By Proposition 2.6, ρ is faithful and we may assume that $\rho(\Gamma)$ is a subgroup of $\mathsf{SL}(4q+2,\mathbb{R})$. If not, we replace ρ with the representation

$$\widehat{\rho}\colon \Gamma \to \mathsf{SL}^{\pm}(n,\mathbb{R}), \quad \widehat{\rho}(\gamma) = |\det(\rho(\gamma))|^{-1/(4q+2)}\rho(\gamma)$$

and Γ with a finite-index subgroup Γ_0 such that $\widehat{\rho}(\Gamma_0)$ is a subgroup of $\mathsf{SL}(4q+2,\mathbb{R})$. Notice that $\widehat{\rho}$ has to be faithful since ξ is $\widehat{\rho}$-equivariant and dynamics preserving for $\widehat{\rho}$.

Let $V_q = \wedge^{2q+1}\mathbb{R}^{4q+2}$, and notice by assumption that $\xi_q = \tau^{+}_{2q+1}\circ\xi$ is $\wedge^{2k+1}\rho$-equivariant and dynamics preserving. We consider the following two cases:

Case 1. Suppose that Γ has infinitely many ends. Then we show that Γ is free. If not, by Stallings' theorem [21], there exists a splitting

$$\Gamma = \Gamma_1 * \cdots * \Gamma_k * F_s,$$

where $s \geqslant 0$ and for $1 \leqslant i \leqslant k$, Γ_i is an one-ended word hyperbolic group. In particular, there exists a quasiconvex subgroup of Γ of the form $\Delta * \mathbb{Z}$, with Δ one-ended. Lemma 3.1, shows that there exists a quasiconvex free subgroup H_0 of Δ_2 such that $\wedge^{2q+1}\rho(H_0)$ is P_1-Anosov in $\mathsf{SL}(V_q)$ and its limit map is the restriction

$$\xi_q\colon \partial_\infty H_0 \to \mathbb{P}(V_q).$$

Since $\wedge^{2q+1}\rho(\delta)$ is proximal for every $\delta \in H_0 \subset \Delta_2$, by Lemma 3.2,

$$\ell_1\big(\wedge^{2q+1}(\rho(\delta))\big) > 0.$$

The representation $\rho\colon H_0 \to \mathsf{SL}(4q+2,\mathbb{R})$ is P_{2q+1}-Anosov and $\wedge^{2q+1}\rho(\gamma)$ is positively proximal for every non-trivial $\gamma \in H_0$. By Theorem 2.1 (iii), we can find a path connected open neighbourhood U of $\rho_0 := \rho|_{H_0}$ in $\mathrm{Hom}(H_0, \mathsf{SL}(4q+2,\mathbb{R}))$ consisting of entirely of P_{2q+1}-Anosov representations. Proposition 2.7 guarantees that there exists $\rho_1 \in U$ such that $\rho_1(F_k)$ is Zariski dense in $\mathsf{SL}(4q+2,\mathbb{R})$. Let $\{\rho_t\}_{0\leqslant t\leqslant 1}$ be a continuous path between ρ_0 and ρ_1 contained entirely in U. By Fact 2.2, for every $\gamma \in H_0$, the map $t \mapsto \ell_1(\wedge^{2q+1}\rho_t(\gamma))$ is continuous with real values and nowhere vanishing. Hence

$$\ell_1\big(\wedge^{2q+1}\rho_1(\gamma)\big) > 0$$

for every $\gamma \in H_0$. Therefore, since $\wedge^{2k+1}$ is an irreducible representation, the group $\wedge^{2q+1}\rho_1(H_0)$ is a strongly irreducible subgroup of $\mathsf{SL}(V_q)$ which is positively proximal. By Theorem 2.4, the group $\wedge^{2q+1}\rho_1(H_0)$ preserves a properly convex cone and hence a properly convex domain of $\mathbb{P}(V_q)$. On the other hand, the group $\wedge^{2q+1}\mathsf{SL}(4q+2,\mathbb{R})$ (and hence $\wedge^{2q+1}\rho_1(H_0)$) preserves the symplectic non-degenerate form $\omega_q\colon V_q \times V_q \to \mathbb{R}$ given by the formula

$$\omega_q(a,b) = a \wedge b \in \langle e_1 \wedge \cdots \wedge e_{4q+2}\rangle.$$

However, by [2, Corollary 3.5], a strongly irreducible subgroup of $\mathsf{SL}(d,\mathbb{R})$ which preserves a symplectic form cannot preserve a properly convex domain of $\mathbb{P}(\mathbb{R}^d)$. We have reached a contradiction, so Γ cannot contain any non-trivial one-ended factors in its free product decomposition. Therefore, Γ is free.

Case 2. Suppose that Γ is one-ended and not virtually a surface group. Wilton's result [23, Corollary B] ensures that Γ contains a quasiconvex subgroup Δ which is either isomorphic to a surface group or rigid. If Δ has infinite index in Γ, then there exists a quasiconvex subgroup of Γ isomorphic to $\Delta * \mathbb{Z}$. However, by the previous case we obtain a contradiction. Therefore, we may assume that Δ is rigid and has finite index in Γ. By Lemma 3.1, there exists H_1 a quasiconvex free subgroup of Δ_2 such that the restriction $\wedge^{2q+1}\rho|_{H_1}$ is P_1-Anosov. By Lemma 3.3, for every $h \in H_1$, $\wedge^{2q+1}\rho(h)$ is positively proximal in $\mathbb{P}(V_q)$. By continuing as previously, we obtain a P_{2q+1}-Anosov, Zariski dense deformation ρ_1 of $\rho|_{H_1}$ such that $\wedge^{2q+1}\rho_1(H_1)$ is positively proximal. Again, by Theorem 2.4, $\wedge^{2q+1}\rho_1(H_1)$ preserves a properly convex domain and the symplectic form ω_q, a contradiction.

We now consider the general case where Γ might have torsion or ρ is not faithful. If ρ is not faithful, Proposition 2.6 shows that $\ker(\rho)$ is finite. The group $\Gamma' = \Gamma/\ker\rho$ is word hyperbolic, $\partial_\infty\Gamma' = \partial_\infty\Gamma$, so ξ is a ρ'-equivariant dynamics preserving map, where $\rho'\colon \Gamma' \to \mathsf{GL}(4q+2,\mathbb{R})$ is the faithful representation induced by ρ. By Selberg's lemma, there exists a torsion free finite-index subgroup Γ_1 of Γ'. The previous arguments imply that Γ_1 is either a surface group or a free group. Therefore, Γ is either a finite extension of a virtually free group or a virtually surface group. In the second case, its boundary is the circle and by [12], Γ is virtually a surface group. In the first case, by [11], Γ has infinitely many ends and splits as the fundamental group of a finite graph of groups with finite edge groups and vertex groups of at most one end. The vertex groups of this splitting are also finite extensions of a virtually free group hence finite. It follows that Γ is virtually free. □

By following the argument of case 1 in the proof of Theorem 1.1 we obtain the following conclusion:

Theorem 3.4. *Let F_2 be the free group on two generators and $\rho\colon F_2 \to \mathsf{GL}(4q+2,\mathbb{R})$ a representation. Suppose that ρ is P_{2q+1}-Anosov. Then $\wedge^{2q+1}\rho(F_2)$ is not a positively proximal subgroup of* $\mathsf{GL}(\wedge^{2q+1}\mathbb{R}^{4q+2})$.

For the proof of Corollary 1.2 we need the following proposition for the existence of lifts of P_{2k+1}-Anosov representations into $\mathsf{PGL}(d,\mathbb{R})$. The proof is similar to

Lemma 3.2 and 3.3. In the case ρ is irreducible and $k = 0$, Zimmer has proved the existence of lifts in [24, Theorem 3.1].

Proposition 3.5. *Let Γ be a torsion free word hyperbolic group and $\rho\colon \Gamma \to \mathsf{PGL}(d,\mathbb{R})$ is a P_{2k+1}-Anosov representation, where $0 \leqslant k \leqslant \frac{d-1}{4}$.*

(i) *Suppose that Δ is an infinite index, one-ended quasiconvex subgroup of Γ and ρ_0 is the restriction of ρ on Δ. There exists a lift $\widetilde{\rho_0}\colon \Delta \to \mathsf{GL}(d,\mathbb{R})$ such that $\wedge^{2k+1}\widetilde{\rho_0}(\Delta)$ is positively proximal.*

(ii) *If Γ is a rigid word hyperbolic group then there exists a lift $\widetilde{\rho}\colon \Gamma \to \mathsf{GL}(d,\mathbb{R})$ of ρ such that $\wedge^{2k+1}\rho(\Gamma)$ is positively proximal.*

Proof. We begin with the following observation: suppose that $\varphi\colon \Gamma \to \mathsf{PGL}(V_1 \oplus V_2)$ is a representation such that $\varphi(\gamma)$ preserves V_1 for every $\gamma \in \Gamma$. If $\rho(\gamma) = [g_\gamma]$ then the map $\varphi_0(\gamma) = [g_\gamma|_{V_1}]$ is a well defined representation $\varphi_0\colon \Gamma \to \mathsf{PGL}(V_1)$. If φ_0 admits a lift $\widetilde{\varphi}_0$, then there exists a lift $\widetilde{\varphi}$ of φ such that

$$\widetilde{\varphi}(\gamma)|_{V_1} = \widetilde{\varphi}_0(\gamma)$$

for every $\gamma \in \Gamma$. The lift $\widetilde{\varphi}$ is defined as follows: for $\gamma \in \Gamma$, $\widetilde{\varphi}(\gamma)$ is the unique element $h_\gamma \in \mathsf{GL}(V_1 \oplus V_2)$ such that the restriction of h_γ on V_1 is $\widetilde{\varphi}_0(\gamma)$ and $\varphi(\gamma) = [h_\gamma]$.

Notice that we may assume that $k = 0$, because the exterior power

$$\wedge^{2k+1}\colon \mathsf{GL}(d,\mathbb{R}) \to \mathsf{GL}\big(\wedge^{2k+1}\mathbb{R}^d\big)$$

is faithful. For part (i), we may consider $\delta \in \Gamma$ with $\delta^{\pm} \notin \partial_\infty\Delta$ and $\xi(\partial_\infty\Delta)$ is a connected compact subset of the affine chart

$$\mathbb{P}(\mathbb{R}^d) - \mathbb{P}\big(V^-_{\rho(\delta)}\big).$$

In particular, $\xi(\partial_\infty\Delta)$ lies in the affine chart

$$A = \mathbb{P}(V) - \mathbb{P}\big(V \cap V^-_{\rho(\delta)}\big)$$

of $\mathbb{P}(V)$, where $V = \langle \xi(\partial_\infty\Delta)\rangle$. Since $\rho_0(\Delta)$ preserves V there exists a well defined representation $\rho_1\colon \Delta \to \mathsf{PGL}(V)$. The image $\rho_1(\Delta)$ preserves the connected compact set $\xi(\partial_\infty\Delta)$ and hence the interior of the convex hull of $\xi(\partial_\infty\Delta)$ in A. There exists a lift $\widetilde{\rho_1}$ of ρ_1 into $\mathsf{GL}(V)$ such that $\widetilde{\rho_1}(\Delta)$ preserves a properly convex cone C of V. The representation $\widetilde{\rho_1}$ is P_1-Anosov, faithful and by Lemma 2.3, $\widetilde{\rho_1}(\gamma)$ is positively proximal for every $\gamma \in \Delta$ non-trivial. By our initial observation we obtain a lift $\widetilde{\rho_0}\colon \Delta \to \mathsf{GL}(d,\mathbb{R})$ of ρ_0 with $\widetilde{\rho_0}(\gamma)|_V = \widetilde{\rho_1}(\gamma)$. The representation $\widetilde{\rho_1}$ is P_1-Anosov with Anosov limit map ξ. For every non-trivial $\gamma \in \Delta$, the attracting fixed point of $\widetilde{\rho_0}(\gamma)$ is in V and

$$\ell_1(\widetilde{\rho_0}(\gamma)) = \ell_1(\widetilde{\rho_1}(\gamma)) > 0.$$

The proof of (ii) follows by observing, as in Lemma 3.3, that the image of $\partial_\infty\Gamma$ under the Anosov limit map ξ lies in an affine chart of $\mathbb{P}(\mathbb{R}^d)$. Then we continue as previously to obtain the lift $\widetilde{\rho}$. □

Proof of Corollary 1.2. We first assume that Γ is torsion free. If Γ contains a quasiconvex infinite index one-ended subgroup Γ_0, there exists a lift $\tilde{\rho}_0$ of $\rho|_{\Gamma_0}$ such that the group $\wedge^{2k+1}\tilde{\rho}_0(\Gamma_0)$ is positively proximal, contradicting Theorem 3.4. Also Γ cannot be rigid again by part (ii) of the previous proposition. Therefore, Γ is either free or has one end and by [23, Corollary B] there exists a quasiconvex surface subgroup which has to be of finite index in Γ.

Now suppose that Γ is not torsion free or $\ker\rho$ is non-trivial. We may find a torsion free finite-index subgroup Γ_1 of $\Gamma' = \Gamma/\ker(\rho)$ so that ρ induces the faithful P_{2q+1}-Anosov representation $\rho'\colon \Gamma_1 \to \mathsf{G}_{4q+2}$. The previous step shows that $\partial_\infty\Gamma_1 = \partial_\infty\Gamma$ is either a circle or totally disconnected. By working as in the last paragraph of Theorem 1.1 we conclude that Γ is virtually free or virtually a surface group. □

Proof of Corollary 1.3. Let $\xi\colon \partial_\infty\Gamma \to \mathsf{Gr}_{2q+1}(\mathbb{R}^{4q+2})$ be a continuous ρ-equivariant map. We first show that ξ is nowhere dynamics preserving. Suppose not, i.e. there exists a P_{2q+1}-proximal element $\rho(\gamma) \in \rho(\Gamma)$ with

$$\xi(\gamma^+) = x^+_{\rho(\gamma)} \quad \text{and} \quad \xi(\gamma^-) = x^-_{\rho(\gamma)}.$$

The map

$$\xi^+ := \tau^+_{2q+1} \circ \xi$$

is $\wedge^{2q+1}\rho$-equivariant and by Lemma 3.1 there exist a free quasiconvex subgroup H of Γ_2 such that $\wedge^{2q+1}\rho|_H$ is P_1-Anosov. Lemma 3.3 shows that $\wedge^{2q+1}\rho(H)$ is positively proximal, a contradiction by Theorem 3.4.

Let

$$V_q = \wedge^{2q+1}\mathbb{R}^{4q+2} \quad \text{and} \quad \xi^- = \tau^-_{2q+1} \circ \xi.$$

We show that the map ξ^+ cannot be spanning. Suppose that ξ^+ is spanning and $x_1, \dots, x_r \in \partial_\infty\Gamma$ with

$$V_q = \oplus_{i=1}^r \xi^+(x_i), \quad r = \dim(V_q).$$

Since Γ acts minimally on $\partial_\infty\Gamma$, for every open subset U of $\partial_\infty\Gamma$, $\xi^+(U)$ spans V_q and the union $\cup_{i=1}^r \xi^-(x_i)$ cannot contain $\xi^+(\partial_\infty\Gamma)$. There exists $y \in \partial_\infty\Gamma$ and $1 \leqslant j \leqslant r$ with

$$V_q = \xi^+(x_j) \oplus \xi^-(y) = \xi^+(y) \oplus \xi^-(x_j).$$

By the density of pairs $\{(\delta^+, \delta^-) : \delta \in \Gamma\}$ in the set of 2-tuples of $\partial_\infty\Gamma$, we can find $\gamma \in \Gamma$ such that

$$V_q = \xi(\gamma^+) \oplus \xi^-(\gamma^-) = \xi^+(\gamma^-) \oplus \xi^-(\gamma^+).$$

Then we claim that $g = \wedge^{2q+1}\rho(\gamma)$ is a biproximal matrix. Up to conjugating g we may assume that

$$\xi^+(\gamma^+) = \left[e_1 \wedge \cdots \wedge e_{2q+1}\right] \quad \text{and} \quad \xi^-(\gamma^-) = \left[W_{2q+1}\right],$$

where W_{2q+1} is defined as in Subsection 2.1. We may write

$$g = \begin{bmatrix} a(g) & 0 \\ 0 & A \end{bmatrix}$$

for some matrix $A \in \mathsf{GL}(W_{2q+1})$. Suppose that $\lambda_1(A) \geqslant |a(g)|$. Let $p \geqslant 1$ be the largest possible dimension of a complex Jordan block corresponding to an eigenvalue of maximum modulus of A. Then there exists a subsequence $(k_n)_{n\in\mathbb{N}}$, A_∞ a non-zero matrix and $b \in \mathbb{R}$ with

$$\lim_{n\to\infty} \frac{1}{k_n^{p-1}\lambda_1(A)^{k_n}} g^{k_n} = \begin{bmatrix} b & 0 \\ 0 & A_\infty \end{bmatrix}.$$

Since $\partial_\infty\Gamma$ is perfect and $\xi^+(\partial_\infty\Gamma)$ spans V_q, we may choose $x \in \partial_\infty\Gamma - \{\gamma^-\}$ such that the projection of $\xi^+(x)$ into W_{2q+1} is not in $\ker(A_\infty)$. Thus,

$$\lim_n g^{k_n}\xi^+(x) = \lim_n \xi^+(\gamma^{k_n}x) = \xi^+(\gamma^+)$$

cannot be the line $[e_1 \wedge \cdots \wedge e_{2q+1}]$, a contradiction. It follows that $|a(g)| > \lambda_1(A)$ and $\wedge^{2q+1}\rho(\gamma)$ is proximal with attracting fixed point $\xi^+(\gamma^+)$. Since

$$V_q = \xi^+(\gamma^-) \oplus \xi^-(\gamma^+),$$

the same argument shows that $\wedge^{2q+1}\rho(\gamma^{-1})$ is proximal with attracting fixed point $\xi^+(\gamma^-)$. The map ξ^+ (and hence ξ) preserves the dynamics of $\{\gamma^-, \gamma^+\}$. This contradicts the fact that ξ is nowhere dynamics preserving. Therefore, $\tau^+_{2q+1}(\xi(\partial_\infty\Gamma))$ lies in some proper vector subspace of V_q. □

4. Examples

In this section, we provide an example showing that the analogue of Theorem 1.1 does not hold in dimensions which are multiples of 4. Also, we give an example of a surface group representation ρ into $\mathsf{SL}(4q+2,\mathbb{R})$ which is not P_{2q+1}-Anosov but admits a ρ-equivariant continuous dynamics preserving map ξ into $\mathsf{Gr}_{2q+1}(\mathbb{R}^{4q+2})$.

Let S be a closed orientable hyperbolic surface and $\tau_2\colon \mathsf{SL}(2,\mathbb{C}) \to \mathsf{SL}(4,\mathbb{R})$ be the standard inclusion defined as

$$\tau_2(g) = \begin{bmatrix} \mathrm{Re}(g) & -\mathrm{Im}(g) \\ \mathrm{Im}(g) & \mathrm{Re}(g) \end{bmatrix}$$

for $g \in \mathsf{SL}(2,\mathbb{C})$.

Example 4.1. Let F_2 be the free group on two generators. The group $\Gamma = \pi_1(S) * F_2$ admits an Anosov representation ρ into $\mathsf{SL}(2,\mathbb{C})$ and hence $\tau_2 \circ \rho$ is a P_2-Anosov representation into $\mathsf{SL}(4,\mathbb{R})$. For $k \in \mathbb{N}$, the representation $\rho_k = \oplus_{i=1}^{k}(\tau_2 \circ \rho)$ of Γ into $\mathsf{SL}(4k,\mathbb{R})$ is P_{2k}-Anosov. In fact, by Theorem 2.1 (iii) and Proposition 2.7 there exists a deformation ρ_k' of ρ_k which is Zariski dense and P_{2k}-Anosov.

Example 4.2. Let M be the mapping torus of the closed hyperbolic surface S with respect to a fixed pseudo-Anosov homeomorphism $\phi\colon S \to S$. The group $\pi_1(M)$ contains a normal and infinite index subgroup Γ isomorphic with $\pi_1(S)$. By a theorem of Thurston [22] (see also Otal [20]), the group $\pi_1(M)$ admits a convex cocompact representation ι into $\mathsf{PSL}(2,\mathbb{C})$. In fact, by [10], ι lifts to a quasi-isometric embedding

$$\tilde{\iota}\colon \pi_1(M) \to \mathsf{SL}(2,\mathbb{C}).$$

By composing τ_2 with $\tilde{\iota}$, we obtain a P_2-Anosov representation $\rho_1\colon \pi_1(M) \to \mathsf{SL}(4,\mathbb{R})$. The Cannon–Thurston map (see [9]), $\theta\colon \partial_\infty \pi_1(S) \to \partial_\infty \pi_1(M)$ composed with the Anosov limit map $\xi_{\rho_1}^2\colon \partial_\infty \pi_1(M) \to \mathsf{Gr}_2(\mathbb{R}^4)$ provides a $\rho_1|_\Gamma$-equivariant dynamics preserving map

$$\xi_0\colon \partial_\infty \Gamma \to \mathsf{Gr}_2(\mathbb{R}^4).$$

Note that the representation $\rho_1|_\Gamma$ is not a quasi-isometric embedding, in particular not P_2-Anosov, since Γ is not a quasiconvex subgroup of $\pi_1(M)$. Let $\rho_F\colon \Gamma \to \mathsf{SL}(2,\mathbb{R})$ be a Fuchsian representation with limit map $\xi_{\rho_F}^1$. The representation

$$\rho = (\oplus_{i=1}^{q} \rho_1|_\Gamma) \oplus \rho_F$$

into $\mathsf{SL}(4q+2,\mathbb{R})$ is not P_{2q+1}-Anosov, however the ρ-equivariant map

$$\xi = (\oplus_{i=1}^{q} \xi_0) \oplus \xi_{\rho_F}^1 \colon \partial_\infty \Gamma \to \mathsf{Gr}_{2q+1}\big(\mathbb{R}^{4q+2}\big)$$

is dynamics preserving.

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