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Rigidity of center Lyapunov exponents and su-integrability

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Abstract. Let f be a conservative partially hyperbolic diffeomorphism, which is homotopic to an Anosov automorphism A on $\mathbb{T}^3$. We show that the stable and unstable bundles of f are jointly integrable if and only if every periodic point of f admits the same center Lyapunov exponent with A. This implies every conservative partially hyperbolic diffeomorphism, which is homotopic to an Anosov automorphism on $\mathbb{T}^3$, is ergodic. This proves the Ergodic Conjecture proposed by Hertz–Hertz–Ures on $\mathbb{T}^3$.

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1. Introduction

A diffeomorphism f on a closed Riemannian manifold M is partially hyperbolic if there exists a continuous Df-invariant splitting $TM = E^s \oplus E^c \oplus E^u$ and continuous functions $\sigma, \mu: M \to \mathbb{R}$, such that $0 < \sigma < 1 < \mu$ and

$$\|Df(v^s)\| < \sigma(p) < \|Df(v^c)\| < \mu(p) < \|Df(v^u)\|$$

for every $p \in M$ and unit vector $v^* \in E^*(p)$, for $* = s, c, u$.

Since Pugh and Shub [18] conjectured that stably ergodic diffeomorphisms are dense in the space of C^2 conservative partially hyperbolic diffeomorphisms, ergodicity of partially hyperbolic diffeomorphisms has been one of the main topics of research in differentiable dynamics. A key ingredient of proving ergodicity for partially hyperbolic diffeomorphisms is a property called accessibility. In dimension 3, for instance, it has been showed [4,22] that every conservative accessible partially hyperbolic diffeomorphism is ergodic. Moreover, accessibility [22] is an open dense property for partially hyperbolic diffeomorphisms with one-dimensional center bundle. It seems promising that we can classify 3 dimensional non-ergodic partially hyperbolic diffeomorphisms. Actually, Hertz–Hertz–Ures proposed the following Ergodic Conjecture [21,23]:

Conjecture 1. *If a conservative partially hyperbolic diffeomorphism of a 3-manifold is non-ergodic, then there is a 2-torus tangential to $E^s \oplus E^u$. This implies the only* *orientable 3-manifolds that admit a non-ergodic conservative partially hyperbolic diffeomorphism are:*

1. *the 3-torus* $\mathbb{T}^3$*;*
2. *the mapping torus of* $-\mathrm{Id}$*; or*
3. *the mapping torus of a hyperbolic automorphism of the 2-torus.*

The simplest 3-manifold supporting partially hyperbolic diffeomorphisms is 3-torus $\mathbb{T}^3$. It has been proven in [3, 17] that if $f: \mathbb{T}^3 \to \mathbb{T}^3$ is partially hyperbolic, then the action $f_*: \pi_1(\mathbb{T}^3) = \mathbb{Z}^3 \to \mathbb{Z}^3$ is also partially hyperbolic. This means $f_* \in \mathrm{GL}(3, \mathbb{Z})$ has three real eigenvalues with different moduli. One eigenvalue has modulus larger than 1, and one has modulus smaller than one. So there are two classes of partially hyperbolic diffeomorphisms on $\mathbb{T}^3$:

- either $f_* \in \mathrm{GL}(3, \mathbb{Z})$ has an eigenvalue equal to -1 or 1;
- or $f_* \in \mathrm{GL}(3, \mathbb{Z})$ is Anosov, i.e. every eigenvalue of f_* has modulus not equal to 1.

In the first case, there are partially hyperbolic diffeomorphisms which are non-ergodic. For instance, an Anosov automorphism on 2-torus $\mathbb{T}^2$ times identity map on $\mathbb{S}^1$ is not ergodic. Moreover, it has been shown [12] that if such f is not ergodic, then it admits 2-torus tangent to $E^s \oplus E^u$.

For the second case, it has been shown [14] that there is no 2-torus tangent to $E^s \oplus E^u$. Thus if we want to prove the Ergodic Conjecture on $\mathbb{T}^3$, we need to show that every C^2 conservative partially hyperbolic diffeomorphism, homotopic to an Anosov automorphism on $\mathbb{T}^3$, is ergodic. See also [14, Conjecture 1.11].

In order to prove ergodicity for partially hyperbolic diffeomorphisms on 3-manifolds, the only obstruction is non-accessibility. If f is conservative, partially hyperbolic, and homotopic to an Anosov automorphism on $\mathbb{T}^3$, then f is non-accessible implies that the stable and unstable bundles of f are jointly integrable [14]. This is equivalent to f admits a 2-dimensional invariant foliation tangent to the union of stable and unstable bundles everywhere. We say that such an f is su-integrable.

Hammerlindl and Ures proved the following theorem.

Theorem ([14]). *Let f be a $C^{1+\alpha}$ conservative partially hyperbolic diffeomorphism, which is homotopic to an Anosov automorphism A on $\mathbb{T}^3$. If f is not ergodic, it is topologically conjugate to A.*

Here f is not ergodic is equivalent to f is su-integrable and the integral su-foliation is minimal on $\mathbb{T}^3$. Moreover, Hammerlindl and Ures proved that the topological conjugacy preserves all invariant foliations of f, see Lemma 2.2.

In this paper, we give a necessary and sufficient condition for su-integrability of this kind of diffeomorphisms. Moreover, such kind of f is Anosov by applying Lemma 2.5.

Theorem 1.1. *Let f be a $C^{1+\alpha}$ conservative partially hyperbolic diffeomorphism, which is homotopic to an Anosov automorphism A on $\mathbb{T}^3$. The stable and unstable bundles of f are jointly integrable, if and only if, every periodic point of f admits the same center Lyapunov exponent as A. In particular, either of these conditions implies f is Anosov.*

Remark 1.1. In Theorem 1.1, the condition that f is conservative can be replaced by assuming the non-wandering set $\Omega(f) = \mathbb{T}^3$. Both properties imply that the su-foliation of f is minimal and the conjugacy preserves the su-foliation.

Combined with the work of Hammerlindl and Ures, we have the following corollary. This proves the Ergodic Conjecture proposed by Hertz–Hertz–Ures on $\mathbb{T}^3$.

Corollary 1.2. *Every $C^{1+\alpha}$ conservative partially hyperbolic diffeomorphism, which is homotopic to an Anosov automorphism on $\mathbb{T}^3$, is ergodic.*

From the previous work of Ren, Gan, and Zhang [20], if f is a $C^{1+\alpha}$ partially hyperbolic and Anosov diffeomorphism on $\mathbb{T}^3$, then there exist a series of equivalent conditions to su-integrability of f. We state them in Theorem 5.1.

Organization of the paper. In Section 2, we recall some properties of partially hyperbolic diffeomorphisms homotopic to an Anosov automorphism on $\mathbb{T}^3$. In Section 3, we prove the "sufficient" part of Theorem 1.1, which states the fact that all periodic points have the same center Lyapunov exponent implies f is su-integrable. In Section 4, we show that if such kind of f is su-integrable, then every periodic point of f admits the same center Lyapunov exponent as A. This proves the "necessary" part of Theorem 1.1. Finally, in Section 5, we give a series of equivalent conditions for su-integrability when f is partially hyperbolic and Anosov on $\mathbb{T}^3$.

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2. Conjugacy and su-integrability

Let f be a partially hyperbolic diffeomorphism which is homotopic to an Anosov automorphism A on $\mathbb{T}^3$. Then A is also partially hyperbolic [3, 17]

$$T\mathbb{T}^3 = E^s_A \oplus E^c_A \oplus E^u_A.$$

These three invariant bundles are linear and correspond to the three eigenvalues $\lambda_s, \lambda_c, \lambda_u$ of A respectively. From now on, we assume that the center bundle of A is expanding, i.e.

$$|\lambda_s| < 1 < |\lambda_c| < |\lambda_u|.$$

Denote by $\mathcal{F}^s_A, \mathcal{F}^c_A, \mathcal{F}^u_A$ the invariant foliations tangent to E^s_A, E^c_A, E^u_A respectively. Since A is linear, all bundles

$$E^{cs}_A = E^s_A \oplus E^c_A, \quad E^{cu}_A = E^c_A \oplus E^u_A, \quad \text{and} \quad E^{su}_A = E^s_A \oplus E^u_A$$

are integrable. Denote by $\mathcal{F}^{cs}_A, \mathcal{F}^{cu}_A, \mathcal{F}^{su}_A$ the foliations tangent to them respectively.

Since f is partially hyperbolic, then f has stable and unstable foliations $\mathcal{F}^s_f$ and $\mathcal{F}^u_f$ tangent to E^s_f and E^u_f respectively. It has been proved by R. Potrie [17] that f is dynamically coherent, i.e. there exist f-invariant foliations $\mathcal{F}^{cs}_f$ and $\mathcal{F}^{cu}_f$ tangent to E^{cs}_f and E^{cu}_f respectively. Moreover, $\mathcal{F}^{cs}_f$ intersects $\mathcal{F}^{cu}_f$ in an one-dimensional f-invariant foliation $\mathcal{F}^c_f$, which is tangent to E^c_f everywhere. We denote by $d_{\mathcal{F}^*_f}(\cdot,\cdot)$ and $d_{\mathcal{F}^*_A}(\cdot,\cdot)$ be the distance induced by the inherited Riemannian metric on leaves of $\mathcal{F}^*_f$ and $\mathcal{F}^*_A$, respectively, for $* = s, c, u, cs, cu$.

We denote by $\widetilde{\mathcal{F}}^*_f$ and $\widetilde{\mathcal{F}}^*_A$ the lifting foliations of $\mathcal{F}^*_f$ and $\mathcal{F}^*_A$ in $\mathbb{R}^3$ for $* = s, c, u, cs, cu$. We denote by $d_{\widetilde{\mathcal{F}}^*_f}(\cdot,\cdot)$ and $d_{\widetilde{\mathcal{F}}^*_A}(\cdot,\cdot)$ the distances induced by the inherited Riemannian metric on leaves of $\widetilde{\mathcal{F}}^*_f$ and $\widetilde{\mathcal{F}}^*_A$, respectively, for $* = s, c, u, cs, cu$.

The following lemma was proved in [13,17]. See also [2,11] when f is absolutely partially hyperbolic.

Lemma 2.1 ([13,17]). *The two foliations $\widetilde{\mathcal{F}}^s_f$ and $\widetilde{\mathcal{F}}^{cu}_f$ have global product structure: $\widetilde{\mathcal{F}}^s_f(x)$ intersects $\widetilde{\mathcal{F}}^{cu}_f(y)$ in exactly one point, for every $x, y \in \mathbb{R}^3$. The two foliations $\widetilde{\mathcal{F}}^u_f$ and $\widetilde{\mathcal{F}}^{cs}_f$ have also global product structure.*

The lifting foliation $\widetilde{\mathcal{F}}^*_f$, $* = s, c, u$ is quasi-isometric in $\mathbb{R}^3$: there exist constants $a, b > 0$, such that for any $y \in \widetilde{\mathcal{F}}^*_f(x)$ with $* = s, c, u$,

$$d_{\widetilde{\mathcal{F}}^*_f}(x, y) \le a \cdot |x - y| + b.$$

Lemma 2.2 ([5,14,17,24]). *Let f be a $C^{1+\alpha}$ partially hyperbolic diffeomorphism which is homotopic to an Anosov automorphism A on $\mathbb{T}^3$. There exists a continuous surjective map $h\colon \mathbb{T}^3 \to \mathbb{T}^3$ satisfying:*

1. *$h \circ f = A \circ h$, taking a lift F of f, there exists a lift H of h, such that $H \circ F = A \circ H$.*
2. *h is homotopic to identity, and for every lift H of h, there exists $L > 0$, such that $\|H - \mathrm{Id}\| < L$.*
3. *For every $\tilde{x} \in \mathbb{R}^3$, $H\colon \widetilde{\mathcal{F}}^s_f(\tilde{x}) \to \widetilde{\mathcal{F}}^s_A(H(\tilde{x}))$ is a homeomorphism.*

4. *For every* $\widetilde{x} \in \mathbb{R}^3$, $H(\widetilde{\mathcal{F}}_f^*(\widetilde{x})) = \widetilde{\mathcal{F}}_A^*(H(\widetilde{x}))$ *for* $* = c, cs, cu$.

5. *For every* $x \in \mathbb{T}^3$, $h^{-1}(h(x))$ *is a compact center arc with length at most* $2aL + b$.

If f *is* su*-integrable and* h *is a homeomorphism, i.e.* f *is topologically conjugate to* A *by* h*, then* h *preserves all invariant foliations*

$$h(\mathcal{F}_f^*) = \mathcal{F}_A^*, \quad \forall * = c, s, u, cs, cu, su.$$

Proof. Item 1 and 2 are well-known results by Franks [5]. Item 3, 4, and 5 were proved by Potrie in [17]. Item 5 see also [24] for absolutely partially hyperbolic diffeomorphisms. The fact that h is a conjugacy preserving all invariant foliations when f is su-integrable was proved by Hammerlindl and Ures [14]. □

In general, if f is topologically conjugate to A but not Anosov, then the conjugacy h^{-1} is not Hölder continuous. However, we can show that h^{-1} is Hölder continuous when restricted to every leaf of $\mathcal{F}_A^s$ and $\mathcal{F}_A^u$.

Lemma 2.3. *Under the assumption in Lemma 2.2, there exist constants* $C > 0$ *and* $0 < \beta < 1$*, such that for every* $x \in \mathbb{T}^3$ *and* $y \in \mathcal{F}_A^*(x)$, $* = s, u$*, we have*

$$d_{\mathcal{F}_f^*}(h^{-1}(x), h^{-1}(y)) \leq C \cdot d_{\mathcal{F}_A^*}(x, y)^{\beta}.$$

Proof. We first prove this fact for $y \in \mathcal{F}_A^u(x)$. We fix $\varepsilon_0, \delta_0 > 0$, such that locally if $d_{\mathcal{F}_A^u}(x, y) < \delta_0$, then $d_{\mathcal{F}_f^u}(h^{-1}(x), h^{-1}(y)) < \varepsilon_0$ for every $x \in \mathbb{T}^3$ and $y \in \mathcal{F}_A^u(x)$. Now we assume that

$$d_{\mathcal{F}_A^u}(x, y) \ll \delta_0.$$

Let k be the largest positive integer such that $d_{\mathcal{F}_A^u}(A^k x, A^k y) < \delta_0$, then we have

$$d_{\mathcal{F}_A^u}(x, y) > |\lambda_u|^{-(k+1)} \cdot \delta_0.$$

On the other hand, we have

$$d_{\mathcal{F}_f^u}(f^k \circ h^{-1}(x), f^k \circ h^{-1}(y)) = d_{\mathcal{F}_f^u}(h^{-1} \circ A^k(x), h^{-1} \circ A^k(y)) < \varepsilon_0.$$

This implies

$$d_{\mathcal{F}_f^u}(h^{-1}(x), h^{-1}(y)) < \mu^{-k} \cdot \varepsilon_0,$$

where $\mu = \inf_{z \in \mathbb{T}^3} m(Df|_{E_f^u(z)}) > 1$.

If $\mu \geq |\lambda_u|$, then we have

$$d_{\mathcal{F}_f^u}(h^{-1}(x), h^{-1}(y)) < \frac{|\lambda_u \cdot \varepsilon_0|}{\delta_0} \cdot d_{\mathcal{F}_A^u}(x, y).$$

Otherwise, we take $0 < \beta < 1$ such that $|\lambda_u|^\beta < \mu$. Then we have

$$d_{\mathcal{F}_f^u}(h^{-1}(x), h^{-1}(y)) < \mu^{-k} \cdot \varepsilon_0 < |\lambda_u|^{-k\beta} \cdot \varepsilon_0 < \frac{\varepsilon_0 |\lambda_u|^\beta}{\delta_0^\beta} \cdot d_{\mathcal{F}_A^u}(x, y)^\beta.$$

This proves that h^{-1} is Hölder continuous on every leaf of $\mathcal{F}_A^u$. The proof for $y \in \mathcal{F}_A^s(x)$ is the same. □

Notation. *Let $p \in \mathrm{Per}(f)$ be a periodic point of f with period $\pi(p)$. We denote by*

$$\lambda_c(p) = \| Df^{\pi(p)}|_{E_f^c(p)} \|^{\frac{1}{\pi(p)}}.$$

Then $\log \lambda_c(p)$ is equal to the center Lyapunov exponent of p. Moreover, we denote $\lambda_c(A) = |\lambda_c| > 1$, and $\log \lambda_c(A)$ is equal to the center Lyapunov exponent of A.

Lemma 2.4. *Let f be a C^1 partially hyperbolic diffeomorphism which is homotopic to an Anosov automorphism A on $\mathbb{T}^3$. Then there exists a sequence of periodic points $\{p_n\}$ of f, such that $\lim_{n\to\infty} \lambda_c(p_n) \geq \lambda_c(A)$.*

Proof. From Lemma 2.2, let $F: \mathbb{R}^3 \to \mathbb{R}^3$ be a lift of f and $H: \mathbb{R}^3 \to \mathbb{R}^3$ be a lift of the semi-conjugacy h. The map H satisfies $|H(\tilde{x}) - \tilde{x}| \leq L$ for every $\tilde{x} \in \mathbb{R}^3$. We can choose two points $\tilde{x}, \tilde{y} \in \mathbb{R}^3$, such that $\tilde{y} \in \widetilde{\mathcal{F}}_f^c(\tilde{x})$ and $|\tilde{x} - \tilde{y}| = 3L$. Then

$$|H(\tilde{x}) - H(\tilde{y})| \geq L > 0 \quad \text{and} \quad H(\tilde{y}) \in \widetilde{\mathcal{F}}_A^c(H(\tilde{x})).$$

Denote by J_f^c the the arc connecting $\tilde{x}$ and $\tilde{y}$ in $\widetilde{\mathcal{F}}_f^c(\tilde{x})$, and J_A^c the arc connecting $H(\tilde{x})$ and $H(\tilde{y})$ in $\widetilde{\mathcal{F}}_A^c(H(\tilde{x}))$, then we have

$$H(F^n(J_f^c)) = A^n(J_A^c), \quad \forall n \geq 0.$$

Then for every n large enough, we have

$$|F^n(J_f^c)| \geq |A^n(J_A^c)| - 2L > \frac{|J_A^c|}{2} \cdot \lambda_c(A)^n$$

(for a smooth arc J, $|J|$ denotes the arc length of J.) This implies that for every n large enough, there exists $\tilde{x}_n \in J_f^c$, such that for $x_n = \pi(\tilde{x}_n)$,

$$\begin{aligned}\frac{1}{n}\sum_{i=0}^{n-1} \log \|Df|_{E^c(f^i(x_n))}\| &= \frac{1}{n}\sum_{i=0}^{n-1} \log \|DF|_{E^c(F^i(\tilde{x}_n))}\| \\ &> \log \lambda_c(A) + \frac{\log |J_A^c| - \log 2|J_f^c|}{n}.\end{aligned}$$

Taking an accumulation point μ_0 of the sequence of measures $\{\sum_{i=0}^{n-1} \delta_{f^i(x_n)}/n\}$, we get that μ_0 is an invariant probability measure of f and

$$\int \log \|Df|_{E^c(x)}\| \,\mathrm{d}\mu_0(x) \geq \log \lambda_c(A).$$

By ergodic decomposition theorem, we can assume μ_0 is ergodic. Since μ_0 is a hyperbolic measure, by Liao's shadowing lemma (e.g., see [6, 7, 15]), there exists a sequence of periodic points $\{p_n\}$ of f, such that $\lim_{n\to\infty} \lambda_c(p_n) \geq \lambda_c(A)$. □

Theorem 2.1 ([1]). *Let p be a hyperbolic periodic point of a diffeomorphism f on a compact manifold. Assume that its homoclinic class $H(p)$ admits a dominated splitting $T_{H(p)}M = E \oplus F$ with E contracting and $\dim(E) = \operatorname{ind}(p)$. If f is uniformly F-expanding at the period on the set of periodic points q homoclinically related to p, then F is uniformly expanding on $H(p)$.*

Lemma 2.5. *Let f be a C^1 partially hyperbolic diffeomorphism which is homotopic to an Anosov automorphism A on $\mathbb{T}^3$. If $\lambda_c(p) = \lambda_c(q)$ for every $p, q \in \operatorname{Per}(f)$, then f is Anosov.*

Proof. From Lemma 2.4, we know that $\lambda_c(p) \geq \lambda_c(A) > 1$ for every $p \in \operatorname{Per}(f)$. From the semi-conjugacy $h\colon \mathbb{T}^3 \to \mathbb{T}^3$ in Lemma 2.2, $h(p)$ is a periodic point of A for every $p \in \operatorname{Per}(f)$. Moreover, we have $h^{-1}(h(p)) = \{p\}$. Otherwise, $h^{-1}(h(p))$ is an f-periodic center arc, which must contain a periodic point of f admitting non-positive center Lyapunov exponents.

This implies that for every $p \in \operatorname{Per}(f)$, the unstable manifold $W_f^u(p) = \mathcal{F}_p^{cu}(f)$ which is dense in $\mathbb{T}^3$ and tangent to E_f^{cu} everywhere. On the other hand, h restricted to every stable leaf $\mathcal{F}_f^s(x)$ is a homeomorphism to $\mathcal{F}_A^s(h(x))$. If h is injective at a point p, then h is injective at every point of $\mathcal{F}_f^s(p)$. Actually, if h is not injective at a point $y \in \mathcal{F}_f^s(p)$, then there exists $z \in \mathcal{F}_f^c(y)$ satisfying $h(y) = h(z)$. Let $w(\neq p)$ be the unique intersecting point of $\mathcal{F}_f^s(z)$ and $\mathcal{F}_f^c(p)$, then we have

$$h(\mathcal{F}_f^s(p)) = h(\mathcal{F}_f^s(y)) = \mathcal{F}_A^s(h(y)) = \mathcal{F}_A^s(h(z)) = h(\mathcal{F}_f^s(z)) = h(\mathcal{F}_f^s(w)).$$

Recall that h is a homeomorphism from $\mathcal{F}_f^s(p)$ to $\mathcal{F}_A^s(h(p))$, and a homeomorphism from $\mathcal{F}_f^s(w)$ to $\mathcal{F}_A^s(h(w))$. Since $\{w\} = \mathcal{F}_f^s(z) \cap \mathcal{F}_f^c(p)$, we have $h(p) = h(w)$, which contradicts the fact that h is injective at p.

Let $H_f(p) = \overline{W_f^s(p) \pitchfork W_f^u(p)}$ be the homoclinic class of p w.r.t. f. Then we have

$$\begin{aligned} h(H_f(p)) &= h(\overline{W_f^s(p) \pitchfork W_f^u(p)}) \\ &= \overline{h(W_f^s(p) \pitchfork W_f^u(p))} \\ &= \overline{W_A^s(h(p)) \pitchfork W_A^u(h(p))} = H_A(h(p)) = \mathbb{T}^3. \end{aligned}$$

Now we consider the partially hyperbolic splitting $T_{H_f(p)}\mathbb{T}^3 = E^s_f \oplus E^{cu}_f$. Since $\lambda_c(p) \geq \lambda_c(A) > 1$ for every $p \in \mathrm{Per}(f)$, f is uniformly E^{cu}_f-expanding at the period on all the periodic points in $H_f(p)$. Applying Theorem 2.1, E^{cu}_f is uniformly expanding and $H_f(p)$ is a hyperbolic set of f. Since h is injective at every point of $W^s_f(p)$, $W^s_f(p) \subset H_f(p)$. If $H_f(p) \neq \mathbb{T}^3$, $H_f(p)$ would be a proper repeller, which is contradictory to the conservativity of f. This proves that f is Anosov. $\square$

Corollary 2.6. *Let f be a C^1 partially hyperbolic diffeomorphism which is homotopic to an Anosov automorphism A on $\mathbb{T}^3$. If $\lambda_c(p) = \lambda_c(q)$ for every $p, q \in \mathrm{Per}(f)$, then $\lambda_c(p) = \lambda_c(A)$.*

Proof. We only have to show that there exists a sequence of periodic points $\{q_n\}$ of f, such that

$$\lim_{n\to\infty} \lambda_c(q_n) \leq \lambda_c(A).$$

This proof goes similarly with Lemma 2.4. In fact, since $\widetilde{\mathcal{F}}^c_f$ is quasi-isometric, there exist constants $a, b > 0$, such that for every n large enough,

$$\begin{aligned} |F^n(J^c_f)| &\leq a \cdot |F^n(\widetilde{x}) - F^n(\widetilde{y})| + b \\ &\leq a \cdot \big(|A^n(J^c_A)| + 2L\big) + b < 2a|J^c_A| \cdot \lambda_c(A)^n. \end{aligned}$$

(for the definition of notations, see the proof of Lemma 2.4.) So there exists $\widetilde{y}_n \in J^c_f$, such that for $y_n = \pi(\widetilde{y}_n)$,

$$\begin{aligned} \frac{1}{n}\sum_{i=0}^{n-1} \log \|Df|_{E^c(f^i(y_n))}\| &= \frac{1}{n}\sum_{i=0}^{n-1} \log \|DF|_{E^c(F^i(\widetilde{y}_n))}\| \\ &< \log \lambda_c(A) + \frac{\log 2a|J^c_A| - \log |J^c_f|}{n}. \end{aligned}$$

Taking an accumulation point μ_1 of the sequence of measures $\{\sum_{i=0}^{n-1} \delta_{f^i(y_n)}/n\}$, we have that μ_1 is an invariant probability measure of f and

$$\int \log \|Df|_{E^c(x)}\| \, \mathrm{d}\mu_1(x) \leq \log \lambda_c(A).$$

By ergodic decomposition theorem, we can assume μ_1 is ergodic. Since f is Anosov, there exists a sequence of periodic points $\{q_n\}$ of f, such that $\lim_{n\to\infty} \lambda_c(q_n) \leq \lambda_c(A)$. $\square$

The following theorem was essentially proved in the classical paper by Pugh–Shub–Wilkinson [19]. We will need it in Section 4.

Theorem 2.2 ([19]). *Suppose that $f: M \to M$ is a $C^{1+\alpha}$ partially hyperbolic diffeomorphism with one-dimensional center bundle. If f is dynamically coherent, then the local unstable and local stable holonomy maps are uniformly C^1 when restricted to each center unstable and each center stable leaf, respectively.*

3. Joint su-integrability

In this section, we prove that if f is a $C^{1+\alpha}$ conservative partially hyperbolic diffeomorphism on $\mathbb{T}^3$ which is homotopic to an Anosov automorphism, and the center Lyapunov exponent of every periodic point of f is equal to $\log \lambda_c(A)$, then f is su-integrable.

Firstly, we need the following lemma.

Lemma 3.1. *Let f be a $C^{1+\alpha}$ partially hyperbolic diffeomorphism which is homotopic to an Anosov automorphism A on $\mathbb{T}^3$. If $\lambda_c(p) = \lambda_c(A)$ for every periodic point $p \in \mathrm{Per}(f)$, then there exists a continuous metric $d^c(\cdot,\cdot)$ defined on every leaf of center foliation $\mathcal{F}^c_f$, such that:*

- *There exists $K > 1$, satisfying $1/K \cdot d_{\mathcal{F}^c_f}(x,y) < d^c(x,y) < K \cdot d_{\mathcal{F}^c_f}(x,y)$, for every $y \in \mathcal{F}^c_f(x)$;*
- *$d^c(f(x), f(y)) = \lambda_c(A) \cdot d^c(x,y)$, for every $y \in \mathcal{F}^c_f(x)$;*
- *The stable and unstable holonomy maps between center leaves are isometries under $d^c(\cdot,\cdot)$ when restricted to each center stable and center unstable leaf, respectively.*

Proof. From Lemma 2.5 and Corollary 2.6, we know that f is Anosov and $\lambda_c(p) = \lambda_c(A)$ for every $p \in \mathrm{Per}(f)$. Then Livschitz Theorem implies that there exists a Hölder continuous function $\phi\colon \mathbb{T}^3 \to \mathbb{R}$, such that

$$\log \|Df|_{E^c_f(x)}\| = \phi(x) - \phi \circ f(x) + \log \lambda_c(A), \quad \forall x \in \mathbb{T}^3.$$

This implies that

$$\lambda_c(A) \cdot \exp(\phi(x)) = \|Df|_{E^c_f(x)}\| \cdot \exp(f \circ \phi(x)), \quad \forall x \in \mathbb{T}^3.$$

Now we can define a metric on every leaf of $\mathcal{F}^c_f$ as the following: for every $y \in \mathcal{F}^c_f(x)$, let $\gamma\colon [0,1] \to \mathcal{F}^c_f(x)$ be a C^1-parametrization with $\gamma(0) = x$ and $\gamma(1) = y$, then

$$d^c(x,y) := \int_0^1 \exp(\phi \circ \gamma(t)) \cdot |\gamma'(t)| \, \mathrm{d}t.$$

Since ϕ is bounded, there exists $K > 1$, such that

$$\frac{1}{K} \cdot d_{\mathcal{F}^c_f}(x,y) < d^c(x,y) < K \cdot d_{\mathcal{F}^c_f}(x,y), \quad \forall y \in \mathcal{F}^c_f(x).$$

Moreover, the cohomology equation implies f is conformal on $\mathcal{F}^c_f$ under this metric:

$$d^c(f(x), f(y)) = \lambda_c(A) \cdot d^c(x,y), \quad \forall y \in \mathcal{F}^c_f(x).$$

From this conformal structure, we know that for every $x \in \mathbb{T}^3$ and $z \in \mathcal{F}_f^u(x)$, we denote $h_{x,z}^u : \mathcal{F}_f^c(x) \to \mathcal{F}_f^c(z)$ the holonomy map induced by the unstable foliation $\mathcal{F}_f^u$ in $\mathcal{F}_f^{cu}(x)$, then

$$d^c(h_{x,z}^u(y_1), h_{x,z}^u(y_2)) = d^c(y_1, y_2), \quad \forall y_1, y_2 \in \mathcal{F}_f^c(x).$$

The same property holds for $z \in \mathcal{F}_f^s(x)$ and the holonomy map $h_{x,z}^s : \mathcal{F}_f^c(x) \to \mathcal{F}_f^c(z)$ induced by stable foliation $\mathcal{F}_f^s$ in $\mathcal{F}_f^{cs}(x)$. □

Remark 3.2. If the function ϕ is a solution of the cohomology equation

$$\log \|Df|_{E_f^c}\| = \phi - \phi \circ f + \log \lambda_c(A),$$

then $\phi + \kappa$ is also a solution for every $\kappa \in \mathbb{R}$. The corresponding center metric $d_1^c(\cdot,\cdot)$ defined by $\phi + \kappa$ also satisfies all the properties in Lemma 3.1. Actually, they satisfy

$$d_1^c(x, y) = e^{\kappa} \cdot d^c(x, y), \quad \forall y \in \mathcal{F}_f^c(x).$$

Proposition 3.3. *Let f be a $C^{1+\alpha}$ partially hyperbolic diffeomorphism which is homotopic to an Anosov automorphism A on $\mathbb{T}^3$. If $\lambda_c(p) = \lambda_c(A)$ for every periodic point $p \in$ Per(f), then the stable and unstable bundles of f are jointly integrable.*

Proof. Since $\lambda_c(p) = \lambda_c(A)$ for every periodic point $p \in \mathrm{Per}(f)$, let $d^c(\cdot,\cdot)$ be the metric on $\mathcal{F}_f^c$ which is defined in Lemma 3.1.

If E_f^s and E_f^u are not jointly integrable, then we have 4-legs local twisting, i.e. there exist $x_0 \in \mathbb{T}^3$, $y_0 \in \mathcal{F}_f^s(x_0)$ and $z_0 \in \mathcal{F}_f^u(x_0)$ which are very close to x_0 in the stable and unstable leaves of x_0, such that locally there exist $w_1 \in \mathcal{F}_f^u(y_0)$ and $w_2 \in \mathcal{F}_f^s(z_0)$ satisfying

$$w_1 \neq w_2 \quad \text{and} \quad w_2 \in \mathcal{F}_f^c(w_1).$$

We denote $d^c(w_1, w_2) = \kappa_0 > 0$.

Claim 3.4. *There exists a family of arcs $\mathcal{I}^s = \{I^s(x) : x \in \mathbb{T}^3\}$ satisfying:*

- *$I^s(x) \subset \mathcal{F}_f^s(x)$ admits x as the start-point and varies continuously with respect to x.*
- *$I^s(x_0)$ admits y_0 as the end-point, and $I^s(z_0)$ admits w_2 as the end-point.*
- *Every $x_2 \in \mathcal{F}_f^{cu}(x_1)$ satisfies that $I^s(x_2) = h_{x_1,x_2}^{cu}(I^s(x_1))$.*
- *There exist constants $0 < l_1 < l_2$, such that $l_1 \leq |I^s(x)| \leq l_2$ for every $x \in \mathbb{T}^3$.*

Proof of the claim. Let $I^s(x_0)$ be the arc from x_0 to y_0 in $\mathcal{F}_f^s(x_0)$, and $I^s(z_0)$ be the arc from z_0 to w_2 in $\mathcal{F}_f^s(z_0)$, then we can see that

$$I^s(z_0) = h^{cu}_{x_0,z_0}(I^s(x_0)),$$

where $h^{cu}_{x_0,z_0}\colon \mathcal{F}_f^s(x_0) \to \mathcal{F}_f^s(z_0)$ is the local holonomy map induced by $\mathcal{F}_f^{cu}$. Then for every point $x \in \mathcal{F}_f^{cu}(x_0)$, we can define

$$I^s(x) = h^{cu}_{x_0,x}(I^s(x_0)) \subset \mathcal{F}_f^s(x).$$

Since every leaf of $\mathcal{F}_f^{cu}$ is homeomorphic to $\mathbb{R}^2$, and the lifting foliations $\widetilde{\mathcal{F}}_f^{cu}$, $\widetilde{\mathcal{F}}_f^s$ admit a global product structure, this tells us that $I^s(x)$ is well-defined for every point $x \in \mathcal{F}_f^{cu}(x_0)$. Moreover, the topological conjugacy h maps $\mathcal{F}_f^{cu}$ into the linear $\mathcal{F}_A^{cu}$ implies that we can extend this family of stable arcs to $\mathbb{T}^3$:

$$\mathcal{I}^s = \{I^s(x) : x \in \mathbb{T}^3\}.$$

Finally, since $\mathcal{F}_A^s$ and $\mathcal{F}_A^{cu}$ are linear foliations, the uniform continuity of h gives us the constants $0 < l_1 < l_2$ such that $l_1 \le |I^s(x)| \le l_2$ for every $x \in \mathbb{T}^3$. □

Symmetrically, we have the following claim.

Claim 3.5. *There exists a family of arcs $\mathcal{I}^u = \{I^u(x) : x \in \mathbb{T}^3\}$ satisfying:*

- *$I^u(x) \subset \mathcal{F}_f^u(x)$ admits x as the start-point and varies continuously with respect to x.*
- *$I^u(x_0)$ admits z_0 as the end-point, and $I^u(z_0)$ admits w_1 as the end-point.*
- *Every $x_2 \in \mathcal{F}_f^{cs}(x_1)$ satisfies that $I^u(x_2) = h^{cs}_{x_1,x_2}(I^u(x_1))$.*
- *There exist constants $0 < l_3 < l_4$, such that $l_3 \le |I^u(x)| \le l_4$ for every $x \in \mathbb{T}^3$.*

We fix the orientation of $I^s(x_0)$ from x_0 to y_0 to be positive and assume it coincides with the positive orientation of $\mathcal{F}_f^s$. Since $\mathcal{F}_f^{cu}(x_0)$ is dense and $\mathcal{F}_f^s$ is orientable, the orientation can be continuously extended to $\mathcal{I}^s$. Symmetrically, we fix the orientation of $\mathcal{I}^u$ which is positive from x_0 to z_0 at $I^u(x_0)$, and assume it coincides with the positive orientation of $\mathcal{F}^u$. Moreover, we assume that the arc from w_1 to w_2 has the same orientation with $\mathcal{F}_f^c$.

For every $x \in \mathbb{T}^3$, we define the su-path $J^{su}(x)$ to be the path that goes through $I^s(x)$ to the end-point y of $I^s(x)$, then go through $I^u(y)$ to the end-point w'. We call w' the end-point of $J^{su}(x)$. Symmetrically, we can define the us-path $J^{us}(x)$ by going through $I^u(x)$ to the end-point z, then go through $I^s(z)$ to the end-point w''. We call w'' the end-point of $J^{us}(x)$.

Claim 3.6. *There exists a family of arcs $\mathcal{I}^c = \{I^c(x) : x \in \mathbb{T}^3\}$ satisfying:*

- *$I^c(x) \subset \mathcal{F}_f^c(x)$ admits x as the start-point and varies continuously with respect to x;*
- *For every $x \in \mathbb{T}^3$, denote w' to be the end-point of $J^{su}(x)$ and w'' to be the end-point of $J^{us}(x)$, then w'' is the end-point of the arc $I^c(w')$. In particular, w_2 is the end-point of $I^c(w_1)$.*
- *For every $w' \in \mathbb{T}^3$ with $\partial I^c(w') = \{w', w''\}$, it satisfies*
$$d^c(w', w'') = d^c(w_1, w_2) = \kappa_0 > 0,$$
and $I^c(w')$ from w' to w'' has the same orientation as $\mathcal{F}_f^c$.

Proof of the claim. The definition of $\mathcal{I}^c$ comes from the second item of the claim. From the continuity of $\mathcal{I}^s$ and $\mathcal{I}^u$, and their holonomy invariance by $\mathcal{F}_f^{cu}$ and $\mathcal{F}_f^{cs}$, $\mathcal{I}^c$ is well defined and varies continuously. We only need to check the last item.

For every $x \in \mathcal{F}_f^c(x_0)$, we denote w' and w'' be the other endpoints of su-path $J^{su}(x)$ and us-path $J^{us}(x)$ respectively. The holonomy invariance of $\mathcal{I}^s$ and $\mathcal{I}^u$ implies $w', w'' \in \mathcal{F}_f^c(w_1)$. Moreover, we consider the composition of holonomy maps $h^s_{x_0,y_0} : \mathcal{F}_f^c(x_0) \to \mathcal{F}_f^c(y_0)$ and $h^u_{y_0,w_1} : \mathcal{F}_f^c(y_0) \to \mathcal{F}_f^c(w_1)$, it is defined as

$$h^{su}_{J^{su}(x_0)} := h^u_{y_0,w_1} \circ h^s_{x_0,y_0} : \mathcal{F}_f^c(x_0) \to \mathcal{F}_f^c(w_1),$$

where $h^{su}_{J^{su}(x_0)}(x) = w'$.

Similarly, we have the holonomy map

$$h^{us}_{J^{us}(x_0)} := h^u_{z_0,w_2} \circ h^s_{x_0,z_0} : \mathcal{F}_f^c(x_0) \to \mathcal{F}_f^c(w_2) = \mathcal{F}_f^c(w_1),$$

which is the composition of the holonomy maps

$$h^u_{x_0,z_0} : \mathcal{F}_f^c(x_0) \to \mathcal{F}_f^c(z_0) \quad \text{and} \quad h^s_{z_0,w_2} : \mathcal{F}_f^c(z_0) \to \mathcal{F}_f^c(w_2)$$

and satisfies $h^{us}_{J^{us}(x_0)}(x) = w''$.

Since the holonomy maps of stable and unstable foliations between center leaves are isometries under the metric $d^c(\cdot,\cdot)$ when restricted in each center-stable and center-unstable leaves, both $h^{su}_{J^{su}(x_0)}$ and $h^{us}_{J^{us}(x_0)}$ are isometries between $\mathcal{F}_f^c(x_0)$ and $\mathcal{F}_f^c(w_1)$ under the metric $d^c(\cdot,\cdot)$. This implies

$$d^c(w_1, w') = d^c(x_0, x) = d^c(w_2, w'') = \kappa_0.$$

So we have $d^c(w_1, w_2) = d^c(w', w'')$, that is $I^c(x)$ has the same length under the metric $d^c(\cdot,\cdot)$ for every $x \in \mathcal{F}_f^c(x_0)$. From the density of $\mathcal{F}_f^c(x_0)$ and continuity of $\mathcal{I}^c$, we prove the claim. $\square$

Now we lift these three family of arcs $\mathcal{I}^s$, $\mathcal{I}^u$ and $\mathcal{I}^c$ to the universal cover $\mathbb{R}^3$. We use the same notation for convenience.

Now we fix $x^0 \in \mathbb{R}^3$ and denote z^0 the end-point of $I^u(x^0)$. Define inductively:

- $x^{i+1} \in \widetilde{\mathcal{F}}^s_f(x^0)$ to be the end-point of $I^s(x^i)$ for $i = 0, 1, \ldots, n-1$;
- $z^{i+1} \in \widetilde{\mathcal{F}}^s_f(z^0)$ to be the end-point of $I^s(z^i)$ for $i = 0, 1, \ldots, n-1$.

Then we consider the end-point w^0 of $I^u(x^n)$, we can see that that $w^0 \in \widetilde{\mathcal{F}}^c_f(z^n)$. Moreover, there exists a sequence of points $\{w^0, w^1, \ldots, w^n\} \subset \widetilde{\mathcal{F}}^c_f(z^n)$, such that

- w^{i+1} is the end-point of $I^c(w^i)$ for $i = 0, 1, \ldots, n-1$;
- $w^n = z^n$ and $d^c(w^0, z^n) = n \cdot \kappa_0$.

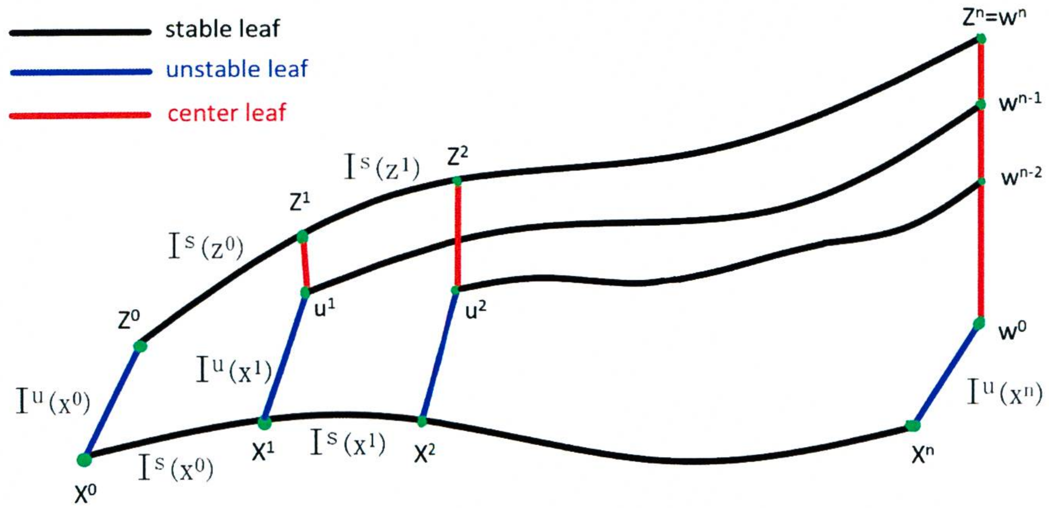

Figure 1. Global twisting.

Actually, if we denote u^i to be the end-point of $I^u(x^i)$ for $i = 1, \ldots, n-1$, we have

$$w^i = \widetilde{\mathcal{F}}^s_f(u^{n-i}) \cap \widetilde{\mathcal{F}}^c_f(w^0) \in \widetilde{\mathcal{F}}^{cs}_f(w^0), \quad i = 1, \ldots, n-1.$$

This implies $d^c(w^0, z^n) = n \cdot \kappa_0$. Since $\widetilde{\mathcal{F}}^c_f$ is quasi-isometric, there exists $a > 0$, such that

$$n \cdot a\kappa_0 \leq |w^0 - z^n| \to \infty \quad \text{as } n \to \infty.$$

Since $|x^n - w^0| \leq |I^u(x^n)| < l_4$, this implies

$$|z^n - x^n| \longrightarrow \infty, \quad \text{as } n \to \infty.$$

Let $F: \mathbb{R}^3 \to \mathbb{R}^3$ be a lift of f, and $H: \mathbb{R}^3 \to \mathbb{R}^3$ be the conjugacy satisfying $A \circ H = H \circ f$. Then there exists $L > 0$, such that $|H(\widetilde{x}) - \widetilde{x}| < L$.

Since $H(\widetilde{\mathcal{F}}_f^s) = \widetilde{\mathcal{F}}_A^s$ and $H(\widetilde{\mathcal{F}}_f^{cu}) = \widetilde{\mathcal{F}}_A^{cu}$, we have

$$H(z^0) \in \widetilde{\mathcal{F}}_A^{cu}(H(x^0)), \quad H(x^n) \in \widetilde{\mathcal{F}}_A^s(H(x^0)),$$

and

$$H(z^n) = \widetilde{\mathcal{F}}_A^s(H(z^0)) \pitchfork \widetilde{\mathcal{F}}_A^{cu}(H(x^n)).$$

If we denote $\widetilde{h}^s : \widetilde{\mathcal{F}}_A^{cu}(x^0) \to \widetilde{\mathcal{F}}_A^{cu}(x^n)$ as the holonomy map induced by the stable foliation $\widetilde{\mathcal{F}}_A^s$, then we have

$$H(x^n) = \widetilde{h}^s(H(x^0)) \quad \text{and} \quad H(z^n) = \widetilde{h}^s(H(z^0)).$$

However, since both $\widetilde{\mathcal{F}}_A^s$ and $\widetilde{\mathcal{F}}_A^{cu}$ are linear, we have

$$|H(z^n) - H(x^n)| = |H(z^0) - H(x^0)| < |z^0 - x^0| + 2L \leq l_4 + 2L.$$

This implies that for every n, we have

$$|z^n - x^n| \leq |z^n - H(z^n)| + |x^n - H(x^n)| + (l_4 + 2L) < l_4 + 4L.$$

This is a contradiction. □

4. Rigidity of center Lyapunov exponents

In this section, we prove that if f is a $C^{1+\alpha}$ conservative partially hyperbolic diffeomorphism on $\mathbb{T}^3$ which is homotopic to an Anosov automorphism and admits jointly integrable su-foliation $\mathcal{F}_f^{su}$, then the center Lyapunov exponent of every periodic orbit of f is equal to $\log \lambda_c(A)$.

From the work of Hammerlindl and Ures, the following proposition implies the "necessary" part of Theorem 1.1. The idea of our proof originates from the work of A. Gogolev [8].

Proposition 4.1. *Let f be a $C^{1+\alpha}$ partially hyperbolic diffeomorphism which is homotopic to an Anosov automorphism A on $\mathbb{T}^3$. If the stable and unstable bundles of f are jointly integrable and f is topologically conjugate to A, then*

$$\lambda_c(p) = \lambda_c(A), \quad \forall p \in \operatorname{Per}(f).$$

Thus f is Anosov.

Proof. Recall we assumed that $\lambda_c(A) > 1$. Since f is topologically conjugate to A, the topological expansion in the center direction implies $\lambda_c(p) \geq 1$ for every periodic point p of f.

From Lemma 2.5, we only need to show that $\lambda_c(p) = \lambda_c(q)$ for any periodic points $p, q \in \operatorname{Per}(f)$. The topological conjugacy property implies that f also satisfies the Shadowing Lemma. If there exist $p_1, p_2 \in \mathrm{Per}(f)$, such that $\lambda_c(p_1) < \lambda_c(p_2)$, then the set

$$\overline{\{\lambda_c(p) : p \in \mathrm{Per}(f)\}} = [\lambda_-, \lambda_+]$$

is a nontrivial interval contained in $[1, +\infty)$. By applying the Shadowing Lemma, we can take a smooth adapted Riemannian metric, such that

$$\frac{\lambda_-}{1+\delta} < \|Df|_{E^c_f(x)}\| < \lambda_+ \cdot (1+\delta), \quad \forall x \in \mathbb{T}^3.$$

Here δ could be arbitrarily small, and we will fix it later.

Now we choose periodic points p, q of f, such that

$$\frac{\lambda_c(p)}{\lambda_-} \leq 1+\delta \quad \text{and} \quad \frac{\lambda_+}{\lambda_c(q)} \leq 1+\delta.$$

Denote by n_0 the minimal common period of p and q.

Claim 4.2. *There exist two constants $C_3 > 0$ and $0 < \theta < 1/2$, such that for every $\eta > 0$, there exist points $x \in \mathcal{F}^u_f(p)$, $y \in \mathcal{F}^s_f(x)$ with $q \in \mathcal{F}^c_f(y)$, such that*

$$d_{\mathcal{F}^c_f}(y, q) \leq \eta \quad \textit{and} \quad d_{\mathcal{F}^s_f}(x, y) \leq \frac{C_3}{D^\theta}, \quad \textit{where } D = d_{\mathcal{F}^u_f}(p, x).$$

Proof of the claim. Denote by $p' = h(p)$ and $q' = h(q)$ the conjugating periodic points of A. Then the strong unstable manifold $\mathcal{F}^u_A(p')$ is a line with irrational direction. This implies that an arc of $\mathcal{F}^u_A(p')$ with length D' is $C_1/\sqrt{D'}$-dense in $\mathbb{T}^3$ for some $C_1 > 0$. From the local product structure, there exist a constant C_2, and two sequences of points $x'_n \in \mathcal{F}^u_A(p')$, $y'_n \in \mathcal{F}^s_A(x'_n)$, such that $q' \in \mathcal{F}^c_A(y'_n)$,

$$D'_n = d_{\mathcal{F}^u_A}(p', x'_n) \longrightarrow \infty \quad (n \to \infty),$$
$$d_{\mathcal{F}^s_A}(x'_n, y'_n) \leq \frac{C_2}{\sqrt{D'_n}} \quad \text{and} \quad d_{\mathcal{F}^c_A}(y', q') \leq \frac{C_2}{\sqrt{D'_n}}.$$

Let $\widetilde{p}'$ be a lifting point of p' in $\mathbb{R}^3$, and $\widetilde{x}'_n \in \widetilde{\mathcal{F}}^u_A(\widetilde{p}')$ be the corresponding lifting point of x'_n. Then we have

$$|\widetilde{p}' - \widetilde{x}'_n| = d_{\widetilde{\mathcal{F}}^u_A}(\widetilde{p}', \widetilde{x}'_n) = d_{\mathcal{F}^u_A}(p', x'_n) = D'_n.$$

Recall that the conjugacy h preserves the stable, unstable, and center foliations: $h(\mathcal{F}^*_f) = \mathcal{F}^*_A$ for $* = s, u, c$. Denote $x_n = h^{-1}(x'_n)$ and $y_n = h^{-1}(y'_n)$. We have $x_n \in \mathcal{F}^u_f(p)$, $y_n \in \mathcal{F}^s_f(x_n)$, and $q \in \mathcal{F}^c_f(y_n)$. From the continuity of the conjugacy h, for every $\eta > 0$, there exists $n_1 > 0$, such that

$$d_{\mathcal{F}^c_f}(y_n, q) \leq \eta, \quad \forall n \geq n_1.$$

We denote $D_n = d_{\mathcal{F}^u_f}(p, x_n)$.

Let $H:\mathbb{R}^3 \to \mathbb{R}^3$ be a lift of h. Lemma 2.2 shows that there exists $L > 0$ satisfying

$$|\widetilde{x} - H(\widetilde{x})| \le L, \quad \forall \widetilde{x} \in \mathbb{R}^3.$$

So we have

$$\begin{aligned} D_n = d_{\mathcal{F}^u_f}(p, x_n) &= d_{\widetilde{\mathcal{F}}^u_f}(H^{-1}(\widetilde{p}'), H^{-1}(\widetilde{x}'_n)) \\ &\ge |H^{-1}(\widetilde{p}') - H^{-1}(\widetilde{x}'_n)| \\ &\ge |\widetilde{p}' - \widetilde{x}'_n| - 2L = D'_n - 2L \longrightarrow \infty \quad (n \to \infty). \end{aligned}$$

On the other hand, since the lifting foliation $\widetilde{\mathcal{F}}^u_f$ is quasi-isometric, we have

$$\begin{aligned} D_n = d_{\mathcal{F}^u_f}(p, x_n) &= d_{\widetilde{\mathcal{F}}^u_f}(H^{-1}(\widetilde{p}'), H^{-1}(\widetilde{x}'_n)) \\ &\le a \cdot |H^{-1}(\widetilde{p}') - H^{-1}(\widetilde{x}'_n)| + b \\ &\le a \cdot \big(|\widetilde{p}' - \widetilde{x}'_n| + 2L\big) + b = a \cdot D'_n + 2aL + b. \end{aligned}$$

Here the constants a, b are quasi-isometric constants in Lemma 2.1. So there exists $n_2 > 0$, such that $D_n < 2a \cdot D'_n$ for every $n \ge n_2$.

By Lemma 2.3, there exists $C'_3 > 0$ and $0 < \theta < 1/2$, such that

$$d_{\mathcal{F}^s_f}(x_n, y_n) \le \frac{C'_3}{(D'_n)^\theta} < \frac{(2a)^\theta C'_3}{D_n^\theta}, \quad \forall n \ge n_2.$$

Let $C_3 = (2a)^\theta C'_3$.

Let $n_0 = \max\{n_1, n_2\}$, and take $x = x_{n_0} \in \mathcal{F}^u_f(p)$, $y = y_{n_0} \in \mathcal{F}^s_f(x)$ with $q \in \mathcal{F}^c_f(y)$. They satisfy

$$d_{\mathcal{F}^c_f}(y, q) \le \eta \quad \text{and} \quad d_{\mathcal{F}^s_f}(x, y) \le \frac{C_3}{D^\theta}, \quad \text{where } D = d_{\mathcal{F}^u_f}(p, x).$$

This finishes the proof of the claim. □

Notice that here constants C_3 and θ only depend on the contracting and expanding rates of f on E^s_f and E^u_f. Moreover, the points x and y also change here when D changes. We will let D tends to infinity in the future.

Let $\eta_0 > 0$, such that for every $z_1, z_2 \in \mathbb{T}^3$ satisfying $d(z_1, z_2) \le 3\eta_0$, we have

$$\frac{\|Df|_{E^c(z_1)}\|}{(1+\delta)} < \|Df|_{E^c(z_2)}\| < (1+\delta) \cdot \|Df|_{E^c(z_1)}\|.$$

From the fact that f is conjugate to A and from the uniform continuity of the conjugacy, there exists $0 < \eta_1 \le \eta_0$, such that for any arc J contained in a leaf of $\mathcal{F}^c_f$ with length $|J| \le \eta_1$, it satisfies

$$|f^{-n}(J)| \le \eta_0, \quad \forall n \ge 0.$$

Moreover, since the su-foliation $\mathcal{F}_A^{su}$ is linear, the uniform continuity of the conjugacy also shows that there exists $0 < \eta_2 \le \eta_1$, such that for any arc J contained in a leaf of $\mathcal{F}_f^c$ with length $|J| \le \eta_2$, if $J' = h_f^{su}(J)$ is an arc contained in a leaf of $\mathcal{F}_f^c$ induced by the holonomy map h^{su} of $\mathcal{F}_f^{su}$, it satisfies $|J'| \le \eta_1$.

Now we consider an arc $J_0 \subset \mathcal{F}_f^c(p)$ with one endpoint p and satisfying $|J_0| = \eta_2$, and we take D large enough such that there exist $x \in \mathcal{F}_f^u(p)$ and $y \in \mathcal{F}_f^s(x)$ such that $q \in \mathcal{F}_f^c(y)$ and satisfy the following estimations:

$$d_{\mathcal{F}_f^u}(p,x) = D, \quad d_{\mathcal{F}_f^s}(x,y) \le \frac{C_3}{D^\theta} \ll \eta_0, \quad \text{and} \quad d_{\mathcal{F}_f^c}(y,q) \le \eta_1.$$

Let $J_1 = h^{su}(J_0)$ admitting x as one endpoint. This implies $|J_1| \le \eta_1$. And we denote by $J^s(x,y)$ the arc contained in $\mathcal{F}_f^s(x)$ with endpoints x and y; $J^c(y,q)$ the arc contained in $\mathcal{F}_f^c(y)$ with endpoints y and q. Notice that when D goes to infinity, all these estimations still hold.

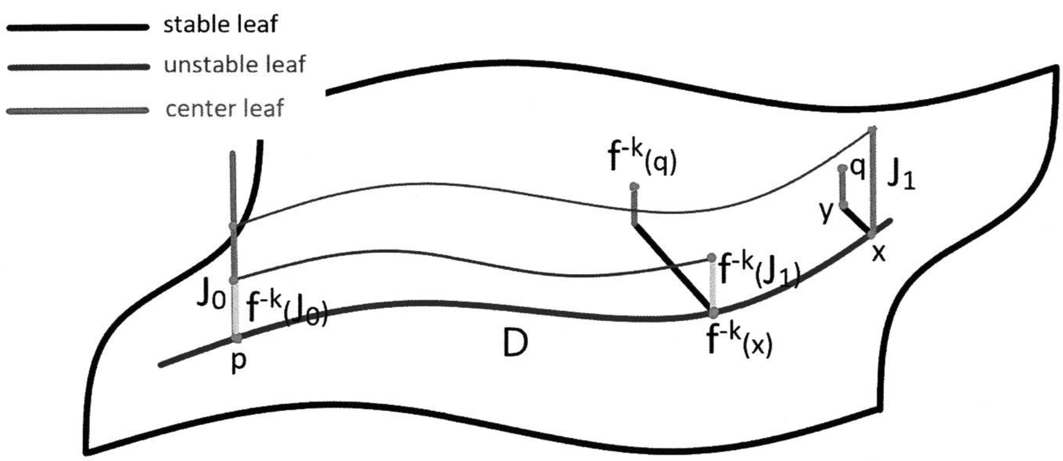

Figure 2. Holonomy map.

Denote by N_0 the first positive integer where $f^{-n_0N_0}(x)$ satisfies

$$d_{\mathcal{F}_f^u}(p, f^{-n_0N_0}(x)) \le 1.$$

Let $\mu = \sup_{x\in\mathbb{T}^3} \|Df^{-1}|_{E_f^u(x)}\| < 1$, then

$$N_0 \le \frac{\log D}{-n_0 \log \mu} + 1.$$

And we have

$$|f^{-n_0N_0}(J_0)| \ge \frac{\lambda_c(p)^{-n_0N_0}}{(1+\delta)^{n_0N_0}} \cdot |J_0| \ge \frac{\lambda_-^{-n_0N_0}}{(1+\delta)^{2n_0N_0}} \cdot |J_0|.$$

On the other hand, denote

$$\gamma = \sup_{x\in\mathbb{T}^3} \|Df^{-1}|_{E^s_f(x)}\| > 1.$$

We split $N_0 = N_1 + N_2$, such that N_1 is the largest integer satisfying

$$|f^{-n_0N_1}(J^s(x,y))| \le \eta_0.$$

Since $|J^s(x,y)| = d_{\mathcal{F}^s_f}(x,y) \le C_3/D^\theta$, we have

$$N_1 \ge \frac{\theta \log D + \log \eta_0 - \log C_3}{n_0 \log \gamma}.$$

Let

$$\beta = \frac{1}{2} \cdot \frac{-\theta \log \mu}{\log \gamma},$$

which only depends on the contracting and expanding rates of f on stable and unstable bundles.

Now we fix the constant δ so that it satisfies

$$(1+\delta)^{[\frac{5}{\beta}]+1} \cdot \lambda_- < \lambda_+.$$

Then we have

$$\begin{aligned}\frac{N_1}{N_0} &\ge \frac{\theta \log D + \log \eta_0 - \log C_3}{n_0 \log \gamma} \cdot \frac{-n_0 \log \mu}{\log D - n_0 \log \mu}\\ &= \frac{-\theta \log \mu}{\log \gamma} \cdot \frac{1 + \frac{\log \eta_0}{\theta \log D} - \frac{\log C_3}{\theta \log D}}{1 - \frac{n_0 \log \mu}{\log D}}\\ &= 2\beta \cdot \frac{1 + \frac{\log \eta_0}{\theta \log D} - \frac{\log C_3}{\theta \log D}}{1 - \frac{n_0 \log \mu}{\log D}}.\end{aligned}$$

So there exists $D_0 > 0$, such that if $D \ge D_0$, then we have

$$N_1 > \beta \cdot N_0.$$

We can estimate the growth rate of $|J_1|$ now. For every $z \in J_1$, we have for every $0 \le k \le n_0 N_1$,

$$d(f^{-k}(z), f^{-k}(q)) \le |f^{-k}(J_1)| + |f^{-k}(J^s(x,y))| + |f^{-k}(J^c(y,q))| \le 3\eta_0.$$

This implies that

$$|f^{-n_0N_1}(J_1)| \le (1+\delta)^{n_0N_1} \lambda_c(q)^{-n_0N_1} |J_1| \le (1+\delta)^{2n_0N_1} \lambda_+^{-n_0N_1} |J_1|.$$

Since $\|Df|_{E^c_f(x)}\| > \lambda_-/(1+\delta)$ for every $x \in \mathbb{T}^3$, we have

$$\begin{aligned}|f^{-n_0N_0}(J_1)| &\leq (1+\delta)^{n_0N_2}\lambda_-^{-n_0N_2}\cdot|f^{-n_0N_1}(J_1)|\\ &< (1+\delta)^{n_0N_2}\lambda_-^{-n_0N_2}\cdot(1+\delta)^{2n_0N_1}\lambda_+^{-n_0N_1}|J_1|\\ &< (1+\delta)^{2n_0N_0}\lambda_-^{-n_0N_2}\lambda_+^{-n_0N_1}|J_1|.\end{aligned}$$

Thus we have

$$\begin{aligned}\frac{|f^{-n_0N_0}(J_1)|}{|f^{-n_0N_0}(J_0)|} &< \frac{(1+\delta)^{2n_0N_0}\lambda_-^{-n_0N_2}\lambda_+^{-n_0N_1}}{\lambda_-^{-n_0N_0}\cdot(1+\delta)^{-2n_0N_0}}\cdot\frac{|J_1|}{|J_0|}\\ &\leq (1+\delta)^{4n_0N_0}\cdot\Big(\frac{\lambda_-}{\lambda_+}\Big)^{\beta n_0N_0}\cdot\frac{|J_1|}{|J_0|}\\ &< (1+\delta)^{-n_0N_0}\cdot\frac{\eta_1}{\eta_2}.\end{aligned}$$

When D tends to infinity, N_0 tends to infinity, and $|f^{-n_0N_0}(J_1)|/|f^{-n_0N_0}(J_0)|$ tends to zero. Since $d_{\mathcal{F}^u_f}(p, f^{-n_0N_0}(x)) \leq 1$, this implies that the holonomy map of unstable foliations restricted in $\mathcal{F}^{cu}_f(p)$ is not C^1-smooth. This contradicts Theorem 2.2, which states that these holonomy maps are locally uniformly C^1-smooth. $\square$

5. Equivalent conditions for su-integrability

From the proof of Proposition 3.3 and Proposition 4.1, we can see that if $f : \mathbb{T}^3 \to \mathbb{T}^3$ is partially hyperbolic and Anosov, then f is su-integrable as a partially hyperbolic diffeomorphism if and only if $\lambda_c(p) = \lambda_c(A)$ for every $p \in \mathrm{Per}(f)$. Combined with the Main Theorem of [20], we have a series of equivalent conditions to su-integrability of f.

Theorem 5.1. *Let f be a $C^{1+\alpha}$ partially hyperbolic and Anosov diffeomorphism, which is topologically conjugate to an Anosov automorphism A, on $\mathbb{T}^3$. The following conditions are equivalent:*

1. *f is su-integrable;*
2. *f is not accessible;*
3. *The topological conjugacy h ($h \circ f = A \circ h$) preserves unstable foliation of f: $h(\mathcal{F}^u_f) = \mathcal{F}^u_A$;*
4. *The lifting unstable foliation $\tilde{\mathcal{F}}^u_f$ is homology bounded in $\mathbb{R}^3$, i.e. $\tilde{\mathcal{F}}^u_f(x)$ is uniformly bounded with $\tilde{\mathcal{F}}^u_A(x)$ for every $x \in \mathbb{R}^3$;*
5. *$\lambda_c(p) = \lambda_c(A)$ for every periodic point $p \in \mathrm{Per}(f)$;*
6. *The topological conjugacy h is differentiable along $\mathcal{F}^c_f$.*

Proof. The equivalence from Item 1 to Item 4 has been proved in [20, Main Theorem]. The equivalence between Item 1 to Item 5 has been proved Proposition 3.3 and Proposition 4.1. We only need to prove the equivalence between Item 5 to Item 6.

Item 5 $\Longrightarrow$ Item 6. Let p be a fixed point of f. The point $p' = h(p)$ is a fixed point of A. Now we choose a point $x \in \mathcal{F}_f^c(p)$, and denote by $J \subset \mathcal{F}_f^c(p)$ the center arc admitting p, x as two endpoints. Then the points $p', x' = h(x)$ are endpoints of $J' = h(J) \subset \mathcal{F}_A^c(p')$.

From Lemma 3.1 and Remark 3.2, there exists a continuous metric $d^c(\cdot,\cdot)$ defined on every leaf of $\mathcal{F}_f^c$, satisfying all three properties in Lemma 3.1 and

$$d^c(p,x) = |J'| = d_{\mathcal{F}_A^c}(p',x').$$

Here $|J'|$ is the length of arc J'.

Claim 5.1. *The conjugacy $h|_J : J \to J'$ is an isometry between $d^c(\cdot,\cdot)$ on J and $d_{\mathcal{F}_A^c}(\cdot,\cdot)$ on J'.*

Proof of the claim. Denote by $x_{1/2} \in J$ be the middle point between p and x under $d^c(\cdot,\cdot)$, i.e.

$$d^c(p,x_{1/2}) = d^c(x_{1/2},x).$$

We want to show that

$$d_{\mathcal{F}_A^c}(p',h(x_{1/2})) = d_{\mathcal{F}_A^c}(h(x_{1/2}),x').$$

Since $\mathcal{F}_f^s(p)$ is dense in $\mathbb{T}^3$, there exists $y_n \in \mathcal{F}_f^s(p)$ such that $y_n \to x_{1/2}$ as $n \to \infty$. Now we consider the holonomy map

$$h^s_{p,y_n} : \mathcal{F}_f^c(p) \to \mathcal{F}_f^c(y_n).$$

Since h^s_{p,y_n} is an isometry under the metric $d^c(\cdot,\cdot)$ and $d^c(p,x_{1/2}) = d^c(x_{1/2},x)$, we have

$$h^s_{p,y_n}(x_{1/2}) \to x \quad \text{as } n \to \infty.$$

On the other hand, $h(\mathcal{F}_f^s(p)) = \mathcal{F}_A^s(p')$ implies $h(y_n) \in \mathcal{F}_A^s(p')$ and

$$h(y_n) \to h(x_{1/2}) \quad \text{as } n \to \infty.$$

Moreover, we have

$$h \circ h^s_{p,y_n}(x_{1/2}) \to x' \quad \text{as } n \to \infty.$$

This implies $d_{\mathcal{F}_A^c}(p',h(x_{1/2})) = d_{\mathcal{F}_A^c}(h(x_{1/2}),x')$.

Repeating this procedure:

- denote by $x_{1/4}$ the middle point between p and $x_{1/2}$ under $d^c(\cdot,\cdot)$, then we have

$$d_{\mathcal{F}_A^c}(p',h(x_{1/4})) = d_{\mathcal{F}_A^c}(h(x_{1/4}),h(x_{1/2}));$$

- denote by $x_{3/4}$ the middle point between $x_{1/2}$ and x' under $d^c(\cdot,\cdot)$, then we have

$$d_{\mathcal{F}_A^c}(h(x_{1/2}), h(x_{3/4})) = d_{\mathcal{F}_A^c}(h(x_{3/4}), x').$$

Again, we take the middle points between p and $x_{1/4}$, $x_{1/4}$ and $x_{1/2}$, $x_{1/2}$ and $x_{3/4}$, $x_{3/4}$ and x', respectively. The same argument shows that h preserves all the middle points between these intervals and their images by h. Repeating this procedure, form the density of these middle points, $h|_J : J \to J'$ is an isometry between $d^c(\cdot,\cdot)$ on J and $d_{\mathcal{F}_A^c}(\cdot,\cdot)$ on J'. $\square$

Recall that

$$d^c(f(x_1), f(x_2)) = \lambda_c(A) \cdot d^c(x_1, x_2)$$

for every $x_1, x_2 \in J$ and $\|DA|_{E_A^c}\| \equiv \lambda_c(A)$. Since h is an isometry between $d^c(\cdot,\cdot)$ on J and the natural distance on J', it is an isometry between $d^c(\cdot,\cdot)$ on $\mathcal{F}_f^c(p)$ and the natural distance on $\mathcal{F}_A^c(p')$. From the density of $\mathcal{F}_f^c(p)$ in $\mathbb{T}^3$, this shows that h is an isometry between $d^c(\cdot,\cdot)$ on every leaf of $\mathcal{F}_f^c$ and the natural distance on every leaf of $\mathcal{F}_A^c$.

Finally, for every $z \in \mathcal{F}_f^c(y)$ and $y \in \mathbb{T}^3$, let $\gamma : [0,1] \to \mathcal{F}_f^c(y)$ be a C^1-curve connecting y and z, then

$$d^c(y,z) := \int_0^1 \exp(\phi \circ \gamma(t)) \cdot |\gamma'(t)| \, \mathrm{d}t.$$

Let $z \to y$, it implies

$$\|Dh|_{E_f^c(y)}\| = e^{\phi(y)}, \quad \forall y \in \mathbb{T}^3,$$

which proves that h is differentiable in the center direction.

Item 6 $\Longrightarrow$ Item 5. Let $p \in \mathrm{Per}(f)$ be a periodic point of f with period $\pi(p)$. Since h is differentiable along $\mathcal{F}_f^c$, there exists a small arc $J \subset \mathcal{F}_f^c(p)$ containing p and a constant $C > 1$, such that for any subarc $I \subseteq J$, it satisfies

$$\frac{1}{C} \le \frac{|h(I)|}{|I|} \le C.$$

From the conjugacy, we have

$$h \circ f^{-k\cdot\pi(p)}(J) = A^{-k\cdot\pi(p)} \circ h(J) \subseteq h(J), \quad \forall k \ge 0.$$

Since f is $C^{1+\alpha}$-smooth and both f^{-1} is uniformly contracting in the center direction, the distortion control techniques shows that there exists another constant $K > 1$, such that

$$\frac{1}{K} \cdot \lambda_c(p)^{-k\cdot\pi(p)} < \frac{|f^{-k\cdot\pi(p)}(J)|}{|J|} < K \cdot \lambda_c(p)^{-k\cdot\pi(p)}, \quad \forall k \ge 0.$$

On the other hand, we have

$$|A^{-k\cdot\pi(p)}(h(J))| = \lambda_c(A)^{-k\cdot\pi(p)} \cdot |h(J)|$$

for every $k \geq 0$.

This shows that

$$\begin{aligned}\frac{1}{K} \cdot \frac{\lambda_c(A)^{-k\cdot\pi(p)} \cdot |h(J)|}{\lambda_c(p)^{-k\cdot\pi(p)} \cdot |J|} &< \frac{|A^{-k\cdot\pi(p)}(h(J))|}{|f^{-k\cdot\pi(p)}(J)|} \\ &< K \cdot \frac{\lambda_c(A)^{-k\cdot\pi(p)} \cdot |h(J)|}{\lambda_c(p)^{-k\cdot\pi(p)} \cdot |J|}, \qquad \forall k \geq 0.\end{aligned}$$

Since $1/C \leq |h(I)|/|I| \leq C$ for every $I \subseteq J$, we have

$$\frac{1}{K \cdot C^2} < \frac{\lambda_c(A)^{-k\cdot\pi(p)}}{\lambda_c(p)^{-k\cdot\pi(p)}} < K \cdot C^2, \quad \forall k \geq 0.$$

This proves $\lambda_c(p) = \lambda_c(A)$. □

Remark 5.2. It should notice that we can build an f such that its topological conjugacy is differentiable only in the center direction. Let $p \in \mathbb{T}^3$ be a fixed point of A. We compose with a rotation around p in the stable and unstable plane. For the new diffeomorphism, the stable and unstable Lyapunov exponents of p are different from A. The topological conjugacy is differentiable in the center foliation.

However, when f is C^1-close to A, it has been showed by Gogolev and Guysinsky [9, 10] that the topological conjugacy is smooth if and only if all periodic points of f admit the same three Lyapunov exponents as A. Thus the topological conjugacy is not differentiable.

References

[1] C. Bonatti, S. Gan, and D. Yang, On the hyperbolicity of homoclinic classes, *Discrete Contin. Dyn. Syst.*, **25** (2009), no. 4, 1143–1162. Zbl 1200.37023 MR 2552132

[2] M. Brin, D. Burago, and S. Ivanov, Dynamical coherence of partially hyperbolic diffeomorphisms of the 3-torus, *J. Mod. Dyn.*, **3** (2009), no. 1, 1–11. Zbl 1190.37026 MR 2481329

[3] D. Burago and S. Ivanov, Partially hyperbolic diffeomorphisms of 3-manifolds with abelian fundamental groups, *J. Mod. Dyn.*, **2** (2008), no. 4, 541–580. Zbl 1157.37006 MR 2449138

[4] K. Burns and A. Wilkinson, On the ergodicity of partially hyperbolic systems, *Ann. of Math. (2)*, **171** (2010), no. 1, 451–489. Zbl 1196.37057 MR 2630044

[5] J. Franks, Anosov diffeomorphisms, in *Global Analysis (Proc. Sympos. Pure Math., Vol. XIV, Berkeley, Calif., 1968)*, 61–93, Amer. Math. Soc., Providence, R.I., 1970. Zbl 0207.54304 MR 271990

[6] S. Gan, A generalized shadowing lemma, *Discrete Contin. Dyn. Syst.*, **8** (2002), no. 3, 627–632. Zbl 0996.37032 MR 1897871

[7] S. Gan, Horseshoe and entropy for C^1 surface diffeomorphisms, *Nonlinearity*, **15** (2002), no. 3, 841–848. Zbl 1017.37006 MR 1901109

[8] A. Gogolev, How typical are pathological foliations in partially hyperbolic dynamics: an example, *Israel J. Math.*, **187** (2012), 493–507. Zbl 1268.37029 MR 2891713

[9] A. Gogolev, Bootstrap for local rigidity of Anosov automorphisms on the 3-torus, *Comm. Math. Phys.*, **352** (2017), no. 2, 439–455. Zbl 1401.37036 MR 3627403

[10] A. Gogolev and M. Guysinsky, C^1-differentiable conjugacy of Anosov diffeomorphisms on three dimensional torus, *Discrete Contin. Dyn. Syst.*, **22** (2008), no. 1-2, 183–200. Zbl 1153.37341 MR 2410954

[11] A. Hammerlindl, Leaf conjugacies on the torus, *Ergodic Theory Dynam. Systems*, **33** (2013), no. 3, 896–933. Zbl 1390.37051 MR 3062906

[12] A. Hammerlindl, Ergodic components of partially hyperbolic systems, *Comment. Math. Helv.*, **92** (2017), no. 1, 131–184. Zbl 1379.37071 MR 3615038

[13] A. Hammerlindl and R. Potrie, Pointwise partial hyperbolicity in three-dimensional nilmanifolds, *J. Lond. Math. Soc. (2)*, **89** (2014), no. 3, 853–875. Zbl 1309.37033 MR 3217653

[14] A. Hammerlindl and R. Ures, Ergodicity and partially hyperbolicity on the 3-torus, *Commun. Contemp. Math.*, **16** (2014), no. 4, 1350038, 22pp. Zbl 1347.37060 MR 3231058

[15] A. Katok, Lyapunov exponents, entropy and periodic orbits for diffeomorphisms, *Inst. Hautes Études Sci. Publ. Math.*, (1980), no. 51, 137–173. Zbl 0445.58015 MR 573822

[16] R. Potrie, A few remarks on partially hyperbolic diffeomorphisms of $\mathbb{T}^3$ isotopic to Anosov, *J. Dynam. Differential Equations*, **26** (2014), no. 3, 805–815. Zbl 1320.37015 MR 3274442

[17] R. Potrie, Partial hyperbolicity and foliations in $\mathbb{T}^3$, *J. Mod. Dyn.*, **9** (2015), 81–121. Zbl 1352.37055 MR 3395262

[18] C. Pugh and M. Shub, Stable ergodicity and partial hyperbolicity, in *International Conference on Dynamical Systems (Montevideo, 1995)*, 182–187, Pitman Res. Notes Math. Ser., 362, Longman, Harlow, 1996. Zbl 0867.58049 MR 1460804

[19] C. Pugh, M. Shub, and A. Wilkinson, Hölder foliations, *Duke Math. J.*, **86** (1997), no. 3, 517–546. Zbl 0877.58045 MR 1432307

[20] Y. Ren, S. Gan, and P. Zhang, Accessibility and homology bounded strong unstable foliation for Anosov diffeomorphisms on 3-torus, *Acta Math. Sin. (Engl. Ser.)*, **33** (2017), no. 1, 71–76. Zbl 1370.37065 MR 3581607

[21] F. Rodriguez Hertz, J. Rodriguez Hertz, and R. Ures, Partial hyperbolicity and ergodicity in dimension three, *J. Mod. Dyn.*, **2** (2008), no. 2, 187–208. Zbl 1148.37019 MR 2383266

[22] F. Rodriguez Hertz, J. Rodriguez Hertz, and R. Ures, Accessibility and stable ergodicity for partially hyperbolic diffeomorphisms with 1D-center bundle, *Invent. Math.*, **172** (2008), no. 2, 353–381. Zbl 1136.37020 MR 2390288

[23] F. Rodriguez Hertz, J. Rodriguez Hertz, and R. Ures, Tori with hyperbolic dynamics in 3-manifolds, *J. Mod. Dyn.*, **5** (2011), no. 1, 185–202. Zbl 1221.37056 MR 2787601

[24] R. Ures, Intrinsic ergodicity of partially hyperbolic diffeomorphisms with a hyperbolic linear part, *Proc. Amer. Math. Soc.*, **140** (2012), no. 6, 1973–1985. Zbl 1258.37033 MR 2888185

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