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Squeezing Lagrangian tori in dimension 4

Richard Hind* and Emmanuel Opshtein

Abstract. Let $\mathbb{C}^2$ be the standard symplectic vector space and $L(a,b) \subset \mathbb{C}^2$ be the product Lagrangian torus, that is, a product of two circles of areas a and b in $\mathbb{C}$. We give a complete answer to the question of finding the minimal ball into which these Lagrangians may be squeezed by a Hamiltonian flow. The result is that there is full rigidity when $a \leq b \leq 2a$, which disappears almost completely when $b > 2a$.

Mathematics Subject Classification (2010). 53D12; 53D35.

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1. Introduction

In this paper we investigate the extent to which product Lagrangian tori can be 'squeezed' by Hamiltonian diffeomorphisms. To be precise, we determine when such a torus can be mapped into a ball or a polydisk. To fix notation, we work in the vector space $\mathbb{R}^4 \approx \mathbb{C}^2$ equipped with its standard symplectic form $\omega = \sum_{i=1}^{2} dx_i \wedge dy_i$. The Lagrangian product tori are defined by

$$L(a,b) = \{\pi|z_1|^2 = a, \pi|z_2|^2 = b\}.$$

The open ball of capacity R is given by

$$B(R) = \{\pi(|z_1|^2 + |z_2|^2) < R\},$$

and our polydisks are defined by

$$P(a,b) = \{\pi|z_1|^2 < a, \pi|z_2|^2 < b\}.$$

Hence $L(a,b)$ is the singular part of the boundary of $P(a,b)$. Up to renormalizing the symplectic form, it is enough to study the special case of squeezing Lagrangian tori $L(1,x)$, $x \geq 1$. The current paper contains the first results about non-monotone tori, but the monotone case is already known: $L(1,1)$ (or any torus with monotonicity constant 2) cannot be squeezed into a ball of size 2, see [3]. Our main result is the following:

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Theorem 1. *For $x \geq 1$, there exists a Hamiltonian diffeomorphism of $\mathbb{C}^2$ that takes $L(1,x)$ into $B(R)$ if and only if $R > \min(1+x, 3)$.*

In other words, the Lagrangian torus $L(1,x)$ belongs to the boundary of the ball $B(1+x)$, and when $x \leq 2$ it cannot be squeezed into a smaller ball, while when $x > 2$ it can be squeezed into a ball of size $3+\varepsilon$ if and only if $\varepsilon > 0$. By the Arnold conjecture, taking $L(1,x)$ even into $B(1+x)$ cannot be done by a Hamiltonian with support in a neighbourhood of the torus. Indeed the flows we construct may have large support in $\mathbb{C}^2$.

Our result when $x \geq 2$ holds in fact in a slightly more general setting that we review now. For a Lagrangian $L \subset \mathbb{C}^2$, there are two homomorphisms

$$\Omega, \mu \colon H_1(L, \mathbb{Z}) \to \mathbb{R}$$

describing how L is embedded in $\mathbb{C}^2$. The first homomorphism is the area class and is defined by

$$\Omega(e) = [\lambda](e)$$

where λ is a Liouville form, that is, a primitive of ω. Equivalently,

$$\Omega(e) = \int_D u^*\omega$$

where D is a disk and $u \colon (D, \partial D) \to (\mathbb{C}^2, L)$ verifies $u_*[\partial D] = e$. The second homomorphism is the Maslov class. If $u \colon S^1 \to L$ with $u_*[S^1] = e$ then $\mu(e)$ is the Maslov class of the loop of Lagrangian subspaces $T_{u(t)}L \subset \mathbb{C}^2$.

Theorem 2. *Suppose $L \subset B(R)$ is a Lagrangian torus and e_1, e_2 is an integral basis of $H_1(L, \mathbb{Z})$ satisfying:*

1. $\Omega(e_1) = 1$, $\Omega(e_2) \geq 2$,
2. $\mu(e_1) = \mu(e_2) = 2$.

Then $R > 3$.

In general squeezing a Lagrangian torus does not depend on the area and Maslov classes alone. For example, $L(1,2)$ and $L(2,3)$ both have integral bases with Maslov class 2 and area classes 1 and 2. However $L(1,2)$ lies in the boundary of $B(3)$ while a rescaling and Theorem 1 implies that $L(2,3)$ cannot be mapped to the interior of $B(5)$.

To the best of our knowledge, although any non-monotone Lagrangian torus in $\mathbb{C}^2$ may be conjectured to be Hamiltonian isotopic to a product one, there is no proof available at this time. In [6], Dimitroglou Rizell–Goodman–Ivrii prove a weaker unknottedness result, namely that any two Lagrangian tori are Lagrangian isotopic. The conjecture would imply that Theorem 2 is a consequence of Theorem 1.

Theorem 1 has an analogue for Lagrangian tori inside polydisks.

Theorem 3. *For $x \geq 1$ and $a \leq b$ there exists a Hamiltonian diffeomorphism of $\mathbb{C}^2$ that takes $L(1,x)$ into $P(a,b)$ if and only if either $a > 2$ or both $a > 1$ and $b > x$.*

In particular Theorems 1 and 3 show that embeddings of Lagrangian tori into balls or polydisks do not necessarily extend to the corresponding polydisks, where there is a volume obstruction. Even more, there are *symplectic* obstructions to embedding a polydisk into a ball which do not obstruct squeezing its Lagrangian 'singular boundary' (see [5, 12] and compare to Theorem 1). Therefore our results do not seem immediately relevant for studying the squeezing of polydisks $P(1,x)$, which remains open for $x > (\sqrt{7}-1)/(\sqrt{7}-2)$. Nevertheless our approach can be applied to the stabilized polydisk embedding problem $P(1,x) \times \mathbb{C}^n \hookrightarrow B(R) \times \mathbb{C}^n$, see [8].

Outline of the paper. Theorems 1 and 3 have two facets. On the one hand, they assert some obstructions for squeezing a Lagrangian torus into a small ball or polydisc. On the other hand, they claim the existence of some embeddings, that we need to construct explicitly.

Our obstructions ultimately come from holomorphic planes in the complement of a Lagrangian torus in $\mathbb{C}P^2$ which have large 'degree' and are asymptotic at their puncture to a large multiple of a closed geodesic for a flat metric on the torus. It seems necessary to study such high degree curves in order to derive sharp bounds. Our approach is to begin with finite energy planes asymptotic to thin ellipsoids which were introduced in [9]. In a sense we can think of these seed curves as substitutes for high degree rational curves which are very singular at a point of our Lagrangian (and by the adjunction formula do not exist). We place the ellipsoid in a tubular neighborhood of the torus and then carry out a neck-stretching process along the unit circle bundle.

We fix notation and describe the seed curves and neck-stretching arrangement in Section 2. Then in Section 3 we collect several lemmas on properties of curves in the holomorphic buildings appearing as limits after stretching, in particular obtaining restrictions on their Fredholm indices.

In Section 4, we prove Theorem 2 and the obstruction part of Theorem 1. Embeddings into balls are actually technically more difficult to study than into polydiscs (where one can work throughout with embedded curves). In Section 5, we outline the adjustments needed to deal with embeddings into polydisks.

Finally, we deal with the constructive part of Theorems 1 and 3 at the same time, by proving the following:

Theorem 4. *If $x > 2$, for any $\lambda > 1$, there exists a Hamiltonian diffeomorphism of $\mathbb{C}^2$ that takes the product torus $L(1,x)$ into $B(3\lambda) \cap P(2\lambda, 2\lambda)$.*

This is described in Section 6. Our approach is to explicitly write down the image of the Lagrangian torus, which can readily be seen to have Maslov and area classes

corresponding to those of $L(1,x)$. We will then rely on techniques used in [6] to show that what we have described is in fact the image of a product torus under a Hamiltonian diffeomorphism, rather than a possible exotic nonmonotone torus in $\mathbb{C}^2$ (which conjecturally do not exist).

2. Geometric framework

The proof of Theorem 1 is based on neck-stretching arguments in the following setting. Let $L \subset B^4(R)$ be an embedded Lagrangian torus in the ball of capacity R. We can compactify $B(R)$ to a projective plane $\mathbb{CP}^2(R)$ with lines of area R and will denote by S_∞ the line at infinity. The cotangent bundle of $\mathbb{T}^2$ can be symplectically identified with $\mathbb{R}^4/\mathbb{Z}^2$ where $\mathbb{Z}^2$ acts by translations in the (x_1, x_2)-plane. We fix a very large integer d and an irrational number $S > 3d-1$. Then, for ε small enough, the neighbourhood

$$V_{\varepsilon,S} := \{|y_1| \le \varepsilon/2, |y_2| \le \varepsilon S/2\}$$

of the zero section symplectically embeds into $B(R)$ as a Weinstein neighbourhood of L, denoted by V. Now $V_{\varepsilon,S}$ contains a symplectic bidisk $\frac{1}{2}P(\varepsilon, \varepsilon S)$, which by inclusion contains an ellipsoid $\varepsilon E(1,S)$. By definition $\varepsilon E(1,S) = E(\varepsilon, \varepsilon S)$ and the ellipsoids are defined by

$$E(a,b) = \left\{\pi\left(\frac{|z_1|^2}{a} + \frac{|z_2|^2}{b}\right) < 1\right\}.$$

Putting everything together, we therefore have inclusions

$$\varepsilon E(1,S) \subset V \subset B(R) \subset \mathbb{CP}^2(R).$$

We will consider some holomorphic curves in $\mathbb{CP}^2(R)$ that we wish to stretch along the boundary of V. But since this boundary has corners, we first replace $V_{\varepsilon,S}$ by a smooth approximation. Following [11], we define

$$U^p_{\varepsilon,S} := \left\{\left\|\left(y_1, \frac{y_2}{S}\right)\right\|_p < \frac{\varepsilon}{2}\right\} \subset T^*\mathbb{T}^2,$$

where $\|(\xi_1,\xi_2)\|_p = \left(|\xi_1|^p + |\xi_2|^p\right)^{\frac{1}{p}}$. Then, $U^p_{\varepsilon,S}$ is a smoothly bounded fiberwise convex subset of T^*L that contains $V_{\varepsilon,S}$, and is close to $V_{\varepsilon,S}$ in the Hausdorff topology when p is large. For large enough p, we therefore have:

$$E \subset U \subset B(R) \subset \mathbb{CP}^2(R),$$

where U is a symplectic embedding of $U^p_{\varepsilon,S}$ and E stands for our inclusion of $\varepsilon E(1,S)$. We then have a symplectic cobordism $X = \mathbb{CP}^2(R) \setminus E$ which supports tame

almost-complex structures with cylindrical ends compatible with the Liouville contact structure on ∂E. We study finite energy J-holomorphic curves in X. These will be J-holomorphic maps

$$u: \mathbb{CP}^1 \setminus \Gamma \to X,$$

where Γ is a finite set of punctures and u is asymptotic to closed Reeb orbits on ∂E at each puncture. We define the degree of these maps to be simply their intersection number with the line at infinity $S_\infty = \mathbb{CP}^2(R) \setminus B(R)$, and their asymptotics are iterates of the two closed Reeb orbits on ∂E, namely (the images of)

$$\gamma_1 = \partial E \cap \{x_2 = y_2 = 0\}$$

and

$$\gamma_2 = \partial E \cap \{x_1 = y_1 = 0\}.$$

Some special finite-energy curves in X. Our starting point is an existence theorem for some curves of degree d in this cobordism:

Theorem 2.1 (Hind–Kerman [9, 10]). *There exists an infinite subset $A \subset \mathbb{N}$ such that, for any $d \in A$, $S > 3d - 1$ irrational and $\varepsilon E(1, S) \subset \mathbb{CP}^2(R)$ and for any generic J, there exists a rigid finite energy plane $u: \mathbb{C} \to X$ of degree d asymptotic to the Reeb orbit γ_1^{3d-1}.*

This theorem was claimed for any d in [9, Theorem 2.36]. Unfortunately there was a mistake in the proof, which is corrected in [10] at the expense of establishing the result only for d belonging to a sequence of natural numbers that diverge to $+\infty$. In [14], McDuff proves the statement claimed in [9], that the previous result holds with $A = \mathbb{N}$. The version considered here is however enough for the purpose of the current paper.

Neck-stretching. Let now J_n be a sequence of almost complex structures on X that are cylindrical near ∂E and that stretch the neck along ∂U. Let $u_n: \mathbb{C} \to X$ be J_n-holomorphic finite energy planes provided by Theorem 2.1. They have degree d, and are asymptotic at their puncture to γ_1^{3d-1}. By [2], this set of curves enjoys a compactness property. In a now well known sense, our sequence of curves converges (modulo extraction) as $n \to \infty$ to a holomorphic building B made of finite energy holomorphic curves in $S\partial E$, $U \setminus E$, $S\partial U$, and $\mathbb{CP}^2 \setminus U$. (Here $S\partial E$ and $S\partial U$ denote the symplectization of ∂E and ∂U respectively, with cylindrical almost-complex structures.) These curves have positive and negative ends that are asymptotic to Reeb orbits of ∂U or ∂E. All these ends but one match together pairwise. The unmatched end is asymptotic to γ_1^{3d-1} on ∂E. Moreover, gluing the different components along their matching ends provide a topological surface which is a bunch of spheres (that may appear because of bubbling phenomenon) and one plane that contains the unmatched end. We need to gather information on the limit building. To make our analysis manageable we will identify various sets of limit curves with matching ends and consider them as single components of the limit. Once the ends are identified,

we can still talk about the index of such a glued component (see Section 3.1). The identifications are made as follows.

(I) The limiting building has a unique curve u_0 in its lowest level with negative end asymptotic to γ_1^{3d-1}. All curves which can be connected to u_0 through a chain of curves with matching ends lying in $U \setminus E$ or the symplectization layers are identified along their matching ends to form our first component F_0.

(II) Suppose F_0 has T unmatched ends. Then the complement of F_0 in our limiting building, after identifying matching ends, has exactly T components which we denote $F_1, \dots, F_T$.

3. Preliminaries

3.1. Reeb orbits and index formulas. The Reeb flow on ∂U is conjugated to that of $\partial U^p_{\varepsilon,S}$, which can be easily computed. It preserves the tori $\{(y_1, y_2) = c\}$, and is a linear flow on each such torus, whose slope depends on c. When this slope is rational, the torus is foliated by a 1-dimensional family of periodic orbits.

Proposition 3.1. *Fix an integral basis of $H_1(L, \mathbb{Z})$. For each pair of integers (k, l) except $(0, 0)$ there is a 1-parameter family of closed Reeb orbits on ∂U which project to a curve in L in the class (k, l). We say these Reeb orbits are of type (k, l) and denote them by $\gamma_{k,l}$. The orbits are embedded if and only if k, l have no common factor. Orbits $\gamma_{rk,rl}$ are r-times covers of the orbits $\gamma_{k,l}$.*

We now recall the index formulas for the different curves that may appear after the neck stretching process. Before specializing to our situation, let us consider the general setting of holomorphic curves in symplectic cobordisms, and their index [1, 18]. Let $\dot{\Sigma}$ be a punctured surface of genus g with s punctures, and W a symplectic cobordism. Recall that this means that W is a symplectic manifold with boundaries $\partial W^+ \sqcup \partial W^-$, which are equipped with locally defined outward (inward respectively) pointing Liouville vector fields. The Liouville vector fields define contact forms on the boundary components of W, and we assume that their Reeb vector fields are Morse–Bott: the closed Reeb orbits on ∂W may come in smooth families, along which the transverse Poincaré return maps are non-degenerate. Let also J be an almost complex structure adapted to our cobordism (compatible with the symplectic structure and cylindrical near the ends). Given a finite energy J-holomorphic curve

$$u\colon \dot{\Sigma} \to W,$$

we denote by

$$\gamma_i^+,\ i = 1 \dots s_+ \quad \text{and} \quad \gamma_j^-,\ j = 1 \dots s_-$$

the positive and negative limiting Reeb orbits in ∂W^+ and ∂W^-, respectively. By the Morse–Bott condition, these asymptotics belong to families of closed Reeb orbits,

denoted S_i^+, S_j^-. We also fix a symplectic trivialization τ of u^*TW along these asymptotics. Then, $c_1^\tau(u^*TW)$ denotes the algebraic number of zeros of a generic section of the vector bundle $\Lambda^2 u^*TW$ which is constant with respect to τ on the boundary. Following [1, 18], the formula for the expected dimension of the moduli space of holomorphic curves (moduli reparameterizations) in the same homology class and having the same asymptotics as u, called below the index of u, is given by

$$\begin{aligned}\operatorname{index}(u) = (n-3)\chi(\dot{\Sigma}) + 2c_1^\tau(u^*TX) &+ \sum_{i=1}^{s_+}\Big(\mu_{\mathrm{CZ}}(\gamma_i^+) + \frac{1}{2}\dim S_i^+\Big)\\ &- \sum_{j=1}^{s_-}\Big(\mu_{\mathrm{CZ}}(\gamma_j^-) - \frac{1}{2}\dim S_j^-\Big). \qquad (3.1)\end{aligned}$$

In this formula, $\mu_{\mathrm{CZ}}(\gamma)$ represents the classical Conley–Zehnder index of γ when it is non-degenerate (and in this case $\dim S(\gamma)$ vanishes), or the generalized Maslov (Robin–Salamon) index of γ in the general case [7, 16]. Note that this dimension formula takes care of the Teichmuller space of $\dot{\Sigma}$, or its automorphisms group.

As we explained above, we will need for practical computations to (abstractly) glue several curves of the building, and consider the resulting subbuilding as a single entity. We then consider the punctures of this subbuilding to be those of its constituent curves *that do not serve as matching ends*. We can then define a notion of index for such a building:

Definition 3.2. Let B be a building made of curves $(u_1, \dots, u_k)$ (in various layers) that match along asymptotic orbits $(\gamma_1, \dots, \gamma_l)$ belonging to spaces $S_1, \dots, S_l$ of closed Reeb orbits (we assume the Morse–Bott situation, where these spaces are manifolds). We define

$$\operatorname{index}(B) := \sum_{i=1}^{k} \operatorname{index}(u_i) - \sum_{i=1}^{l} \dim S_i.$$

Working with this definition, the following proposition sums up those properties of the index that will be important for us.

Proposition 3.3. *The index formula of the buildings has the following properties:*

1. Recursivity: *Let B be a building obtained by gluing different buildings B_i along matching orbits γ_i that belong to spaces S_i of Reeb orbits. Then,*

$$\operatorname{index}(B) = \sum \operatorname{index}(B_i) - \sum \dim S_i.$$

2. Computability: *Let B be a holomorphic building and let the underlying curve (after gluing) be Σ, the positive punctures be γ_i^+ and negative punctures be γ_i^-*

(recall that those punctures of the constituent curves that have to matched to form B are not considered as punctures of B). Then,

$$\operatorname{index}(B) = (n-3)\chi(\dot{\Sigma}) + 2c_1^\tau(B) + \sum_{i=1}^{s_+}\left(\mu_{\mathrm{CZ}}(\gamma_i^+) + \frac{1}{2}\dim S_i^+\right) - \sum_{j=1}^{s_-}\left(\mu_{\mathrm{CZ}}(\gamma_j^-) - \frac{1}{2}\dim S_j^-\right).$$

Here $c_1^\tau(B)$ simply means the sum of the $c_1^\tau(u^*TX)$ over the constituent curves of B.

3. Continuity: *Let X be a symplectic cobordism, (J_n) a neck-stretching in X (along some hypersurface), and (B_n) a sequence of J_n-holomorphic buildings that converge in the sense of* [2] *to a building B. Then*

$$\operatorname{index}(B) = \lim \operatorname{index}(B_n).$$

Let us now specialize formula (3.1) to our context. We recall that E denotes our embedded ellipsoid $\varepsilon E(1, S)$ in U.

Proposition 3.4. *Let $\dot{\Sigma}$ be a punctured sphere, and $u\colon \dot{\Sigma} \to W$ be a J-holomorphic map asymptotic to γ_i^+, γ_j^-, $i = 1 \dots s_+$, $j = 1 \dots s_-$. Let $s := s_+ + s_-$. We denote the Reeb orbits on ∂U with respect to an integral basis of L consisting of classes with Maslov index 2.*

(a) *If $W = U$,*

$$\operatorname{index}(u) = 2s - 2.$$

(b) *If $W = S\partial U$,*

$$\operatorname{index}(u) = 2s_+ + s_- - 2.$$

(c) *If $W = \mathbb{CP}^2 \backslash U$ (thus $s = s_-$) and γ_j^- is of type $(-k_j, -l_j)$,*

$$\operatorname{index}(u) = s - 2 + 6d + 2\sum(k_j + l_j).$$

(d) *If $W = U \backslash E$, the s_- negative asymptotics can be further split into s_1^- covers of γ_1 and s_2^- covers of γ_2 (we denote r_i^-, t_j^- the multiplicities of these covers). Then,*

$$\operatorname{index}(u) = 2s_+ - 2 - 2\sum_{i=1}^{s_1^-}\left(r_i^- + \left\lfloor \frac{r_i^-}{S} \right\rfloor\right) - 2\sum_{j=1}^{s_2^-}\left(t_j^- + \lfloor t_j^- S \rfloor\right).$$

(e) *If $X = S\partial E$,*

$$\operatorname{index}(u) \geq 0.$$

Proof. In dimension 4, for a punctured sphere, the formula (3.1) gives

$$\begin{aligned}\operatorname{index}(u) = s - 2 + 2c_1^\tau(u^*TX) &+ \sum_{i=1}^{s_+}\Big(\mu_{\mathrm{CZ}}(\gamma_i^+) + \frac{1}{2}\dim S_i^+\Big)\\ &- \sum_{j=1}^{s_-}\Big(\mu_{\mathrm{CZ}}(\gamma_j^-) - \frac{1}{2}\dim S_j^-\Big).\end{aligned}$$

In U, there is a global Lagrangian distribution $\mathcal{L}$ given by the vertical distribution of the cotangent bundle. This Lagrangian distribution can be extended to a symplectic trivialization τ of u^*TU, and for this choice, $c_1^\tau(u) = 0$. In ∂U, each closed orbit comes in a 1-parameter family and its generalized Maslov index is $\frac{1}{2}$. Finally, U is a symplectic cobordism with one positive end ∂U, so $s = s_+$. Thus, for $u\colon \dot{\Sigma} \to U$,

$$\operatorname{index}(u) = s - 2 + 0 + \sum_{i=1}^{s_+}\Big(\frac{1}{2} + \frac{1}{2}\Big) = 2s - 2.$$

The same choice of τ can be made in $S\partial U$, and the previous remarks still hold in this setting, except that there are now positive and negative ends. Thus, for $u\colon \dot{\Sigma} \to S\partial U$, we have

$$\operatorname{index}(u) = s - 2 + 0 + s_+ - \sum_{j=1}^{s_-}\Big(\frac{1}{2} - \frac{1}{2}\Big) = 2s_+ + s_- - 2.$$

In $W = U\backslash E$, we still have the same Lagrangian distribution that we extend to a symplectic trivialization τ of u^*TW. Notice that since E is contractible, we can deform this symplectic trivialization above E (and hence its boundary ∂E) so that it coincides with the trivialization coming from $E \subset \mathbb{R}^{2n}$, with the standard trivialization. Relative to this choice, $c_\tau^1(u)$ still vanishes, and the the Conley–Zehnder indices of the closed orbits of ∂E are well known:

$$\mu_{\mathrm{CZ}}(\gamma_1^r) = 2r + 1 + 2\Big\lfloor \frac{r}{s} \Big\rfloor,$$

while

$$\mu_{\mathrm{CZ}}(\gamma_2^r) = 2t + 1 + 2\lfloor ts \rfloor.$$

As a result, we get for $u\colon \dot{\Sigma} \to U\backslash E$:

$$\operatorname{index}(u) = s - 2 + s_+ - s_- - 2\sum_{i=1}^{s_1^-}\Big(r_i + \Big\lfloor \frac{r_i}{S} \Big\rfloor\Big) - 2\sum_{j=1}^{s_2^-}\big(t_j + \lfloor t_j S \rfloor\big).$$

Similarly in $S\partial E$,

$$\begin{aligned}\operatorname{index}(u) &= s-2+s_+-s_- + 2\sum_{i=1}^{s_1^+}\left(r_i^+ + \left\lfloor \frac{r_i^+}{S}\right\rfloor\right) + 2\sum_{j=1}^{s_2^+}\left(t_j^+ + \lfloor t_j^+ S\rfloor\right)\\ &\qquad - 2\sum_{i=1}^{s_1^-}\left(r_i^- + \left\lfloor \frac{r_i^-}{S}\right\rfloor\right) - 2\sum_{j=1}^{s_2^-}\left(t_j^- + \lfloor t_j^- S\rfloor\right)\\ &= -2 + 2\sum_{i=1}^{s_1^+}\left(r_i^+ + \left\lfloor \frac{r_i^+}{S}\right\rfloor + 1\right) + 2\sum_{j=1}^{s_2^+}\left(t_j^+ + \lfloor t_j^+ S\rfloor + 1\right)\\ &\qquad - 2\sum_{i=1}^{s_1^-}\left(r_i^- + \left\lfloor \frac{r_i^-}{S}\right\rfloor\right) - 2\sum_{j=1}^{s_2^-}\left(t_j^- + \lfloor t_j^- S\rfloor\right).\end{aligned}$$

By positivity of the area, we also have

$$\sum r_i^+ + \sum t_j^+ S \geq \sum r_i^- + \sum t_j^- S,$$

so

$$\sum r_i^+ + \sum \lfloor t_j^+ S\rfloor + 1 \geq \sum r_i^- + \sum \lfloor t_j^- S\rfloor$$

and

$$\sum \left\lfloor \frac{r_i^+}{S}\right\rfloor + 1 + \sum t_j^+ \geq \sum \left\lfloor \frac{r_i^-}{S}\right\rfloor + \sum t_j^-$$

and one of the inequalities must be strict. Hence we have $\operatorname{index}(u) \geq 0$ for any curve $u\colon \dot{\Sigma} \to S\partial E$.

Finally, in $\mathbb{CP}^2\backslash U$, we consider the symplectic trivialization over ∂U that comes from the inclusion of ∂U in the affine chart $\mathbb{CP}^2\backslash S_\infty$. Then $c_1^\tau(u) = 3d$ and u has only negative ends, whose asymptotic limits come in 1 parameter families. The discrepancy between the generalized Maslov index defined with respect to the trivialization generated by $\mathcal{L}$ above and the index with respect to the trivialization coming from the affine chart is the Maslov class of L evaluated on the homology class of the projection of the limiting Reeb orbit to L under the natural projection in T^*L. Hence if the i-th end γ_i is asymptotic to a Reeb orbit of type $(-k_i, -l_i)$ then we substitute

$$\mu_{\mathrm{CZ}}(\gamma_i) = \frac{1}{2} - 2(k_i + l_i)$$

into formula (3.1) to get

$$\operatorname{index}(u) = s - 2 + 6d + 2\sum_i (k_i + l_i). \qquad \square$$

3.2. Nonnegative index and multiple covers. We recall from Section 2 that the F_i are subbuildings of the building B obtained by our neck-stretching process, and that their indices are given by the sum of their constituent curves in the different layers, minus the sum of the dimensions of the space of closed Reeb orbits along which these different curves match. Moreover, F_0 plays a special role: it is the connected subbuilding in $U \setminus E \sqcup S\partial E \sqcup S\partial U$ that is asymptotic to γ_1^{3d-1} at its unmatched negative end. It has T positive ends, at which the F_i, $i = 1 \dots T$ are connected.

Lemma 3.5. $\operatorname{index}(F_i) \geq 1$ *for* $1 \leq i \leq T$.

Proof. These components only have one unmatched negative end in ∂U (corresponding to an end of F_0) and hence by Propositions 3.3 and 3.4(c), they have odd index. Therefore it suffices to show that $\operatorname{index}(F_i) \geq 0$. Decompose F_i into subbuildings

$$\{F_{ij}^+\}_{j=1\dots l_+} \cup \{F_{ij}^-\}_{j=1\dots l_-},$$

where the F_{ij}^+ are just the constituent curves of F_i in $\mathbb{CP}^2 \setminus U$ and the F_{ij}^- their complementary connected subbuildings in F_i (thus F_{ij}^- lies in $U \setminus E \sqcup S\partial U \sqcup S\partial E$). Let s_{ij}^+ be the number of (negative) ends of F_{ij}^+ and s_{ij}^- be the number of (positive) ends of F_{ij}^-. Since moreover F_i has only one (unmatched) negative end in ∂U, we see that

$$\sum s_{ij}^+ = \sum s_{ij}^- + 1.$$

We infer by Propositions 3.3 and 3.4(a) that the index of F_{ij}^- is $2s_{ij}^- - 2$. Now notice that there is no finite energy plane in U ($\approx T^*\mathbb{T}^2$), so $s_{ij}^- \geq 2$ and

$$\operatorname{index}(F_{ij}^-) = 2s_{ij}^- - 2 \geq s_{ij}^-.$$

As a result,

$$\operatorname{index}(F_i) = \sum \operatorname{index}(F_{ij}^+) + \sum \operatorname{index}(F_{ij}^-) - \sum s_{ij}^- \geq \sum \operatorname{index}(F_{ij}^+),$$

and it suffices to prove that $\operatorname{index}(F_{ij}^+) \geq 0$ $\forall i, j$ to conclude our proof. Let therefore u be a constituent curve of F_i in $\mathbb{CP}^2 \setminus U$, and suppose it has degree d and s ends asymptotic to Reeb orbits of type $(-k_i, -l_i)$ as in Proposition 3.4. If u is somewhere injective then for generic J we may assume it has nonnegative index. Otherwise it is a multiple cover of an underlying curve $\tilde{u}$. We may assume that $\tilde{u}$ is somewhere injective and so $\operatorname{index}(\tilde{u}) \geq 0$ for generic almost-complex structures. Suppose this cover is of degree r, so $\tilde{u}$ has degree $\tilde{d} = d/r$, and further that $\tilde{u}$ has $\tilde{s}$ negative ends asymptotic to orbits of type $(-\tilde{k}_j, -\tilde{l}_j)$. The Riemann–Hurwitz formula shows that the domain of $\tilde{u}$ is a punctured sphere, so Proposition 3.4 gives

$$\operatorname{index}(\tilde{u}) = \tilde{s} - 2 + 6\tilde{d} + 2\sum_{j=1}^{\tilde{s}} (\tilde{k}_i + \tilde{l}_i).$$

Hence we have

$$\operatorname{index}(u) = r\operatorname{index}(\widetilde{u}) + 2(r-1) - (r\widetilde{s} - s) \geq 2(r-1) - (r\widetilde{s} - s). \tag{3.2}$$

Let now $\varphi: S^2\backslash\Gamma \to S^2\backslash\widetilde{\Gamma}$ be the holomorphic (ramified) covering such that $u = \widetilde{u} \circ \varphi$. Removing singularities, φ extends to a holomorphic map $\Phi: S^2 \to S^2$ that sends Γ to $\widetilde{\Gamma}$. Then, $r\widetilde{s} - s$ represents the total ramification of φ over the points of Γ, so, letting m_c be the multiplicity of a singular point c, the Riemann–Hurwitz formula gives

$$r\widetilde{s} - s = \sum_{c\in\Gamma}(m_c - 1) \leq \sum_{c\in S^2}(m_c - 1) = 2(r-1).$$

By (3.2), we see that $\operatorname{index}(u) \geq 0$ as required. □

By Proposition 3.4, since F_0 has T positive ends and a single negative end asymptotic to γ_1^{3d-1} we have $\operatorname{index}(F_0) = 2T - 6d$.

Lemma 3.6. $\operatorname{index}(F_0) \geq 0$, *that is* $T \geq 3d$.

Proof. We argue by contradiction and assume that $T < 3d$, that is F_0 has less than $3d$ positive ends. Let us consider the curves of F_0 in $S\partial E$ which fit together to form a connected component G_0 of F_0 including the lowest level curve with negative end asymptotic to γ_1^{3d-1}. Then let $u_i: S^2\backslash\Gamma_i \to U\backslash E$ be the curve of F_0 with a negative end matching the ith positive end of G_0. (Note that u_i may have other negative ends matching with curves in $S\partial E$ not included in G_0 , but only one matches G_0 because the building has genus 0.)

Note that by the maximum principle, each positive end of the u_i is connected through components in $S\partial U$ to positive unmatched ends of F_0. And since our building is obtained by degenerating curves of genus 0, these different positive ends of the u_i are connected to distinct positive unmatched ends of F_0. Denoting by Q_i the number of positive ends of u_i, we therefore get that

$$\sum Q_i \leq T \leq 3d - 1.$$

This, in turn, guarantees that no negative end of the u_i is asymptotic to a cover of γ_2. Indeed, a curve in $U\backslash E$ is (possibly a multiple cover of) a somewhere injective curve $\widetilde{u}$ in $U\backslash E$ with s_+ ends with index

$$\operatorname{index}(\widetilde{u}) = 2s_+ - 2 - 2\sum_{i=1}^{s_1^-}\left(r_i + \left\lfloor\frac{r_i}{S}\right\rfloor\right) - 2\sum_{j=1}^{s_2^-}\left(t_j + \lfloor t_j S\rfloor\right) \geq 0.$$

Since $s_+ \leq 3d - 1$ and $S > 3d - 1$, we see that the t_j must vanish, so $\widetilde{u}$ and therefore u itself has no negative end asymptotic to a cover of γ_2. Altogether, the curves u_i therefore verify the following. They have Q_i positive ends with

$$\sum Q_i = Q \leq 3d - 1,$$

and they have a negative end asymptotic to a $\gamma_1^{q_i}$ which is matched with a positive end of G_0. For area reasons, we then see that

$$\sum q_i \geq 3d - 1.$$

In total, we therefore have

$$\sum Q_i \leq 3d - 1 \leq \sum q_i,$$

so there exists an i such that $Q_i \leq q_i$. We henceforth denote this curve by u, and let q and Q the corresponding numbers; hence $q \geq Q$. The index of this curve is

$$\operatorname{index}(u) = 2Q - 2 - 2\sum\left(r_i + \left\lfloor \frac{r_i}{S} \right\rfloor\right) \leq 2Q - 2 - 2q < 0.$$

Thus u must be an r-covering of a somewhere injective curve $\widetilde{u}\colon S^2\backslash\widetilde{\Gamma} \to U\backslash E$ with say $\widetilde{Q} \leq Q < 3d$ positive ends, $\widetilde{s}_-$ negative ends, the i-th of which is asymptotic to $\gamma_1^{\widetilde{r}_i}$ (none of them is asymptotic to a cover of γ_2), and non-negative index given by

$$\operatorname{index}(\widetilde{u}) = 2\widetilde{Q} - 2 - \sum_{i=1}^{\widetilde{s}^-}\left(2\widetilde{r}_i + 2\left\lfloor \frac{\widetilde{r}_i}{S} \right\rfloor\right).$$

Hence $\widetilde{Q} \geq 1 + \sum \widetilde{r}_i$. Suppose that the end of u asymptotic to γ_1^q covers t times an end of $\widetilde{u}$ asymptotic to $\gamma_1^{\widetilde{q}}$, so $t\widetilde{q} = q$. Consider also the ramified covering

$$\varphi\colon S^2\backslash\Gamma \to S^2\backslash\Gamma$$

defined by $u = \widetilde{u} \circ \varphi$, and remove its singularities to get a map $\varphi\colon S^2 \to S^2$. We recall that Γ splits as $\Gamma^+ \cup \Gamma^-$, where Γ^+ are the positive ends of u and Γ^- the negative ones. One of the negative ends, say c_0, has order t. The Riemann–Hurwitz formula, together with the facts that $\widetilde{Q} \geq 1 + \sum \widetilde{r}_i$, $t \leq r$, and $q \geq Q$, then give:

$$\begin{aligned}
2(r-1) = \sum_{c\in S^2}(m_c - 1) &\geq \sum_{c\in\Gamma^+}(m_c - 1) + (t-1) \\
&= r\widetilde{Q} - Q + t - 1 \\
&\geq r\Big(1 + \sum \widetilde{r}_i\Big) - Q + t - 1 \\
&\geq r(1 + \widetilde{q}) - Q + t - 1 \\
&= r + r\widetilde{q} - Q + t - 1 \\
&= r + t\widetilde{q} - Q + (r-t)\widetilde{q} + t - 1 \\
&\geq r + q - Q + r - t + t - 1 \\
&\geq 2r - 1,
\end{aligned}$$

which is a contradiction. □

3.3. Holomorphic planes in $\mathbb{C}^2\backslash U$. We recall the inclusions $L \subset U \subset B^4(R)$, with U symplectomorphic to $U^p_{\varepsilon,S}$. We consider here the situation where L is the image by a Hamiltonian diffeomorphism of $\mathbb{C}^2$ of the product torus $L(1,x)$ with $x \geq 1$. We moreover assume in this paragraph that $x \in \mathbb{Q}$. The aim of this section is the following result:

Lemma 3.7. *Under the above hypothesis, for a generic almost complex structure and $s \geq 1$, there is no genus 0 finite energy curve in $\mathbb{C}^2\backslash U$ with s negative ends, deformation index at least s and area strictly less than 1.*

Remark 3.8. To avoid complicating our formulas with terms of order ϵ, we define the *area* of a finite energy curve in $\mathbb{C}^2\backslash U$ to be the area of the closed surface formed by topologically gluing a half-cylinder in U to each end of u. The open end of the cylinder is asymptotic to a limiting Reeb orbit matching the corresponding end of u and the boundary of the cylinder is a curve on L. Note that this cylinder is *symplectic*, so starting with a curve in $\mathbb{C}^2\backslash U$ of positive genuine area, the area we define here still remains positive, and this is all we will care about in the sequel.

Before proving Lemma 3.7 we derive the property for the standard Lagrangian torus $L(1,x) \subset \mathbb{C}^2$ and some particular cylindrical almost complex structures.

Lemma 3.9. *Let U be a neighbourhood of $L(1,x) \subset \mathbb{C}^2$ (with $x \geq 1$), symplectomorphic as before to $U^p_{\varepsilon,S}$. Let J be an almost complex structure on $\mathbb{C}^2\backslash U$ cylindrical near ∂U and such that J coincides with the standard complex structure i near the line $\{z_2 = 0\}$. Then, there is no genus 0 finite energy curve in $\mathbb{C}^2\backslash U$ with s negative ends, deformation index at least s and area strictly less than 1.*

Proof. Let $u\colon \mathbb{CP}^1\backslash\{z_1,\dots,z_s\} \to \mathbb{C}^2\backslash U$ be a genus 0 J-holomorphic curve with finite energy and index at least s. Its ends are asymptotic to orbits of type $(-k_j,-l_j)$ as in Section 3 where we now use the standard basis of $H_1(L(1,x),\mathbb{Z})$. Then,

$$\operatorname{index}(u) = s - 2 + 2\sum_{j=1}^{s}(k_j + l_j) \geq s$$

so $\sum k_j + l_j \geq 1$. On the other hand,

$$\operatorname{Area}(u) = \sum_{j=1}^{s}(k_j + l_j x) = \sum_{j=1}^{s}(k_j + l_j + l_j(x-1)) \geq 1 + (x-1)\sum_{j=1}^{s} l_j.$$

Now the sum of the l_j's represents the intersection number between u and $\{z_2 = 0\}$ (parameterized in the obvious way by the z_1-coordinate). Since $J = J_{\mathrm{st}}$ near this

line, and u is holomorphic, each intersection point between these two curves counts positively, so $\sum l_j \geq 0$. Since $x \geq 1$, we indeed get $\mathrm{Area}(u) \geq 1$. □

Proof of Lemma 3.7. Arguing by contradiction, since there exists a Hamiltonian diffeomorphism f mapping $L(1,x)$ to L, if such an almost-complex structure and finite energy curve exists then we can pull-back using f to find holomorphic curves of area less than 1 asymptotic to a neighborhood U of the product torus. Hence it suffices to work with $L = L(1,x)$. By Lemma 3.9 we can find at least one almost-complex structure J_1 for which no such curves exist. Moreover, since the only constraints on J_1 appear near $\{z_2 = 0\}$ we may assume any generality properties of J_1 with respect to curves asymptotic to ∂U.

We need two facts about moduli spaces of finite energy curves in $\mathbb{C}^2 \backslash U$. We denote by $\mathcal{J}$ the collection of compatible almost-complex structures on $\mathbb{C}^2 \backslash U$.

Theorem 3.10 (Ivrii [13, Section 2.4] and Wendl [19].). *Given $J \in \mathcal{J}$, immersed J-holomorphic finite energy curves with index at least the number of negative ends are regular. That is, the normal Cauchy–Riemann operator is surjective and our curves appear in a family of the expected dimension. Such curves are also regular in their moduli space of curves with fixed asymptotic limits (rather than allowing the limits to move in the family of Reeb orbits).*

Theorem 3.11 (see Zehmisch [20] and Oh–Zhu [15]). *There exists a comeagre subset $\mathcal{J}_I \subset \mathcal{J}$ such that if $J \in \mathcal{J}_I$, in any moduli space of somewhere injective J-holomorphic curves the collection of non-immersed curves form a stratified subset of codimension* 2.

*A parametric version of this statement (*i.e. *for families of almost complex structures) also follows by the same arguments.*

Let $\mathcal{J}_R \subset \mathcal{J}$ be the collection of almost-complex structures on $\mathbb{C}^2 \backslash U$ which are regular for all of the (countably many) moduli spaces of finite energy curves with unconstrained asymptotic limits (that is, moduli spaces defined for all choices of Reeb orbit families of asymptotic limits and for all relative homology classes in $H^2(\mathbb{R}^4, L; \mathbb{Z})$). We will work with $\mathcal{J}_0 = \mathcal{J}_R \cap \mathcal{J}_I \subset \mathcal{J}$, which is again a comeagre set. Since J_1 can be freely defined in the complement of $\{z_2 = 0\}$, and since all holomorphic curves with boundary on $L(1,x)$ meet this free region, we may assume $J_1 \in \mathcal{J}_0$.

Given this, we define $\mathcal{A}$ to be the infimum of the areas of J-holomorphic curves having index at least the number of negative ends and $J \in \mathcal{J}_0$. By contradiction we are assuming $\mathcal{A} < 1$. But since $x \in \mathbb{Q}$ there are only finitely many possible areas less than 1 which can be realized by holomorphic curves and so $\mathcal{A} > 0$ is realized by J-holomorphic curves. Amongst all choices for (u, J) with minimal area $\mathcal{A}$, we choose (u, J_0) for which u has the minimal number of negative ends, which we call s. We may further assume that u is somewhere injective from the following lemma.

Lemma 3.12. *Let v be a finite energy curve in $\mathbb{C}^2\backslash U$ with t negative ends and* index $v \geq t$. *Suppose that v is a multiple cover of a curve $\widetilde{v}$ with $\widetilde{t}$ negative ends. Then* $\operatorname{index}(\widetilde{v}) \geq \widetilde{t}$.

Proof. Suppose the ends of $\widetilde{v}$ are asymptotic to orbits of type $(-\widetilde{m}_i, -\widetilde{k}_i)$ and the cover is of degree r. Then we have

$$\operatorname{index}(\widetilde{v}) = \widetilde{t} - 2 + 2\sum_{i=1}^{\widetilde{t}}(\widetilde{m}_i + \widetilde{k}_i)$$

and

$$\begin{aligned}\operatorname{index}(v) &= t - 2 + 2r\sum_{i=1}^{\widetilde{t}}(\widetilde{m}_i + \widetilde{k}_i)\\ &= t - 2 + r(\operatorname{index}(\widetilde{v}) - \widetilde{t} + 2).\end{aligned}$$

Thus

$$r(\operatorname{index}(\widetilde{v}) - \widetilde{t} + 2) - 2 \geq 0$$

and

$$\operatorname{index}(\widetilde{v}) - \widetilde{t} \geq \frac{2}{r} - 2 > -2.$$

As $\operatorname{index}(\widetilde{v}) - \widetilde{t}$ is even this gives our inequality as required. □

Now, by the definition of $\mathcal{J}_0$ the J_0-holomorphic curve u is regular, and by Theorem 3.11 we may further assume it is immersed. Denote its asymptotic limits by $\sigma_1, \ldots, \sigma_s$. In the case when $\operatorname{index}(u) > s$ we also fix $N = \frac{1}{2}(\operatorname{index}(u) - s)$ points $p_1, \ldots, p_N$ in the range of the injective points of u (recall from Proposition 3.4(c) that N must be an integer).

For a family J_t, $0 \leq t \leq 1$ of almost-complex structures interpolating between J_0 and J_1, define the universal moduli space

$$\mathcal{M} := \left\{ \begin{array}{ll} (t,u) \text{ such that} & u\colon \mathbb{CP}^1\backslash\{z_1,\ldots,z_s\} \to \mathbb{C}^2\backslash U,\\ & \overline{\partial}_{J_t} u = 0,\\ & u \text{ somewhere injective},\\ & \operatorname{image}(u) \cap p_i \neq \emptyset \text{ for all } i,\\ & u \text{ is asymptotic to } \sigma_i \text{ at } z_i \end{array} \right\} \Big/ \sim,$$

where we quotient by reparameterizations of the domain. We note that $\mathcal{M}$ has virtual dimension 1. Indeed, fixing the asymptotic limits reduces the virtual dimension by s, and the fixed points further reduce the dimension by $2N$.

Since $J_0, J_1 \in \mathcal{J}_0$, we can find a path $\{J_t\}_{t\in[0,1]}$ between J_0 and J_1 which is both regular for all of the countably many unconstrained moduli spaces of curves asymptotic to L and is also regular for the constrained moduli space $\mathcal{M}$ (so in particular $\mathcal{M}$ has dimension 1). Furthermore, the parametric version of Theorem 3.11

implies that we may also assume that our path is generic in the sense that non-immersed curves in all of these moduli spaces have codimension 2 (and so in particular do not exist at all in $\mathcal{M}$).

Given all of this, since curves in $\mathcal{M}$ are immersed they are also regular for a fixed almost-complex structure by Theorem 3.10 (which also holds when the index is a constrained index for curves passing through fixed points). Therefore the map $\mathcal{M} \to [0,1]$, $(t,u) \mapsto t$, is a submersion. By Lemma 3.9 the fiber over 1 is empty and so we will arrive at a contradiction if we can show $\mathcal{M}$ to be compact.

To this end, let $(t_n, u_n) \in \mathcal{M}$ and suppose that $t_n \to t_\infty$. We claim that a subsequence of the u_n converges to a J_{t_∞}-holomorphic finite energy plane u_∞ such that $(t_\infty, u_\infty) \in \mathcal{M}$. By [2], some subsequence of u_n converges in a suitable sense to a J_{t_∞}-holomorphic building. This building consists of top level curves in $\mathbb{C}^2 \backslash U$ and lower level curves in $S\partial U$. For simplicity we will gather lower level curves with matching asymptotics and consider them as a single curve. We can also assume by adding trivial lower cylinders that there are no unmatched negative ends of the top level curves. Hence, there are s unmatched negative ends to the lower level curves which are asymptotic to $\sigma_1, \ldots, \sigma_s$. Let L_{top} be the number of top level curves and L_{low} the number of low level components (after our identifications). We call I_i, s_i, $i = 1, \ldots, L_{\text{top}}$ the (unconstrained) index and number of negative ends of the i-th top level curve, and (J_i, r_i, t_i), $i = 1, \ldots, L_{\text{low}}$ the index, number of positive and number of negative ends of the i-th low-level component. Then by Proposition 3.4(b),

$$J_i = 2r_i + t_i - 2.$$

Since all ends of the upper level curves match,

$$\sum_{i=1}^{L_{\text{top}}} s_i = \sum_{i=1}^{L_{\text{low}}} r_i,$$

and since the total number of unmatched ends is s,

$$\sum_{i=1}^{L_{\text{low}}} t_i = s.$$

We can associate a graph to a holomorphic building by adding a vertex for each curve and asymptotic limit (just one vertex when an asymptotic limit is a matching asymptotic between two curves), and an edge between the vertex for each curve and its asymptotics. As we take limits of curves of genus 0, this graph is a tree, hence has Euler characteristic 1, so we have

$$L_{\text{top}} + L_{\text{low}} + \sum_{i=1}^{L_{\text{top}}} s_i + \sum_{i=1}^{L_{\text{low}}} t_i - \sum_{i=1}^{L_{\text{top}}} s_i - \sum_{i=1}^{L_{\text{low}}} (r_i + t_i) = L_{\text{top}} + L_{\text{low}} - \sum_{i=1}^{L_{\text{low}}} r_i = 1.$$

Finally, the index of the limiting building, namely the sum of the indices of the constituent curves minus matching conditions, equals the index of the original curve, which by assumption is at least s. Thus, taking all previous equalities into account,

$$\begin{aligned} s \leq \sum_{i=1}^{L_{\text{top}}} I_i + \sum_{i=1}^{L_{\text{low}}} J_i - \sum_{i=1}^{L_{\text{top}}} s_i &= \sum_{i=1}^{L_{\text{top}}} (I_i - s_i) + \sum_{i=1}^{L_{\text{low}}} (2r_i + t_i - 2) \\ &= \sum_{i=1}^{L_{\text{top}}} (I_i - s_i) + 2(L_{\text{top}} + L_{\text{low}} - 1) + s - 2L_{\text{low}} \\ &= s + 2(L_{\text{top}} - 1) + \sum_{i=1}^{L_{\text{top}}} (I_i - s_i) \\ &= s - 2 + \sum_{i=1}^{L_{\text{top}}} (I_i - s_i + 2). \end{aligned}$$

Thus,

$$\sum_{i=1}^{L_{\text{top}}} (I_i - s_i + 2) \geq 2.$$

By Proposition 3.4(c) the differences $I_i - s_i$ are even, so at least one of them is non-negative, and the corresponding top level curve v therefore has index at least the number of its negative ends. As it appears as part of the limit, the area of v is at most the common area $\mathcal{A}$ of the curves u_n.

By our assumptions on $\{J_t\}$ the J_{t_∞}-holomorphic curve v appears in a universal moduli space containing some immersed curves, so without loss of generality we will assume v itself is immersed. Next by Theorem 3.10 such an immersed curve v is regular and so we find holomorphic curves with the same asymptotics if we deform the almost-complex structure. We cannot assume $J_{t_\infty} \in \mathcal{J}_0$, however it can be approximated by $J \in \mathcal{J}_0$ and so we can approximate v by a curve w with the same asymptotics, and hence the same area, which is holomorphic with respect to an almost-complex structure in $\mathcal{J}_0$. By minimality of $\mathcal{A}$ among the areas of such curves, the area of w is at least $\mathcal{A}$, and hence as v has area at most $\mathcal{A}$ we see that v and w have area $\mathcal{A}$ precisely.

In conclusion there was exactly one top level curve (as v occupies all of the area) which must therefore intersect the points p_i. By minimality of s we see that v has at least s negative ends, but as the limit is of curves of genus 0 and there are no holomorphic planes in U (there are no contractible Reeb orbits) the curve v must have exactly s negative ends. It follows that the lower level curves are cylinders which for action reasons must be trivial cylinders asymptotic to the $\sigma_1, \ldots, \sigma_s$. We conclude that v has the correct asymptotics and $(t_\infty, v) \in \mathcal{M}$ as required. □

4. Proof of the obstruction part of Theorem 1

We now gather together the information from the previous section to prove the obstructions claimed in Theorem 1. We can divide the theorem into the cases $x < 2$ and $x \geq 2$. For the latter case we actually prove the more general Theorem 2, where L is not assumed to be Hamiltonian isotopic to the product torus: it may be any Lagrangian torus in $B(R)$, with a basis (e_1, e_2) of the homology with $\mu(e_1) = \mu(e_2) = 2$ and $\Omega(e_1) = 1$, $\Omega(e_2) = x$. We consider in the following the general situation until we get to the proof of the first case of Theorem 1, then we add the assumption of L being Hamiltonian isotopic to a product torus, allowing for the use of Lemma 3.7.

Some restrictions on the F_i. By Lemmas 3.5 and 3.6, the component F_0 has

$$\text{index}(F_0) \geq 0$$

and the F_i for $1 \leq i \leq T$ have

$$\text{index}(F_i) \geq 1.$$

As there are T remaining ends to match and the sum of the indices minus matching is 0, we conclude that $\text{index}(F_0) = 0$ (and hence $T = 3d$) and $\text{index}(F_i) = 1$ for $i \geq 1$.

By Proposition 3.4, the index equality for the F_i with $i \geq 1$ says

$$1 = \text{index}(F_i) = 6d_i + 2(m_i + k_i) - 1, \tag{4.1}$$

where F_i has total degree d_i and is asymptotic to orbits of type $(-m_i, -k_i)$. Meanwhile the action in $B(R)$ of orbits of type $(-m_i, -k_i)$ is $m_i + k_i x$ (recalling Remark 3.8). Therefore by Stokes' Theorem we have

$$\text{Area}(F_i) = Rd_i + (m_i + k_i x).$$

In view of (4.1) we get

$$\text{Area}(F_i) = (R - 3)d_i + (k_i(x - 1) + 1).$$

The case $x \geq 2$ (proof of Theorem 2). We note that by setting $\Omega(e_2) = x$, the more general Theorem 2 implies the obstruction part of Theorem 1 in the case when $x \geq 2$. Also note that proving $R \geq 3$ is enough to deduce $R > 3$. Indeed, since L is compact and $B(R)$ is open, if L were to lie in $B(3)$, then it would lie in a smaller ball $B(3 - \varepsilon)$ for some $\varepsilon > 0$. We argue by contradiction, and assume that $R < 3$ and $x \geq 2$. As $\text{area}(F_i) > 0$ this means $k_i \geq 0$. But as all limiting orbits $\gamma_{-m_i,-k_i}$ bound the component F_0 in U they represent a trivial homology class, and so

$$\sum m_i = \sum k_i = 0.$$

Thus $k_i = 0$ for all i and the $3d$ components F_i are all asymptotic to orbits of type $(-m_i, 0)$.

Now we use Stokes' Theorem to calculate the area of F_0 in U. This has $3d$ positive ends asymptotic to orbits $\gamma_{-m_i,0}$ and a negative end on γ_1^{3d-1}. From our description of U in Section 2 we see that the integral of the standard Liouville form around both $\gamma_{-1,0}$ and $\gamma_{1,0}$ is $\epsilon/2$ and so we get

$$\text{area}(F_0) = \sum_i |m_i| \frac{\epsilon}{2} - (3d-1)\epsilon.$$

As $\sum m_i = 0$ we have

$$\sum_{m_i > 0} m_i \geq 3d - 1.$$

Focus for a moment on the components F_i with $m_i > 0$. By the index formula (4.1) their index is $1 = 6d_i + 2m_i - 1$, so their degrees d_i vanish, while $m_i = 1$. Since the sum of these degrees is at least $3d - 1$, there are exactly $3d - 1$ such components (since there are $3d$ components in total, and one of them at least must have $m_i < 0$). The final picture for the building is therefore a component F_0 with $3d$ positive ends, $3d - 1$ of which are asymptotic to orbits $\gamma_{-1,0}$ that match with components F_i of degree 0, and one positive end asymptotic to an orbit $\gamma_{3d-1,0}$ that matches with a component (say F_{3d}) of degree d. The positive area of this last component is $Rd - (3d-1)$, so $R > 3 - 1/d$. Taking d large gives us the result.

The case $x < 2$. For this part of the proof Theorem 2 does not apply. Specifically, we will apply Lemma 3.7, which requires our tori to be Hamiltonian isotopic to products.

We again argue by contradiction, and assume now that $R \leq 1 + x$. As in the case of $x \geq 2$, this means that we may assume $R < 1 + x$. Note that by Weinstein's neighbourhood theorem, it is enough to prove our result for $x \in \mathbb{Q}$. Note also that if all k_i vanish, the same proof as the above shows that $R \geq 3$, which is already a contradiction. Hence, there must be planes asymptotic to orbits of type (m_i, k_i) with $k_i \neq 0$, and in particular there must be such a plane F asymptotic to an orbit $\gamma_{m,k}$ with $k < 0$. Let d be the degree of F. Then,

$$1 = \text{index}(F) = 6d + 2(m + k) - 1$$

and

$$\text{Area}(F) = Rd + (m + kx).$$

From the first equation we get $m = 1 - 3d - k$, and substituting in the second one gives

$$\text{Area}(F) = Rd + (1 - 3d - k) + kx = (R - 3)d + k(x - 1) + 1 > 0.$$

Since $k \leq -1$, $x < 2$ and $R < 1 + x$, we get

$$(2 - x)d < 2 - x$$

and so $d = 0$. Thus F is a plane with one negative end, has index 1, lies in $B^4(R)\backslash U \subset \mathbb{C}^2\backslash U$, and verifies $m = 1 - k$, so

$$\text{Area}(F) = 1 + k(x - 1) < 1.$$

This is in contradiction with Lemma 3.7 (recall that $x \in \mathbb{Q}$). □

5. Proof of the obstruction part of Theorem 3

We briefly outline the adjustments required to establish the obstruction part of Theorem 3. Note that there does not exist an embedding $L(1, x) \hookrightarrow P(a, b)$ when $a < 1$ since by [4], Proposition 2.1, the Lagrangian torus $L(1, x)$ has displacement energy 1. We still argue by contradiction, assuming that $a \leq 2$ and $b \leq x$, which as the polydisk is open implies we may take $a < 2$ and $b < x$. The proof of Theorem 3 proceeds similarly to that of Theorem 1 except now we compactify $P(a, b)$ to a copy of $S^2 \times S^2$ with factors having areas a and b. The analogue of Proposition 3.4 is that the deformation index of a finite energy curve u of bidegree (d_1, d_2) asymptotic to Reeb orbits of type $(-m_i, -k_i)$ is given by

$$\text{index}(u) = s - 2 + 4(d_1 + d_2) + 2\sum_{i=1}^{s}(m_i + k_i).$$

As before, we consider the situation

$$E \subset U \subset S^2(a) \times S^2(b),$$

where U is our Weinstein neighbourhood of L. We now use the existence of J-holomorphic planes

$$u\colon \mathbb{CP}^1\backslash\{\infty\} \to S^2 \times S^2\backslash E$$

of bidegree $(d, 1)$, asymptotic to γ_1^{2d+1}, which exist for d arbitrarily large by [9], and we stretch the neck of ∂U. As previously, we split the limit building B into subbuildings $F_0, F_1, \ldots, F_T$, where F_0 is the maximal connected subbuilding of B in $S\partial E$, $U\backslash E$ and $S\partial U$ attached to the negative unmatched end, while $F_1, \ldots F_T$ are the connected subbuildings attached to the T positive ends of F_0. Arguing as for balls, we get the following:

1. $\text{index}(F_i) \geq 1$. The proof is exactly the same as Lemma 3.5, replacing the formula for the index of $\widetilde{u}$ by the correct formula in $S^2 \times S^2\backslash U$.

2. $\operatorname{index}(F_0) \geq 0$. Since F_0 lies in U, this is exactly Lemma 3.6. Since now $\operatorname{index}(F_0) = 2T - 4(d+1)$, we conclude that $T \geq 2(d+1)$.

Then again we get restrictions on the F_i:

$$\operatorname{index}(F_0) = 0, \quad \operatorname{index}(F_i) = 1, \quad \text{and} \quad T = 2(d+1).$$

Suppose F_i is asymptotic to an orbit of type $(-m_i, -k_i)$ and has bidegree (d_i^1, d_i^2). The index and area formula for the F_i now give

$$1 = \operatorname{index}(F_i) = -1 + 4(d_i^1 + d_i^2) + 2(m_i + k_i),$$
$$\operatorname{Area}(F_i) = d_i^1 a + d_i^2 b + m_i + k_i x,$$

so

$$\operatorname{Area}(F_i) = d_i^1(a-2) + d_i^2(b-2) + k_i(x-1) + 1 > 0,$$

while the degrees verify $\sum d_i^1 = d$, $\sum d_i^2 = 1$.

When $x > 2$, taking into account our assumptions $a < 2$, $b < x$, we see that the component with $d_i^2 = 1$ verifies

$$(b-1) + k_i(x-1) > 0,$$

so $k_i \geq 0$ because $b < x$. Meanwhile the ones with $d_i^2 = 0$ verify

$$k_i(x-1) + 1 > 0,$$

so $k_i \geq 0$ because $x \geq 2$. Thus $k_i \geq 0$ for all i, and since $\sum k_i = 0$, we get all $k_i = 0$, so all Reeb asymptotics are of type $(m, 0)$. Exactly the same argument as in Section 4 then shows that $2d+1$ of the F_i have bidegree $(0,0)$ and are asymptotic to $\gamma_{(-1,0)}$ while one has bidegree $(d,1)$ and asymptotic $\gamma_{(2d+1,0)}$. The area of this subbuilding is

$$ad + b - 2d - 1 = (a-2)d + b - 1 > 0.$$

Since this inequality has to hold for arbitrarily large d, it contradicts our assumption $a < 2$.

When $x \leq 2$ we see as above that if all k_i vanish, we get a contradiction with $a < 2$. Thus at least one of the F_i must be asymptotic to $\gamma_{(-m,-k)}$, $k < 0$. Its area is

$$\mathcal{A} = d_1(a-2) + d_2(b-2) + k(x-1) + 1, \quad k \leq -1.$$

Now $1 + k(x-1) < 1$, so Lemma 3.7 rules out the possibility that $d_1 = d_2 = 0$. If $d_1 \geq 1$ (and since $a < 2$ and $b < x \leq 2$ by assumption) we get

$$a - 2 - x + 2 = a - x \geq 0,$$

which contradicts the fact that $a \leq b < x$. If $d_1 = 0$, $d_2 = 1$, we get

$$b - 2 - x + 2 \geq 0$$

so $b \geq x$. □

6. Squeezing Lagrangian tori: proof of Theorem 4

The construction is slightly delicate, so we proceed in several steps. We first construct a Lagrangian torus close to $P(2,2)$ or to $B(3)$, with several properties that we use in a second step to show that this Lagrangian torus is in fact Hamiltonian isotopic to a product torus.

Proposition 6.1. *Let $x > 2$ and $\mathcal{U}$ be an arbitrary neighbourhood of $B(3) \cap P(2,2)$. There exists a Lagrangian torus $L \subset \mathcal{U}$ which has an integral basis (e_1, e_2) of its 1-dimensional homology with*

$$\mu(e_1) = \mu(e_2) = 2, \quad \Omega(e_1) = x, \quad \Omega(e_2) = 1.$$

This Lagrangian torus bounds a solid torus $\Sigma \subset \mathbb{C}^2$ with symplectic meridian discs having boundary in the class e_2 and whose characteristic foliation is by closed leaves.

One can further impose that there exists $a, b \in \mathbb{C}$ such that:

(a) $\Sigma \subset \mathbb{C}^2 \backslash \{z = a\} \cup \{w = b\}$ *(hence the same holds for L),*

(b) $\mathrm{lk}(e_2, \{z = a\}) = \mathrm{lk}(e_2, \{w = b\}) = 0$,

(c) $\mathrm{lk}(e_1, \{z = a\}) = 1$, $\mathrm{lk}(e_1, \{w = b\}) = -1$.

Proof. Let $A > 1$ be a real number to be chosen more specifically later. Let γ be the immersed closed loop in the $z = (x_1, y_1)$-plane represented in Figure 1. It is contained in the square

$$S := \{0 < x_1 < A,\ 0 < y_1 < A\}$$

and approximates an n-times cover of ∂S. More precisely, we can parameterize γ as a map

$$[0,1]_{/0\sim 1} \to \mathbb{C}$$

(which we will also denote by γ) such that $\gamma(t) = w_0 + it$ for t small, with $\operatorname{Im} w_0 < \epsilon$ and $\operatorname{Re} w_0 > 1$. The curve γ then winds counterclockwise around ∂S on an interval $[0, 1-\delta]$, remaining embedded by spiraling slightly inwards. Finally, the segment $\gamma\big([1-\delta, 1]\big)$ is roughly horizontal, closing up the curve and intersecting the remainder of the curve $n-1$ times.

The horizontal segment of (the image of) γ which runs from close to $\operatorname{Re} z = 0$ to include $\gamma\big([1-\delta, 1]\big)$ is labelled H. This is the uppermost horizontal segment close to $\{y_1 = 0\}$. The vertical segment extending the part of γ defined by positive parameters close to 0 will be labelled V. This is the rightmost vertical segment close to $\{x_1 = A\}$ (see Figure 2). For technical reasons, we chose $0 < \varepsilon \ll 1$ and assume that $H \subset \{y_1 < \varepsilon\}$ and $V \subset \{x_1 \geq A - \varepsilon/2\}$, while γ lies in the ε-strip around ∂S.

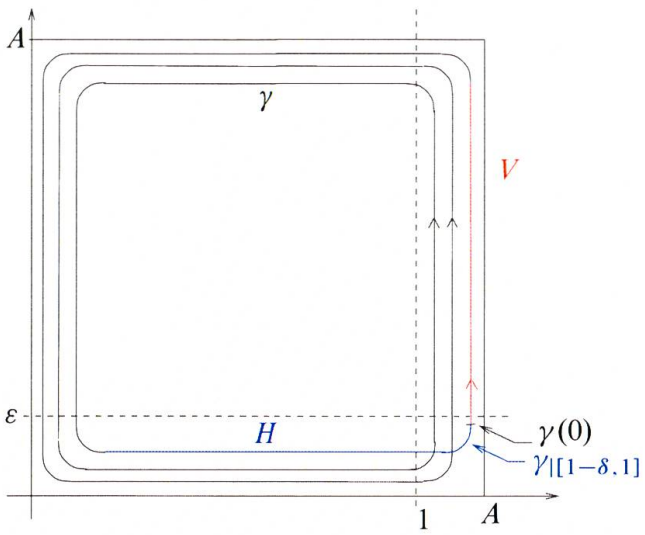

Figure 1. The curve γ.

Let $T := \partial D$ be the circle of unit area in $\mathbb{C}_w$, $w = x_2 + iy_2$. The product of γ with T is an immersed Lagrangian torus in $\mathbb{C}^2$. Denoting by

$$\widetilde{e}_1 = [\gamma \times \{*\}] \in H_1(\gamma \times T) \quad \text{and} \quad e_2 = [\{*\} \times T],$$

we have

$$\mu(\widetilde{e}_1) = 2n, \quad \mu(e_2) = 2, \quad \Omega(\widetilde{e}_1) \approx nA^2, \quad \text{and} \quad \Omega(e_2) = 1.$$

Step 1. We first construct a Lagrangian torus L as required, except for the linking conditions (b) and (c). Let $G(w)$ be a Hamiltonian function which displaces the circle T in a disc of area $2 + \varepsilon$ and satisfies $0 \le G \le 1 + \varepsilon$. Let $\chi\colon \mathbb{R} \to [0, 1]$ be a smooth function with $\chi(t) = 1$ for $t \ge A - \varepsilon$, $\chi(t) = 0$ when $t \le \epsilon$ and $0 \le \chi'(t) \le 1$ (recall $A > 1$). Let also $\Omega \subset \mathbb{C}$ be defined by $\Omega := Q_1 \cup Q_2 \cup Q_3$, with

$$\begin{aligned} Q_1 &:= \{\varepsilon < x_1 < A - \varepsilon\}, \\ Q_2 &:= \{x_1 \ge A - \varepsilon/2\}, \\ Q_3 &:= \{A - \varepsilon \le x_1 \le A\} \cap \{0 < y_1 < \varepsilon\}. \end{aligned}$$

Note that $H \subset Q_1 \cup Q_3$, $V \subset Q_2 \cup Q_3$, and $Q_1 \cap Q_2 = \emptyset$ (see Figure 2). The self-intersections of γ all lie in the region Q_3. By $\gamma \setminus H$, $\gamma \setminus V$, or $\gamma \setminus (H \cup V)$ we will mean the image of the parameterized curve restricted to the complement of the interval defining H, V or $H \cup V$ respectively.

We leave the reader to check the following straightforward facts.

Claim 6.2. *The function $f\colon \Omega \subset \mathbb{C} \to \mathbb{R}$ defined by $f_{|Q_1} := \chi(x_1)$, $f_{|Q_2} := 1 - \chi(y_1)$, and $f_{|Q_3} := 1$ is smooth, and its Hamiltonian flow Φ^t_f verifies:*

1. $\Phi^t_f | Q_3 = \mathrm{Id}$,

2. $\Phi^t_f(H \cap Q_1) \subset\subset Q_1 \cap \{0 < y_1 < A - \varepsilon\}$ *and* $\Phi^t_f(H) \cap (\gamma \backslash H) \cap Q_1 = \emptyset$,
3. $\Phi^t_f(V \cap Q_2) \subset Q_2 \cap \{A - \varepsilon/2 < x_1 < A + 1\}$ *and* $\Phi^t_f(V) \cap (\gamma \backslash V) \cap Q_2 = \emptyset$.

We finally define the function $K(z, w) := f(z)G(w)$ on $\Omega \times \mathbb{C} \subset \mathbb{C}^2$, and

$$L := \gamma \backslash (H \cup V) \times T \ \cup \ \Phi^1_K\big((H \cup V) \times T\big).$$

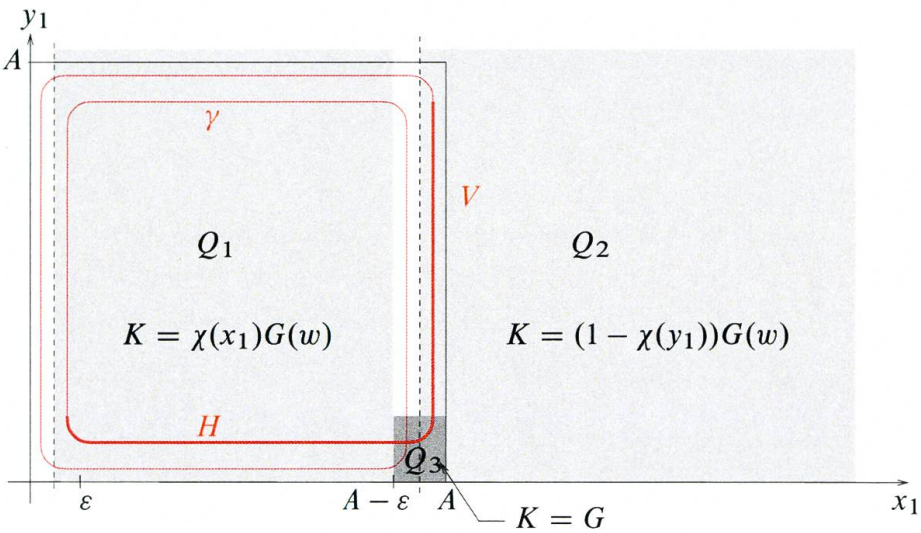

Figure 2. The curve γ, and the function K.

Since $f = 0$ near $\partial(H \cup V)$ (considering $H \cup V$ as a connected arc), and hence $K = 0$ near $\partial(H \cup V) \times T$, we see that L is an immersed torus. Moreover, also since $K = 0$ near $\partial(H \cup V) \times T$, the integral of a primitive λ of the standard symplectic form over a path σ in $(H \cup V) \times T$ is equal to its integral over $\Phi^1_K(\sigma) \subset L$. Thus the area class of L coincides with that of $\gamma \times T$. The same holds for the Maslov class, because L is Lagrangian isotopic to $\gamma \times T$.

Notice also that the Hamiltonian vector field of K is

$$G(w)\vec{X}_f(z) + f(z)\vec{X}_G(w).$$

Thus, Since $0 \leq f \leq 1$ and $0 \leq G \leq 1 + \varepsilon$, for any subset $P \subset \Omega \times \mathbb{C}$,

$$\pi_z(\Phi^t_K(P)) \subset \cup_{s \leq (1+\varepsilon)t} \Phi^s_f(\pi_z P) \quad \text{and} \quad \pi_w \Phi^t_K(P) \subset \cup_{s \leq t} \Phi^s_G(\pi_w P).$$

Hence, by points (2) and (3) of the above claim, L has no self-intersection in $Q_1 \times \mathbb{C}$, nor in $Q_2 \times \mathbb{C}$. Moreover, since $f = 1$ on Q_3, $\Phi^1_K = \mathrm{Id} \times \Phi^1_G$ on $Q_3 \times \mathbb{C}$, so

$$\Phi^1_K(H \cup V \times T) \cap \gamma \times T \cap Q_3 = \emptyset.$$

Therefore, L is embedded.

Let $p \in T$ and $q \in \gamma \setminus \mathrm{Supp}(f)$. Then the homology of L is generated by

$$\widetilde{e}_1 := [\widetilde{\gamma}] := \big[\gamma\backslash(H \cup V) \times \{p\} \cup \Phi_K^1\big(H \cup V \times \{p\}\big)\big] \quad \text{and} \quad e_2 := \big[\{q\} \times T\big].$$

We have $\mu(e_2) = 2$ and $\mu(\widetilde{e}_1) = 2n$, so $(e_1 := \widetilde{e}_1 - (n-1)e_2, e_2)$ is a Maslov 2 integral basis of $H_1(L)$, with areas

$$\mathcal{A}(e_2) = 1, \qquad \mathcal{A}(e_1) = n(A^2 - \delta) - (n-1),$$

where δ is a term that can be chosen arbitrarily small. Given x, one may find A arbitrarily close to 1, $\delta \ll 1$ and n large enough, so that $\mathcal{A}(e_1) = x$, which we assume henceforth. Thus, L is an embedded Lagrangian torus, with a Maslov two basis of areas x and 1, which lies in $[0, A+1] \times [0, A] \times D(2)$, which can be mapped into an arbitrary neighbourhood of $P(2,2)$ provided A is close to 1.

Step 2. We now show that L can in fact be taken into an arbitrary neighbourhood of $B(3) \cap P(2,2)$. Throughout $D(c)$ denotes the standard round disk of area c. Let $\varepsilon \ll 1$ and $\psi\colon [0, A+1] \times [0, A] \to D(A(A+1) + \varepsilon)$ be such that

$$\psi\big([0,A]\times[0,A]\big) \subset D(A^2+\varepsilon) \quad \text{and} \quad \psi\big([A,A+1]\times[0,\rho]\big) \subset D(A^2+\rho+\varepsilon) \quad \forall \rho.$$

Such a symplectic diffeomorphism is easy to construct by stretching the part

$$[A, A+1] \times [0, A]$$

horizontally by a factor 4, shrinking vertically by the same factor, then wrapping the long strip around the square $[0, A]^2$ (see Figure 3). The resulting domain can be mapped into the standard disk $D(A(A+1) + \varepsilon)$ such that the image of points in $[A, A+1] \times [0, A]$ with larger y coordinate (the dark rim in our figure) are mapped towards the boundary of the disk while points with smaller y coordinate (the lighter rims) are mapped closer to the disk $D(A^2)$. We can quantify this precisely by following [17, Lemma 3.3.3], so that points in $[A, A+1] \times [0, b]$ map into $D(A^2 + b + \varepsilon)$.

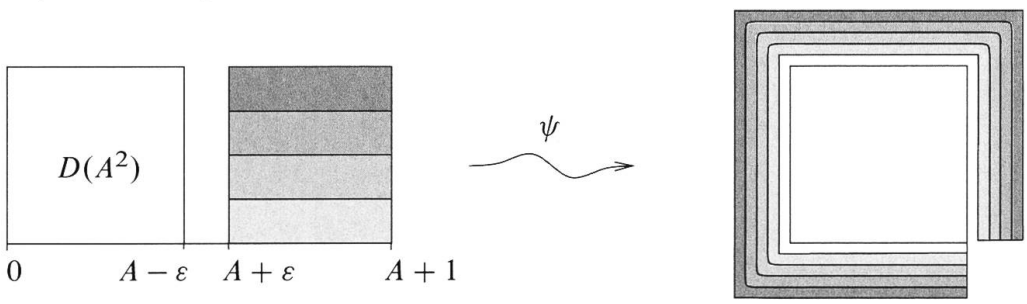

Figure 3. The projection of L onto the z-plane, and its image by ψ.

We claim that $\psi \times \mathrm{Id}(L)$ lies close to $B(3)$ provided A is close to 1. Indeed,

- if $(z, w) \in L$, with $z \in [0, A] \times [0, A]$, then $\psi \times \mathrm{Id}(z, w) = (z', w')$ verifies

$$|z'| \in D(A^2 + \varepsilon) \quad \text{and} \quad |w'| \in D(2),$$

so $(z', w') \in B(A^2 + 2 + \varepsilon)$;

- if $(z, w) \in L$ with $z \in [A, A+1] \times \{\rho\}$, then $(z, w) \in \Phi_K^1(V \times T)$. Since

$$\begin{aligned}\vec{X}_K(z, w) &= f(z)\vec{X}_G(w) + G(w)\vec{X}_f(z)\\ &= [1 - \chi(y_1)]\vec{X}_G(w) + G(w)\vec{X}_f(z) \quad \text{in } Q_2,\end{aligned}$$

$\vec{X}_K$ preserves the hyperplanes $\{y_1 = c\}$ in $Q_2 \times \mathbb{C}$, so $(z, w) = \Phi_K^1(z', w')$ with $\operatorname{Im} z = \operatorname{Im} z' = \rho$. Then,

$$\pi_w \Phi_K^1(z', w') = \Phi_G^{(1-\chi(\rho))}(w'),$$

so we may assume $w \in D(2 - \chi(\rho))$, while $z \in [A, A+1] \times \{\rho\}$. As a result,

$$\psi \times \mathrm{Id}(z, w) \in D(A^2 + \rho + \varepsilon) \times D(2 - \chi(\rho)) \subset B(A^2 + 2 + \rho - \chi(\rho) + \varepsilon).$$

Notice now that since A is close to 1 and $\chi(\rho)$ is chosen at $\mathcal{C}^0$-distance ε from the identity, and this shows that $\psi(z, w)$ lies arbitrarily close to $B(3)$.

Step 3. We now slightly alter the previous construction to achieve the linking condition. Note first that taking a in $\{\operatorname{Re} z < \varepsilon\}$ in such a way that γ winds around a exactly once achieves the correct linking between L and $\{z = a\}$. The problem is therefore with the line of the type $\{w = b\}$.

First, fix $b \in \mathbb{C}$ in the complement of $\cup \Phi_G^t(T)$, in $D(2 + \varepsilon)$. In particular, since Φ_G^1 displaces T, b is in the complement of $D \cup \Phi_G^1(D)$, where $T = \partial D$. There exists a Hamiltonian diffeomorphism Ψ with support in $D(2+\varepsilon)$, disjoint from $D \cup \Phi_G^1(D)$ and such that $\forall p \in T$, the path $\Phi_G^t(p)$ for $0 \le t \le 1$ and $\Psi \circ \Phi_G^{2-t}(p)$ for $1 \le t \le 2$ winds around b with winding -1. Such a diffeomorphism is not easy to describe in words, but easy to draw (see Figure 4).

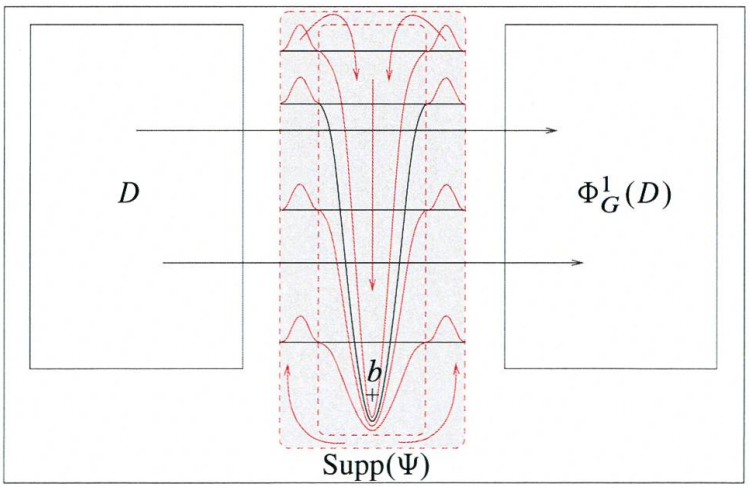

Figure 4. The diffeomorphism Ψ. The red oriented curves represent the flow lines of Ψ. The red unoriented lines represent the image of the horizontal foliation. The support of Ψ can be made arbitrarily thin, so that the whole picture fits into a disc of area $2 + \varepsilon$.

We now define

$$L_\Psi := \gamma\backslash(H \cup V) \times T \cup \Phi_K^1(H \times T) \cup \Psi(\Phi_K^1(V \times T)),$$

where $K(z, w) = f(z)G(w)$ as before and by abuse of notation we are writing $\Psi(z, w) = (z, \Psi(w))$.

Let us explain why L_Ψ satisfies all our requirements. Note first that L_Ψ is obtained by L from cutting the tube $\mathcal{T} := \Phi_K^1(V \times T)$ and pasting $\mathcal{T}_\Psi := \Psi(\Phi_K^1(V \times T))$. As $\pi_w(\Phi_K^1(\partial V \times K)) = D \cup \Phi_G^1(D)$ is disjoint from the support of Ψ we have that $\mathcal{T}_\Psi$ differs from $\mathcal{T}$ by a Hamiltonian generated by a function with compact support. Moreover, since the corresponding Hamiltonian flow is parallel to the w-plane, the flow of $\mathcal{T}$ remains disjoint from the remainder of L and so the generating function can be extended to be identically 0 near the remainder of L. We conclude that L_Ψ differs from L by a Hamiltonian diffeomorphism and in particular L_Ψ has the same Maslov and area classes as L.

We now claim that L_Ψ has the correct linking with the lines $\{z = a\}$ and $\{w = b\}$. It is clear for the former, while the latter needs a justification. First, since e_2 is represented by $\{*\} \times T$ for suitable $\{*\}$, and since $b \notin D$, $\mathrm{lk}(e_2, \{w = b\}) = 0$. Fix now $w_0 \in T$. A representative of the class $\widetilde{e}_1 = e_1 + (n-1)e_2$ is simply

$$\gamma\backslash(H \cup V) \times \{w_0\} \cup \Phi_K^1\big(H \times \{w_0\}\big) \cup \Psi\big(\Phi_K^1\big(V \times \{w_0\}\big)\big).$$

Parametrize H by $\rho \mapsto \rho + ic$, for $\rho \in [\rho_0, \rho_1]$, $\rho_0 < \varepsilon$ and $\rho_1 > A$ and $c < \varepsilon$. Then there exists a function $\widetilde{c}(\rho)$ taking values in $(0, 1)$ with,

$$\Phi_K^1(\rho + ic, w_0) = \big(\rho + i\widetilde{c}(\rho), \Phi_G^{\chi(\rho)}(w_0)\big). \tag{6.1}$$

Similarly V can be parameterized by $\rho \mapsto b + i\rho$ and there exists a $\widetilde{b}(\rho)$ with

$$\Psi\big(\Phi_K^1(b + i\rho, w_0)\big) = \big(\widetilde{b}(\rho) + i\rho, \Psi \circ \Phi_G^{1-\chi(\rho)}(w_0)\big) \tag{6.2}$$

By choice of Ψ, we therefore see that

$$\mathrm{lk}\,\big(\widetilde{e}_1, \{w = b\}\big) = -1,$$

and since $e_1 = \widetilde{e}_1 - (n-1)e_2$, that

$$\mathrm{lk}\,\big(e_1, \{w = b\}\big) = -1.$$

In view of the particular form of Ψ, whose support can clearly be given any arbitrarily small area, the arguments that showed that L can be taken into an arbitrary neighbourhood of $B(3)$ go through. Finally, L_Ψ bounds the solid torus

$$\Sigma_\Psi := \gamma\backslash(H \cup V) \times D \cup \Phi_K^1(H \times D) \cup \Psi\big(\Phi_K^1(V \times D)\big).$$

The characteristic foliation of $\gamma \times D$ is by curves $\gamma \times \{*\}$, and Σ_Ψ is obtained from $\gamma \times D$ by explicit Hamiltonian diffeomorphisms on various pieces, and therefore the characteristic leaves remain closed as required. □

Let us finally now prove that the Lagrangian tori that we have just produced are Hamiltonian isotopic to a product torus.

Proposition 6.3. *A Lagrangian torus that satisfies the requirements of Proposition 6.1 is Hamiltonian isotopic to the product torus $L(1,x)$ in $\mathbb{C}^2$.*

Proof. Let L, Σ, a, b be as in Proposition 6.1, but observe that without loss of generality $a = b = 0$. We denote $X := \mathbb{C}^2 \backslash \{z = 0\} \cup \{w = 0\}$. We wish to prove that L is Hamiltonian isotopic to $L(x,1) = S^1(x) \times S^1(1)$ in $\mathbb{C}^2$. We proceed in two steps.

Step 1 (Lagrangian isotopy). We first reposition the product torus $L(x,1)$ in X and then find a Lagrangian isotopy inside X between L and the repositioned torus.

Consider the linear symplectomorphism

$$A\colon (z,w) \mapsto \frac{1}{\sqrt{1-\lambda^2}}(z + \lambda\overline{w}, w + \lambda\overline{z}),$$

where λ is a constant satisfying $\sqrt{\frac{1}{x}} < \lambda < 1$. We observe that $A(L(x,1)) \subset X$ and denote the image by L_0. We denote the image of the standard basis of $H_1(L(x,1))$ by $f_1, f_2 \in H_1(L_0)$, so

$$\mu(f_1) = \mu(f_2) = 2 \quad \text{and} \quad \Omega(f_1) = x, \quad \Omega(f_2) = 1.$$

Also

$$\operatorname{lk}\big(f_1, \{z = 0\}\big) = 1, \quad \operatorname{lk}\big(f_1, \{w = 0\}\big) = -1,$$

and

$$\operatorname{lk}\big(f_2, \{z = 0\}\big) = 0, \quad \operatorname{lk}\big(f_2, \{w = 0\}\big) = 0.$$

Moreover the solid torus $\Sigma_0 = A\big(\{\pi|z|^2 = x, \pi|w|^2 \leq 1\}\big)$ also lies in X and has trivial characteristic foliation.

The existence of a Lagrangian isotopy between L and L_0 given the existence of Σ and Σ_0 has already been established in [6, Theorem 6.1], (see also [13, Proposition 3.4.6]). For the sake of clarity and completeness, we briefly recall an outline of the argument here.

Let us fix characteristic leaves (called core circles)

$$\Gamma \subset \Sigma \quad \text{and} \quad \Gamma_0 := A\big(S^1(x) \times \{0\}\big) \subset \Sigma_0,$$

and a meridian disk D in Σ. The return map on D of the characteristic foliation in Σ being the identity, any isotopy of circles in D starting from ∂D defines a Lagrangian isotopy of L, by considering the suspension of these loops by the characteristic

foliation in Σ. Hence an isotopy of loops that shrinks ∂D to a small loop around $D \cap \Gamma$ gives a Lagrangian isotopy from L to a small neighborhood of Γ in Σ.

Next, since the linking numbers about the axes are the same, Γ and Γ_0 are smoothly isotopic in X, and the smooth isotopy extends to a *symplectic* isotopy ψ_t (not Hamiltonian), defined in a neighbourhood of Γ, that brings Γ to Γ_0. Since these two curves are characteristic leaves, the image $\Sigma' = \psi_1(\Sigma)$ is an embedding of $S^1 \times D(\varepsilon)$ which is tangent to Σ_0 along Γ_0.

There is an obstruction to the existence of a symplectic isotopy which fixes Γ_0 and maps Σ' into Σ_0. We can find a trivialization of the meridinal disks in Σ' which is invariant under the characteristic flow (since the leaves are closed). This induces a trivialization of the tangent spaces to the meridinal disks in Σ_0 along Γ_0 but this may not be invariant under the characteristic flow here. Indeed, since the leaves of the characteristic flow on Σ_0 are also closed, leaves other than Γ_0 have a winding number about Γ_0 relative to the trivialization. We are able to correct this relative winding however, by a small perturbation of Σ_0, leaving it tangent to Σ' along Γ_0, as follows.

We identify a neighborhood of Γ_0 in X with a neighborhood of $S^1 \times \{0\} \times \{0\}$ in $S^1 \times \mathbb{R} \times \mathbb{C}$ with the product symplectic form from $S^1 \times \mathbb{R} = T^*S^1$ and $\mathbb{C}$. We can make this identification such that Σ_0 is $S^1 \times \{0\} \times \mathbb{C}$. Given this, choose a smooth function $\chi\colon \mathbb{R}^+ \to \mathbb{R}^+$ with small support, small $\mathcal{C}^0$-norm, linear with slope N near 0, and define

$$\widetilde{\Sigma}_0 := \{(\theta, \chi(|w|^2), w) \in S^1 \times \mathbb{R} \times \mathbb{C},\ |w| < 1\}.$$

The characteristic foliation of $\widetilde{\Sigma}_0$ is given by the curves

$$\theta \mapsto \left(\theta, \chi(|w|^2),\ we^{i\chi'(|w|^2)\theta}\right).$$

Thus close to Γ_0 the characteristic foliation of $\widetilde{\Sigma}_0$ can be arranged to have a winding N equal to that of Σ'.

It follows that we can find a symplectic isotopy ζ_t taking a neighborhood of Γ_0 in Σ' into $\widetilde{\Sigma}_0$. To see this, note that a smooth map sending Σ' into $\widetilde{\Sigma}_0$, preserving the characteristic foliations, mapping meridinal disks of Σ' to meridinal disks of $\widetilde{\Sigma}_0$, and acting as a symplectomorphism on a single disk, is automatically a symplectic embedding of Σ', which can be extended locally to a symplectomorphism. Since the characteristic foliations have the same winding, such a symplectomorphism is isotopic to the identity through diffeomorphisms which preserve Γ_0 and whose derivatives along Γ_0 preserve the meridinal disks. By an application of Moser's method these diffeomorphisms can be deformed near Γ_0 into symplectomorphisms.

The image $\widetilde{L} = \zeta_1(\psi_1(L))$ of L is now a circle of characteristic leaves in $\widetilde{\Sigma}_0$ intersecting a particular meridinal disk D_0 in a small circle σ_0 about Γ_0. We may assume that σ_0 lies in the region $\{|w| < \epsilon\}$ where $\chi'(|w|) = N$. Finally we

find a Lagrangian isotopy between $\widetilde{L}$ and L_0. First consider a family ξ_t of circles in D_0 connecting $\xi_0 = \partial D_0$ with $\xi_1 = \sigma_0$. We can choose this family such that if a ξ_t intersects the region where χ' is not constant, then ξ_t is one of the circles $\{|w| = c\} \cap D_0$. We then observe that the return map $D_0 \to D_0$ generated by following the characteristic foliation preserves all ξ_t, and hence the characteristic leaves through these circles form a family of Lagrangian tori giving our isotopy as required.

Step 2 (Hamiltonian isotopy). We now use classical transformations on X to modify the Lagrangian isotopy L_t between L_0 and $L = L_1$ constructed (in reverse) in Step 1, such that it becomes Hamiltonian.

Let (e_1^t, e_2^t) the continuous deformation of basis of $H_1(L_t, \mathbb{Z})$ that ends at $(e_1^1, e_2^1) = (e_1, e_2) \in H_1(L, \mathbb{Z})$, and denote by α_t, β_t the corresponding symplectic actions. Given the construction in Step 1 it is not hard to see that $e_2^0 \in H_1(L_0)$ is the class of a meridian circle $A(\{z = *\})$ and so $\beta_1 = \beta_0 = 1$. Since Lagrangian isotopies preserve the Maslov class, $e_1^0 \in H_1(L_0)$ is a Maslov 2 class with the same linking as e_1. There is only one such class, namely f_1, and so $\alpha_1 = \alpha_0 = x$. Our goal is to deform the isotopy L_t inside X relative to $L_{0,1}$ such that $\alpha_t = x$ and $\beta_t = 1$ for all t. The isotopy will then be Hamiltonian as required.

Recall that $X = \mathbb{C}^2 \backslash \{z = 0\} \cup \{w = 0\}$ is symplectomorphic to a subset of $T^*\mathbb{T}^2$, for instance to $\{(\theta_1, \theta_2, p_1, p_2) \in \mathbb{T}^2 \times (\mathbb{R}_+^*)^2\}$.

Start first by applying dilations

$$d_\lambda \colon (\theta_1, \theta_2, p_1, p_2) \mapsto (\theta_1, \theta_2, \lambda p_1, \lambda p_2).$$

These dilations are conformally symplectic so they preserve the class of Lagrangian submanifolds. They correspond to standard dilations $(\lambda z, \lambda w)$ in $\mathbb{C}^2$ and thus $L_t' := d_{1/\beta_t} L_t$ is a Lagrangian isotopy from L to L_0 in X with actions α_t' and $\beta_t' \equiv 1$.

To correct the action α_t' we use translations

$$\tau_c^1 \colon (\theta_1, \theta_2, p_1, p_2) \mapsto (\theta_1, \theta_2, p_1 + c, p_2)$$

or

$$\tau_c^2 \colon (\theta_1, \theta_2, p_1, p_2) \mapsto (\theta_1, \theta_2, p_1, p_2 + c).$$

These transformations are symplectic, and they preserve the set $\{p_1, p_2 > 0\}$, and hence X, provided $c > 0$. In $\mathbb{C}^2$, they correspond to inflating either the line $\{z = 0\}$ or $\{w = 0\}$, that is removing these lines and pasting $D(c) \times \mathbb{C}$ in their place. Because of the linking condition, we see that the area class of $\tau_c^1 L_t'$ is $(\alpha_t' + c, 1)$ while that of $\tau_c^2 L_t'$ is $(\alpha_t' - c, 1)$. Thus, defining $L_t'' := \tau^1_{x-\alpha_t'} L_t'$ when $\alpha_t' \leq x$ and $L_t'' := \tau^2_{\alpha_t'-x} L_t'$ when $\alpha_t' \geq x$, we get a Lagrangian isotopy from L to L_0 by Lagrangian submanifolds with fixed action in $\mathbb{C}^2$, and hence a Hamiltonian isotopy as required. □

References

[1] F. Bourgeois, A Morse–Bott approach to contact homology, Ph. D. thesis, Stanford University, 2002. MR 2703292

[2] F. Bourgeois, Y. Eliashberg, H. Hofer, K. Wysocki, and E. Zehnder, Compactness results in symplectic field theory, *Geom. Topol.*, **7** (2003), 799–888. Zbl 1131.53312 MR 2026549

[3] K. Cieliebak and K. Mohnke, Punctured holomorphic curves and Lagrangian embeddings, *Invent. Math.*, **212** (2018), no. 1, 213–295. Zbl 1396.53105 MR 3773793

[4] Y. Chekanov and F. Schlenk, Lagrangian product tori in symplectic manifolds, *Comment. Math. Helv.*, **91** (2016), no. 3, 445–475. Zbl 1373.53115 MR 3541716

[5] K. Cristianson and J. Nelson, Symplectic embeddings of four-dimensional polydisks into balls, *Algebr. Geom. Topol.*, **18** (2018), no. 4, 2151–2178. Zbl 1392.53083 MR 3797064

[6] G. Dimitroglou Rizell, E. Goodman, and A. Ivrii, Lagrangian isotopy of tori in $S^2 \times S^2$ and $\mathbb{C}P^2$, *Geom. Funct. Anal.*, **26** (2016), no. 5, 1297–1358. Zbl 1358.53089 MR 3568033

[7] J. Gutt, Generalized Conley–Zehnder index, *Ann. Fac. Sci. Toulouse Math. (6)*, **23** (2014), no. 4, 907–932. Zbl 1330.37020 MR 3270429

[8] R. Hind, Stabilized symplectic embeddings, in *Complex and symplectic geometry*, 85–93, Springer INdAM Ser., 21, Springer, Cham, 2017. Zbl 1387.57042 MR 3645308

[9] R. Hind and E. Kerman, New obstructions to symplectic embeddings, *Invent. Math.*, **196** (2014), no. 2, 383–452. Zbl 1296.53160 MR 3193752

[10] R. Hind and E. Kerman, Correction to: New obstructions to symplectic embeddings [MR 3193752], *Invent. Math.*, **214** (2018), no. 2, 1023–1029. Zbl 1407.53094 MR 3867635

[11] M. Hutchings, Lecture notes on embedded contact homology, in *Contact and symplectic topology*, 389–484, Bolyai Soc. Math. Stud., 26, János Bolyai Math. Soc., Budapest, 2014. Zbl 1432.53126 MR 3220947

[12] M. Hutchings, Beyond ECH capacities, *Geom. Topol.*, **20** (2016), no. 2, 1085–1126. Zbl 1338.53119 MR 3493100

[13] A. Ivrii, Lagrangian unknottedness of tori in certain symplectic 4-manifolds, Ph. D. thesis, Stanford University, 2003. MR 2705130

[14] D. McDuff, A remark on the stabilized symplectic embedding problem for ellipsoids, *Eur. J. Math.*, **4** (2018), no. 1, 356–371. Zbl 1393.53080 MR 3782228

[15] Y-G. Oh and K. Zhu, Floer trajectories with immersed nodes and scale-dependent gluing, *J. Symplectic Geom.*, **9** (2011), no. 4, 483–636. Zbl 1257.53117 MR 2900788

[16] J. Robbin and D. Salamon, The Maslov index for paths, *Topology*, **32** (1993), no. 4, 827–844. Zbl 0798.58018 MR 1241874

[17] F. Schlenk, *Embedding problems in symplectic geometry*, De Gruyter Expositions in Mathematics, 40, Walter de Gruyter GmbH & Co. KG, Berlin, 2005. Zbl 1073.53117 MR 2147307

[18] M. Schwarz, Cohomology operations from S^1-cobordisms in Floer homology, Ph. D. thesis, ETH Zürich, 1995.

[19] C. Wendl, Automatic transversality and orbifolds of punctured holomorphic curves in dimension four, *Comment. Math. Helv.*, **85** (2010), no. 2, 347–407. Zbl 1207.32021 MR 2595183

[20] K. Zehmisch, Holomorphic jets in symplectic manifolds, *J. Fixed Point Theory Appl.*, **17** (2015), no. 2, 379–402. Zbl 1344.53070 MR 3397123

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